Research Article

Assessment of the Exact Solutions of the Space and Time Fractional Benjamin-Bona-Mahony Equation via the G'/G -Expansion Method, Modified Simple Equation Method, and Liu's Theorem

Olusola Kolebaje and Oyebola Popoola

Theoretical Physics Group, Department of Physics, University of Ibadan, Ibadan, Nigeria

Correspondence should be addressed to Olusola Kolebaje; olusolakolebaje2008@gmail.com

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Exact travelling wave solutions to the space and time fractional Benjamin-Bona-Mahony (BBM) equation defined in the sense of Jumarie's modified Riemann-Liouville derivative via the (G'/G) expansion and the modified simple equation methods are presented in this paper. A fractional complex transformation was applied to turn the fractional BBM equation into an equivalent integer order ordinary differential equation. New complex type travelling wave solutions to the space and time fractional BBM equation were obtained with Liu's theorem. The modified simple equation method is not effective for constructing solutions to the fractional BBM equation.

1. Introduction

Nonlinear partial differential equations arise in a large number of physics, mathematics, and engineering problems. In the Soliton theory, the study of exact solutions to these nonlinear equations plays a very germane role, as they provide much information about the physical models they describe. In recent times, it has been found that many physical, chemical, and biological processes are governed by nonlinear partial differential equations of noninteger or fractional order [1–4].

Various powerful methods have been employed to construct exact travelling wave solutions to nonlinear partial differential equations. These methods include the inverse scattering transform [5], the Backlund transform [6, 7], the Darboux transform [8], the Hirota bilinear method [9], the tanh-function method [10, 11], the sine-cosine method [12], the exp-function method [13], the generalized Riccati equation [14], the homogenous balance method [15], the first integral method [16, 17], the (G'/G) expansion method [18, 19], and the modified simple equation method [20–22].

In this paper, we apply the (G'/G) expansion method and the modified simple equation method to construct travelling wave solutions to the space and time fractional Benjamin-Bona-Mahony (BBM) equation in the sense of Jumarie's modified Riemann-Liouville derivative via a fractional complex transformation and we further get new complex type solutions to the equation by applying Liu's theorem [23]. The Benjamin-Bona-Mahony equation is of the following form:

$$\frac{\partial^{\nu} u(x, t)}{\partial t} + \frac{\partial^{\sigma} u(x, t)}{\partial x} + u(x, t) \frac{\partial^{\sigma} u(x, t)}{\partial x} - \frac{\partial^{2\sigma} \partial^{\nu} u(x, t)}{\partial x^2 \partial t} = 0. \quad (1)$$

This equation was introduced in [24] as an improvement of the Korteweg-de Vries equation (KdV equation) for modelling long waves of small amplitude in 1 + 1 dimensions. It is used in modelling surface waves of long wavelength in liquids, acoustic gravity waves in compressible fluids, and acoustic waves in anharmonic crystals.

Jumarie's modified Riemann-Liouville derivative of order σ with respect to x is defined as [25]

$$D_x^\sigma f(x) = \begin{cases} \frac{1}{\Gamma(1-\sigma)} \int_0^x (x-\xi)^{-\sigma-1} [f(\xi) - f(0)] d\xi, & \sigma < 0, \\ \frac{1}{\Gamma(1-\sigma)} \frac{d}{dx} \int_0^x (x-\xi)^{-\sigma} [f(\xi) - f(0)] d\xi, & 0 < \sigma < 1, \\ [f^{(\sigma-n)}(x)]^{(n)}, & n \leq \sigma < n+1, n \geq 1. \end{cases} \quad (2)$$

Some useful properties of the modified Riemann-Liouville derivative are listed below [25, 26]:

$$D_x^\sigma x^k = \frac{\Gamma(1+k)}{\Gamma(1+k-\sigma)} x^{k-\sigma}, \quad (3)$$

$$D_x^\sigma (f(x)g(x)) = g(x)D_x^\sigma f(x) + f(x)D_x^\sigma g(x).$$

2. Description of the (G'/G) Expansion Method

Here, we provide a brief explanation of the (G'/G) expansion method for finding travelling wave solutions of nonlinear fractional partial differential equations. Suppose that the nonlinear evolution equation is in the following form:

$$\wp(u, D_t^\gamma u, D_x^\sigma u, D_t^{2\gamma} u, D_x^{2\sigma} u, D_x^\sigma D_t^\gamma u, \dots) = 0, \quad (4)$$

$$0 < \sigma, \gamma \leq 1,$$

where $\wp$ is a polynomial of $u(x, t)$ and its derivatives (integer and fractional) with respect to x and t . σ and γ are parameters that describe the order of the space and time derivatives, respectively.

Theorem 1. *Fractional Complex Transformation.* To transform (4) into a nonlinear ordinary differential equation (ODE)

of integer order by applying a fractional complex transformation proposed by Li and He [27],

$$u(x, t) = U(\xi), \quad \xi = \frac{x^\sigma}{\Gamma(1+\sigma)} - \frac{ct^\gamma}{\Gamma(1+\gamma)}, \quad (5)$$

where c is an arbitrary constant and (4) reduces to a nonlinear integer order ODE of the following form:

$$P(u, u', u'', u''', \dots) = 0. \quad (6)$$

The method assumes that the solution to (6) can be expressed as a polynomial in (G'/G) :

$$u(\xi) = \sum_{i=0}^m \alpha_i \left[\frac{G'(\xi)}{G(\xi)} \right]^i, \quad \alpha_m \neq 0, \quad (7)$$

where $\alpha_0, \alpha_1, \dots, \alpha_m$ are constants to be determined and $G = G(\xi)$ is a solution of the linear ordinary differential equation of the following form:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (8)$$

where λ and μ are arbitrary constants. The general solution of (8) gives [28] the following:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[\frac{C_1 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right) + C_2 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right)}{C_1 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right) + C_2 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right)} \right] - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0 \text{ Hyperbolic}, \\ \frac{\sqrt{4\mu - \lambda^2}}{2} \left[\frac{-C_1 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right) + C_2 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right)}{C_1 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right) + C_2 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right)} \right] - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0 \text{ Trigonometric}, \\ \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0 \text{ Rational}. \end{cases} \quad (9)$$

Equation (6) is integrated as long as all the terms contain derivatives, where integration constants are considered to be zero. To determine m , one considers the homogenous balance between the highest order derivative and the highest order nonlinear term(s).

Substitute (7) with the determined value of m into (6), and collect all terms with the same order of (G'/G) together. If the coefficients of $(G'/G)^i$ vanish separately, one has a set of algebraic equations in $\alpha_0, \alpha_1, \dots, \alpha_m, c, \lambda$ and μ that is solved with the aid of Mathematica.

Finally, substituting $\alpha_0, \alpha_1, \dots, \alpha_m, c$ and the general solution to (8) into (7) yields the travelling wave solution of (4).

Theorem 2 (Liu's theorem [23]). *If a nonlinear evolution equation has a kink-type solution in the form of*

$$u(\xi) = P_k(\tanh[\psi\xi]), \quad (10)$$

where P_k is a polynomial of degree k , then it has a certain kink-bell-type solution in the following form:

$$u(\xi) = P_k(\tanh[2\psi\xi] \pm i \operatorname{sech}[2\psi\xi]), \quad (11)$$

where i is the imaginary number unit.

3. Description of the Modified Simple Equation Method

Here, we highlight the basic ideas of the modified simple equation method. Suppose that the fractional nonlinear evolution equation is in the following form:

$$\wp(u, D_t^\gamma u, D_x^\sigma u, D_t^{2\gamma} u, D_x^{2\sigma} u, D_x^\sigma D_t^\gamma u, \dots) = 0, \quad (12)$$

where $\wp$ is a polynomial of u and its fractional partial derivatives with respect to x and t . Suppose a fractional complex transformation (see (5)) which allows us to reduce (12) to an ordinary differential equation (ODE) of the following form:

$$P(u, u', u'', \dots) = 0. \quad (13)$$

We suppose that the solution to (13) can be expressed as a polynomial of (ϕ'/ϕ) in the following form:

$$u(\xi) = \sum_{i=0}^N A_i \left[\frac{\phi'(\xi)}{\phi(\xi)} \right]^i, \quad (14)$$

where N is a positive integer obtained by balancing the highest order derivatives and the highest order nonlinear terms in (13). A_i ($i = 0, 1, 2, \dots, N$) are arbitrary constants to be determined such that $A_N \neq 0$ and $\phi(\xi)$ is an unknown function to be determined and is not a solution of any predefined differential equation.

We substitute (14) and its derivatives into (13) taking into account the function $\phi(\xi)$. As a result of this substitution we obtain a polynomial of $\phi^{-j}(\xi)$ with the derivatives of $\phi(\xi)$. Equating all the coefficients of $\phi^{-j}(\xi)$ to zero yields a system of equations which can be solved to obtain A_i and $\phi(\xi)$. Finally, substituting the values of A_i , $\phi(\xi)$ and its derivative $\phi'(\xi)$ into (14) gives the exact solution of (12).

4. Application

4.1. Solutions for Space and Time Fractional BBM Equation via (G'/G) Expansion Method. In this section, we apply the (G'/G) expansion method to construct travelling wave solution of the space and time fractional Benjamin-Bona-Mahony equation. Consider the space and time fractional

BBM equation (1) which can be written in subscript notation as

$$D_t^\gamma u + D_x^\sigma u_x + u D_x^\sigma u_x - D_x^{2\sigma} D_t^\gamma u = 0. \quad (15)$$

We make the transformation $u(x, t) = u(\xi)$, $\xi = x^\sigma/\Gamma(1 + \sigma) - ct^\gamma/\Gamma(1 + \gamma)$. Equation (15) becomes

$$(1 - c)u' + uu' + cu''' = 0. \quad (16)$$

Integrating (16) once with respect to ξ and setting the integration constant to zero yields

$$(1 - c)u + \frac{u^2}{2} + cu'' = 0. \quad (17)$$

To get m , we consider the homogenous balance between the highest order derivative u'' and the highest order nonlinear term u^2 :

$$2m = m + 2 \implies m = 2. \quad (18)$$

Then (7) becomes

$$u(\xi) = \alpha_2 \left(\frac{G'}{G} \right)^2 + \alpha_1 \left(\frac{G'}{G} \right) + \alpha_0, \quad \text{where } \alpha_2 \neq 0, \quad (19)$$

where α_0, α_1 , and α_2 are constants to be determined later. From (19),

$$\begin{aligned} u'(\xi) &= -2\alpha_2 \left(\frac{G'}{G} \right)^3 + (-2\alpha_2\lambda - \alpha_1) \left(\frac{G'}{G} \right)^2 \\ &\quad + (-2\alpha_2\mu - \alpha_1\lambda) \left(\frac{G'}{G} \right) - \alpha_1\mu, \\ u''(\xi) &= 6\alpha_2 \left(\frac{G'}{G} \right)^4 + (10\alpha_2\lambda + 2\alpha_1) \left(\frac{G'}{G} \right)^3 \\ &\quad + (8\alpha_2\mu + 4\alpha_2\lambda^2 + 3\alpha_1\lambda) \left(\frac{G'}{G} \right)^2 \\ &\quad + (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2) \left(\frac{G'}{G} \right) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu. \end{aligned} \quad (20)$$

Substituting (19) and its derivatives into (17) and collecting all terms with the same power of (G'/G) together yields a simultaneous set of nonlinear algebraic equations as follows:

$$\begin{aligned} \left(\frac{G'}{G} \right)^0 : & 2(1 - c)\alpha_0 + \alpha_0^2 + 4\alpha_2 c\mu^2 + 2\alpha_1 c\lambda\mu = 0, \\ \left(\frac{G'}{G} \right)^1 : & (1 - c)\alpha_1 + \alpha_0\alpha_1 + 6\alpha_2 c\lambda\mu + 2\alpha_1 c\mu + \alpha_1 c\lambda^2 = 0, \end{aligned}$$

$$\left(\frac{G'}{G}\right)^2 : 2(1-c)\alpha_2 + 2\alpha_0\alpha_2 + \alpha_1^2 + 16\alpha_2c\mu + 6\alpha_1c\lambda + 8\alpha_2c\lambda^2 = 0,$$

$$\left(\frac{G'}{G}\right)^3 : \alpha_1\alpha_2 + 2\alpha_1c + 10\alpha_2c\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^4 : \alpha_2^2 + 12\alpha_2c = 0.$$

(21)

Solving this algebraic system of equations with the aid of Mathematica yields the following solution:

$$\alpha_2 = -12c, \quad \alpha_1 = -12c\lambda, \quad \alpha_0 = -1 + c - c\lambda^2 - 8c\mu. \quad (22)$$

Substituting the solution to the algebraic equation and the general solution to (8) into (19), we obtain three types of travelling wave solutions of the space and time fractional Benjamin-Bona-Mahony equation.

Case 1. When $\lambda^2 - 4\mu > 0$, we obtain the following hyperbolic function solutions:

$$\begin{aligned} u(\xi) = & -12c \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[\frac{C_1 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right) + C_2 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right)}{C_1 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right) + C_2 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right)} - \frac{\lambda}{2} \right]^2 \right. \\ & - 12c\lambda \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left[\frac{C_1 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right) + C_2 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right)}{C_1 \cosh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right) + C_2 \sinh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)\xi\right)} - \frac{\lambda}{2} \right] \right. \\ & \left. \left. - 1 + c - c\lambda^2 - 8c\mu, \right. \right. \end{aligned} \quad (23)$$

where $\xi = x^\sigma/\Gamma(1+\sigma) - ct^\gamma/\Gamma(1+\gamma)$. If we set $C_1 = 0$ and $C_2 \neq 0$ in (23), we obtain

$$\begin{aligned} u_1(\xi) = & -1 + c + 2c(\lambda^2 - 4\mu) - 3c(\lambda^2 - 4\mu) \\ & \times \coth^2 \left[\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{x^\sigma}{\Gamma(1+\sigma)} - c \frac{t^\gamma}{\Gamma(1+\gamma)} \right) \right]. \end{aligned} \quad (24)$$

If we set $C_1 \neq 0$ and $C_2 = 0$ in (23), we obtain

$$\begin{aligned} u_2(\xi) = & -1 + c + 2c(\lambda^2 - 4\mu) - 3c(\lambda^2 - 4\mu) \\ & \times \tanh^2 \left[\frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{x^\sigma}{\Gamma(1+\sigma)} - c \frac{t^\gamma}{\Gamma(1+\gamma)} \right) \right]. \end{aligned} \quad (25)$$

Applying Theorem 1 to (24) yields two further solutions:

$$\begin{aligned} u_3(\xi) = & -1 + c + 2c(\lambda^2 - 4\mu) - 3c(\lambda^2 - 4\mu) \\ & \times \left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\xi\right) + i \operatorname{sech}\left(\sqrt{\lambda^2 - 4\mu}\xi\right) \right]^{-2}, \\ u_4(\xi) = & -1 + c + 2c(\lambda^2 - 4\mu) - 3c(\lambda^2 - 4\mu) \\ & \times \left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\xi\right) - i \operatorname{sech}\left(\sqrt{\lambda^2 - 4\mu}\xi\right) \right]^{-2}, \end{aligned} \quad (26)$$

where $\xi = x^\sigma/\Gamma(1+\sigma) - ct^\gamma/\Gamma(1+\gamma)$.

Applying Theorem 1 to (25) yields two further solutions:

$$\begin{aligned} u_5(\xi) = & -1 + c + 2c(\lambda^2 - 4\mu) - 3c(\lambda^2 - 4\mu) \\ & \times \left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\xi\right) + i \operatorname{sech}\left(\sqrt{\lambda^2 - 4\mu}\xi\right) \right]^2, \\ u_6(\xi) = & -1 + c + 2c(\lambda^2 - 4\mu) - 3c(\lambda^2 - 4\mu) \\ & \times \left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\xi\right) - i \operatorname{sech}\left(\sqrt{\lambda^2 - 4\mu}\xi\right) \right]^2, \end{aligned} \quad (27)$$

where $\xi = x^\sigma/\Gamma(1+\sigma) - ct^\gamma/\Gamma(1+\gamma)$.

Case 2. When $\lambda^2 - 4\mu < 0$, we obtain the following trigonometric function solutions:

FIGURE 1: Shape of (24) with $c = 1$, $\lambda = 3$, $\mu = 2$, and $0 \leq x, t \leq 10$ for (a) $\sigma = \gamma = 1$, (b) $\sigma = 0.5$ and $\gamma = 1$, (c) $\sigma = 1$ and $\gamma = 0.5$, and (d) $\sigma = \gamma = 0.5$.

$$\begin{aligned}
 u(\xi) = & -12c \left(\frac{\sqrt{4\mu - \lambda^2}}{2} \left[\frac{-C_1 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right) + C_2 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right)}{C_1 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right) + C_2 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right)} - \frac{\lambda}{2} \right]^2 \right. \\
 & - 12c\lambda \left(\frac{\sqrt{4\mu - \lambda^2}}{2} \left[\frac{-C_1 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right) + C_2 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right)}{C_1 \cos\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right) + C_2 \sin\left(\left(\sqrt{4\mu - \lambda^2}/2\right)\xi\right)} - \frac{\lambda}{2} \right] \right. \\
 & \left. \left. - 1 + c - c\lambda^2 - 8c\mu, \right) \right.
 \end{aligned} \tag{28}$$

where $\xi = x^\sigma/\Gamma(1+\sigma) - ct^\gamma/\Gamma(1+\gamma)$. If we set $C_1 = 0$ and $C_2 \neq 0$ in (28), we obtain

If we set $C_1 \neq 0$ and $C_2 = 0$ in (28), we obtain

$$\begin{aligned}
 u_7(\xi) = & -1 + c - 2c(4\mu - \lambda^2) - 3c(4\mu - \lambda^2) \\
 & \times \cot^2 \left[\frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{x^\sigma}{\Gamma(1+\sigma)} - c \frac{t^\gamma}{\Gamma(1+\gamma)} \right) \right].
 \end{aligned} \tag{29}$$

$$\begin{aligned}
 u_8(\xi) = & -1 + c - 2c(4\mu - \lambda^2) - 3c(4\mu - \lambda^2) \\
 & \times \tan^2 \left[\frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{x^\sigma}{\Gamma(1+\sigma)} - c \frac{t^\gamma}{\Gamma(1+\gamma)} \right) \right].
 \end{aligned} \tag{30}$$

FIGURE 2: Shape of (25) with $c = 1$, $\lambda = 3$, $\mu = 2$, and $0 \leq x, t \leq 10$ for (a) $\sigma = \gamma = 1$, (b) $\sigma = 0.5$ and $\gamma = 1$, (c) $\sigma = 1$ and $\gamma = 0.5$, and (d) $\sigma = \gamma = 0.5$.

Case 3. When $\lambda^2 - 4\mu = 0$, we obtain a rational function solution:

$$u(\xi) = -12c \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2} \right)^2 - 12c\lambda \left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2} \right) - 1 + c - c\lambda^2 - 8c\mu, \quad (31)$$

where $\xi = x^\sigma/\Gamma(1+\sigma) - ct^\gamma/\Gamma(1+\gamma)$. If we set $C_1 = 0$ and $C_2 \neq 0$ in (31), we obtain

$$u_9(\xi) = -1 + c - \frac{12}{(x^\sigma/\Gamma(1+\sigma) - c(t^\gamma/\Gamma(1+\gamma)))^2}. \quad (32)$$

If we set $C_1 \neq 0$ and $C_2 = 0$ in (31), we obtain

$$u_{10}(\xi) = -1 + c. \quad (33)$$

4.2. Solutions for Space and Time Fractional BBM Equation via MSE Method. Now, we apply the modified simple equation method to construct travelling wave solutions of the space and time fractional Benjamin-Bona-Mahony equation. The procedure for transforming the nonlinear fractional PDE into an ODE in the MSE method is the same as that of the (G'/G) expansion method. Hence, from (17), we have

$$(1-c)u + \frac{u^2}{2} + cu'' = 0. \quad (34)$$

We get $N = 2$ by considering the homogenous balance between the highest order derivative and the highest order nonlinear term. Then (14) becomes

$$u(\xi) = A_2 \left(\frac{\phi'(\xi)}{\phi(\xi)} \right)^2 + A_1 \left(\frac{\phi'(\xi)}{\phi(\xi)} \right) + A_0, \quad (35)$$

where A_0 , A_1 , and A_2 are constants to be determined such that $A_2 \neq 0$ and $\phi(\xi)$ is an unidentified function to be determined. It is easy to find that

$$\begin{aligned} u' &= A_1 \left(\frac{\phi''}{\phi} \right) - A_1 \left(\frac{\phi'}{\phi} \right)^2 + 2A_2 \left(\frac{\phi' \phi''}{\phi^2} \right) - 2A_2 \left(\frac{\phi'}{\phi} \right)^3, \\ u'' &= A_1 \left(\frac{\phi'''}{\phi} \right) - 3A_1 \left(\frac{\phi' \phi''}{\phi^2} \right) + 2A_1 \left(\frac{\phi'}{\phi} \right)^3 \\ &\quad + 2A_2 \left(\frac{\phi' \phi'''}{\phi^2} \right) + 2A_2 \left(\frac{\phi''}{\phi} \right)^2 \\ &\quad - 10A_2 \left(\frac{\phi'^2 \phi''}{\phi^3} \right) + 6A_2 \left(\frac{\phi'}{\phi} \right)^4, \end{aligned}$$

$$\begin{aligned}
 u^2 = & A_0^2 + 2A_0A_1 \left(\frac{\phi'}{\phi} \right) + (2A_0A_2 + A_1^2) \left(\frac{\phi'}{\phi} \right)^2 \\
 & + 2A_1A_2 \left(\frac{\phi'}{\phi} \right)^3 + A_2^2 \left(\frac{\phi'}{\phi} \right)^4.
 \end{aligned} \quad (36)$$

Substituting these values into (17) and equating the coefficients of ϕ^0 , ϕ^{-1} , ϕ^{-2} , ϕ^{-3} , and ϕ^{-4} , respectively, to zero, we obtain

$$2(1-c)A_0 + A_0^2 = 0, \quad (37)$$

$$2(1-c)A_1\phi' + 2A_0A_1\phi' + 2cA_1\phi''' = 0, \quad (38)$$

$$2(1-c)A_2\phi'^2 + 2A_0A_2\phi'^2 + A_1^2\phi'^2 \quad (39)$$

$$-6cA_1\phi'\phi'' + 4cA_2\phi'\phi''' + 4cA_2\phi''^2 = 0,$$

$$2A_1A_2\phi'^3 + 4cA_1\phi'^3 - 20cA_2\phi'^2\phi'' = 0, \quad (40)$$

$$A_2^2\phi'^4 + 12cA_2\phi'^4 = 0. \quad (41)$$

Solving (37) and (41), respectively, yields

$$\begin{aligned}
 A_0 = 0, 2(c-1), \quad A_2 = -12c \\
 (\text{where } c \neq 0) \text{ since } A_2 \neq 0.
 \end{aligned} \quad (42)$$

From (38), (39), and (40), we have

$$\begin{aligned}
 A_1 = \frac{12c\phi''}{\phi'}, \quad \text{or} \quad A_1 = 0, \quad \phi'' = 0, \\
 \frac{\phi'''}{\phi'} = \frac{c-1-A_0}{c}.
 \end{aligned} \quad (43)$$

Case 1. When $A_1 = 12c\phi''/\phi'$, we get $A_1 = \pm 12\sqrt{c(c-1)}$. This yields an absurd solution and hence this case is discarded.

Case 2. When $A_1 = 0$ and $\phi'' = 0$, we have

$$\phi'(\xi) = \alpha_1, \quad (44)$$

where α_1 is a constant of integration. Integrating (44) with respect to ξ , we obtain

$$\phi(\xi) = \alpha_1\xi + \alpha_2, \quad (45)$$

where α_2 is a constant of integration. Substituting the value of ϕ' and ϕ from (44) and (45), respectively, into (35) yields

$$u(\xi) = -12c \left(\frac{\alpha_1}{\alpha_1\xi + \alpha_2} \right)^2 + A_0, \quad (46)$$

when $A_0 = 0$, we obtain the exact solution of (15) as

$$u(x, t) = -12c \left[\frac{\alpha_1}{\alpha_1\xi + \alpha_2} \right]^2, \quad (47)$$

when $A_0 = 2(c-1)$, we obtain the exact solution of (15) as

$$u(x, t) = -12c \left[\frac{\alpha_1}{\alpha_1\xi + \alpha_2} \right]^2 + 2(c-1). \quad (48)$$

Since α_1 and α_2 are arbitrary constants, therefore, if we set $\alpha_1 = 1$ and $\alpha_2 = 0$, we have the solutions (47) and (48) as

$$u_{11} = -\frac{12c}{(x^\sigma/\Gamma(1+\sigma) - c(t^\gamma/\Gamma(1+\gamma)))^2}, \quad (49)$$

$$u_{12} = -\frac{12c}{(x^\sigma/\Gamma(1+\sigma) - c(t^\gamma/\Gamma(1+\gamma)))^2} + 2(c-1).$$

Bulent and Erdal [29] used direct algebraic method to get new complex travelling wave solutions to the BBM equation (15) by employing a transformation given by $u(x, t) = u(\xi)$, $\xi = ik(x - ct)$. The results reported by [29] are

$$u_{1,1} = -1 + c + 8ck^2\alpha + 12ck^2 \quad (50)$$

$$\times \left[-\sqrt{-\alpha} \tanh(ik\sqrt{-\alpha}(x - ct)) \right]^2, \quad \alpha < 0,$$

$$u_{1,2} = -1 + c + 8ck^2\alpha + 12ck^2 \quad (51)$$

$$\times \left[-\sqrt{-\alpha} \coth(ik\sqrt{-\alpha}(x - ct)) \right]^2, \quad \alpha < 0,$$

$$u_{1,3} = -1 + c + 8ck^2\alpha + 12ck^2 \quad (52)$$

$$\times \left[\sqrt{\alpha} \tan(ik\sqrt{\alpha}(x - ct)) \right]^2, \quad \alpha > 0,$$

$$u_{1,4} = -1 + c + 8ck^2\alpha + 12ck^2 \quad (53)$$

$$\times \left[-\sqrt{\alpha} \cot(ik\sqrt{\alpha}(x - ct)) \right]^2, \quad \alpha > 0,$$

$$u_{1,5} = -\frac{1}{ik(x - ct)}, \quad \alpha = 0. \quad (54)$$

A generalized (G'/G) expansion method which uses the Klein-Gordon equation as the auxiliary equation was applied by Yanhong and Baodan [30] to a form of BBM equation given by

$$u_t + u_x + uu_x - ku_{xxt} = 0. \quad (55)$$

FIGURE 3: Shape of (29) with $c = 1$, $\lambda = 2$, $\mu = 2$, and $0 \leq x, t \leq 10$ for (a) $\sigma = \gamma = 1$, (b) $\sigma = 0.5$ and $\gamma = 1$, (c) $\sigma = 1$ and $\gamma = 0.5$, and (d) $\sigma = \gamma = 0.5$.

The travelling wave solutions to the BBM equation obtained by [30] using a transformation given by $u(x, t) = u(\xi)$, $\xi = x - ct$ are

$$u_{2,1} = -\frac{12kc}{\xi^2} + c - 1, \quad (56)$$

$$u_{2,2} = \frac{48kcB}{[e^\xi + Be^{-\xi}]^2} - 4kc + c - 1, \quad (57)$$

$$u_{2,3} = -12kc \operatorname{cosech}^2 \xi - 4kc + c - 1, \quad (58)$$

$$u_{2,4} = 12kc \operatorname{sech}^2 \xi - 4kc + c - 1, \quad (59)$$

$$u_{2,5} = -12kc \tanh^2 \xi - 12kc \coth^2 \xi + 8kc + c - 1, \quad (60)$$

$$u_{2,6} = -\frac{12kc(1+B^2)}{[B \sin \xi + \cos \xi]^2} + 4kc + c - 1, \quad (61)$$

$$u_{2,7} = -12kc m^2 \cdot \operatorname{sn}^2(\xi) - 12kc \cdot \operatorname{ns}^2(\xi) + 4kc m^2 + 4kc + c - 1, \quad (62)$$

$$u_{2,8} = 12kc m^2 \cdot \operatorname{cn}^2(\xi) - 12kc \cdot \operatorname{dc}^2(\xi) + 4kc m^2 + 4kc + c - 1, \quad (63)$$

$$u_{2,9} = 12kc m^2 \cdot \operatorname{dn}^2(\xi) - 12kc \cdot \operatorname{cd}^2(\xi) + 4kc m^2 + 4kc + c - 1, \quad (64)$$

where $\operatorname{sn}(\xi)$, $\operatorname{cn}(\xi)$, $\operatorname{dn}(\xi)$, $\operatorname{ns}(\xi)$, $\operatorname{dc}(\xi)$, and $\operatorname{cd}(\xi)$ are Jacobi elliptic functions and $0 < m < 1$ gives the modulus of the Jacobi elliptic functions. These functions generate hyperbolic functions when $m \rightarrow 1$ and trigonometric functions when $m \rightarrow 0$ as follows:

$$m \rightarrow 1 \quad \operatorname{sn}(\xi) \rightarrow \tanh(\xi) \quad \operatorname{cn}(\xi) \rightarrow \operatorname{sech}(\xi) \quad \operatorname{dn}(\xi) \rightarrow \operatorname{sech}(\xi)$$

$$m \rightarrow 0 \quad \operatorname{sn}(\xi) \rightarrow \sin(\xi) \quad \operatorname{cn}(\xi) \rightarrow \cos(\xi) \quad \operatorname{dn}(\xi) \rightarrow 1 \quad (65)$$

$$\operatorname{ns}(\xi) = \frac{1}{\operatorname{sn}(\xi)} \quad \operatorname{dc}(\xi) = \frac{\operatorname{dn}(\xi)}{\operatorname{cn}(\xi)} \quad \operatorname{cd}(\xi) = \frac{\operatorname{cn}(\xi)}{\operatorname{dn}(\xi)}.$$

FIGURE 4: Shape of (30) with $c = 1$, $\lambda = 2$, $\mu = 2$, and $0 \leq x, t \leq 10$ for (a) $\sigma = \gamma = 1$, (b) $\sigma = 0.5$ and $\gamma = 1$, (c) $\sigma = 1$ and $\gamma = 0.5$, and (d) $\sigma = \gamma = 0.5$.

Remark 3. When the arbitrary constants C_1 and C_2 are taken to be zero separately in (23), (28), and (31), we get (24), (25), (29), (30), and (32) which are the travelling wave solutions to the space and time fractional BBM equation obtained via the (G'/G) expansion method. Equations (49) are the travelling wave solution to the space and time fractional BBM equation obtained through the modified simple equation method. We see that the modified simple equation is not very effective for constructing travelling wave solutions to the space and time fractional BBM equation because the integer N obtained from considering the homogenous balance between the highest nonlinear terms and the highest derivative is greater than 1.

Remark 4. Applying Liu's theorem to (24) and (25), we obtain new complex travelling wave solutions in the form of (26) and (27).

Remark 5. All the travelling wave solutions to the space and time fractional Benjamin-Bona-Mahony equation obtained were checked by putting them back into (1) with the aid of Mathematica.

Remark 6. For $\sigma = \gamma = 1$, $k = -i$, and $\alpha = (4\mu - \lambda^2)/4$, the complex travelling wave solutions $u_{1,1}$, $u_{1,2}$, $u_{1,3}$, and $u_{1,4}$ in (50)–(54) by [29] reduce to some of the solutions we obtained

via the (G'/G) expansion method; that is, $u_{1,1}$ (see (50)), $u_{1,2}$ (see (51)), $u_{1,3}$ (see (52)), and $u_{1,4}$ (see (53)) are the same as u_2 (see (25)), u_1 (see (24)), u_8 (see (30)), and u_7 (see (29)), respectively. Hence, the complex direct algebra method does not produce new complex solutions as suggested in [29]. Solution $u_{1,5}$ (see (54)) reported by [29] does not satisfy the space and time fractional BBM equation for $\sigma = \gamma = 1$.

Remark 7. We also examine the solutions to the BBM equation from [30] for $k = 1$. For $\sigma = \gamma = 1$ and $c = 1$, $u_{2,1}$ (see (56)) reduces to u_9 (see (32)). For $\lambda^2 - 4\mu = 4$, $u_{2,3}$ (see (58)) and $u_{2,4}$ (see (59)) reduce to u_1 (see (24)) and u_2 (see (25)), respectively. When the modulus of the Jacobi elliptic functions is $m = 1$, then both $u_{2,8}$ (see (63)) and $u_{2,9}$ (see (64)) reduce to u_2 (see (25)) for $\lambda^2 - 4\mu = 4$. We observe that u_7 (see (29)) and u_8 (see (30)) are, respectively, the same as $u_{2,7}$ (see (62)) and $u_{2,8}$ (see (63)) when $\lambda^2 - 4\mu = -4$ and $m = 0$. It should be noted that $u_{2,2}$ (see (57)), $u_{2,5}$ (see (60)), and $u_{2,6}$ (see (61)) are not valid solutions to the space and time fractional BBM equation for $\sigma = \gamma = 1$. Also $u_{2,7}$ (see (62)) and $u_{2,9}$ (see (64)) for $m = 1$ and $m = 0$, respectively, do not satisfy the space and time fractional BBM equation for $\sigma = \gamma = 1$.

The shapes of (24), (25), (29), and (30) for selected values of c , λ , and μ at different σ and γ values are presented in Figures 1–4. The solution u_1 (see (24)) is a singular kink shaped

travelling wave solution of (15). Figure 1 shows the shape of (24) with $c = 1$, $\lambda = 3$, and $\mu = 2$ for the interval $0 \leq x, t \leq 10$ at different σ and γ values. From Figure 1 and $u(x, t) = u(x - ct)$, we observe that, for $c > 0$, the propagation of the wave will be in the positive x -direction. If we take $c < 0$, then the propagation of the wave will be in the negative x -direction. The solution u_2 (see (25)) is a bell-shaped Soliton solution of (15). Figure 2 shows the shape of (25) with $c = 1$, $\lambda = 3$, and $\mu = 2$ for the interval $0 \leq x, t \leq 10$ at different σ and γ values. From Figure 2 and $u(x, t) = u(x - ct)$, we also observe that, for $c > 0$, the propagation of the wave will be in the positive x -direction. If we take $c < 0$, then the propagation of the wave will be in the negative x -direction. Figures 3 and 4 show, respectively, the shape of u_3 (see (29)) and u_4 (see (30)) with $c = 1$ and $\lambda = \mu = 2$ for the interval $0 \leq x, t \leq 10$ at different σ and γ values. u_3 (see (29)) and u_4 (see (30)) are exact periodic travelling wave solutions of the space and time fractional Benjamin-Bona-Mahony equation.

5. Conclusion

Hyperbolic, trigonometric, and rational function travelling wave solutions to the space and time fractional Benjamin-Bona-Mahony equation defined in the sense of Jumarie's modified Riemann-Liouville derivative have been obtained using the (G'/G) expansion method and modified simple equation method. Liu's theorem was also applied to obtain new complex travelling wave solutions to the space and time fractional BBM equation. The results were compared to those obtained by Bulent and Erdal [29] using the direct algebraic method for complex travelling wave solution and those by Yanhong and Baodan [30] using the generalized (G'/G) expansion method with the Klein-Gordon equation as an auxiliary equation. The (G'/G) expansion method is equivalent to the direct algebraic method and the generalized (G'/G) expansion method with some of their reported solutions found not to satisfy the BBM equation. Also, the modified simple equation method is not effective for constructing travelling wave solutions to the space and time fractional BBM equation because the index N obtained from considering the homogenous balance between the highest nonlinear term and the highest derivative is greater than 1. The improvement of the modified simple equation scheme for evolution equations with $N > 1$ is still an open problem.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

References

- [1] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.
- [2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.
- [3] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, New Jersey, NJ, USA, 2000.
- [4] B. J. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators*, Springer, New York, NY, USA, 2003.
- [5] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, Pa, USA, 1981.
- [6] M. R. Miura, *Backlund Transformation*, Springer, Berlin, Germany, 1978.
- [7] C. Rogers and W. F. Shadwick, *Backlund Transformations*, Academic Press, New York, NY, USA, 1982.
- [8] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, Germany, 1991.
- [9] R. Hirota, "Exact envelope-soliton solutions of a nonlinear wave equation," *Journal of Mathematical Physics*, vol. 14, no. 7, pp. 805–809, 1973.
- [10] E. Fan, "Extended tanh-function method and its applications to nonlinear equations," *Physics Letters A*, vol. 277, no. 4-5, pp. 212–218, 2000.
- [11] W. Malfliet, "The tanh method: a tool for solving certain classes of non-linear PDEs," *Mathematical Methods in the Applied Sciences*, vol. 28, no. 17, pp. 2031–2035, 2005.
- [12] C. T. Yan, "A simple transformation for nonlinear waves," *Physics Letters A*, vol. 224, no. 1-2, pp. 77–84, 1996.
- [13] J.-H. He and X.-H. Wu, "Exp-function method for nonlinear wave equations," *Chaos, Solitons and Fractals*, vol. 30, no. 3, pp. 700–708, 2006.
- [14] Z. Y. Yan and H. Q. Zhang, "New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water," *Physics Letters A*, vol. 285, no. 5-6, pp. 355–362, 2001.
- [15] E. G. Fan, "Two new applications of the homogeneous balance method," *Physics Letters A*, vol. 265, no. 5-6, pp. 353–357, 2000.
- [16] Z. S. Feng, "Exact solution to an approximate sine-Gordon equation in $(n + 1)$ -dimensional space," *Physics Letters A*, vol. 302, no. 2-3, pp. 64–76, 2002.
- [17] Z. S. Feng, "The first integral method to the two dimensional Burgers-Korteweg-de-Vries equation," *Physics Letters A*, vol. 308, pp. 173–178, 2003.
- [18] M. L. Wang, X. Li, and J. Zhang, "The (G'/G) -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.
- [19] E. M. E. Zayed, "Traveling wave solutions for higher dimensional nonlinear evolution equations using the (G'/G) -expansion method," *Journal of Applied Mathematics & Informatics*, vol. 28, pp. 383–395, 2010.
- [20] A. J. M. Jawad, M. D. Petković, and A. Biswas, "Modified simple equation method for nonlinear evolution equations," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 869–877, 2010.
- [21] E. M. E. Zayed, "A note on the modified simple equation method applied to Sharma-Tasso-Olver equation," *Applied Mathematics and Computation*, vol. 218, no. 7, pp. 3962–3964, 2011.
- [22] E. M. E. Zayed and S. A. H. Ibrahim, "Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method," *Chinese Physics Letters*, vol. 29, no. 6, Article ID 060201, 2012.
- [23] C. P. Liu, "The relation between the kink-type solution and the kink-bell-type solution of nonlinear evolution equations," *Physics Letters A*, vol. 312, no. 1-2, pp. 41–48, 2003.
- [24] T. B. Benjamin, J. L. Bona, and J. J. Mahony, "Model equations for long waves in nonlinear dispersive systems," *Philosophical Transactions of the Royal Society of London A*, vol. 272, pp. 47–55, 1972.

- [25] G. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results," *Computers and Mathematics with Applications*, vol. 51, no. 9-10, pp. 1367-1376, 2006.
- [26] G. Jumarie, "Fractional partial differential equations and modified riemann-liouville derivative new methods for solution," *Journal of Applied Mathematics and Computing*, vol. 24, no. 1-2, pp. 31-48, 2007.
- [27] Z.-B. Li and J.-H. He, "Fractional complex transform for fractional differential equations," *Mathematical and Computational Applications*, vol. 15, no. 5, pp. 970-973, 2010.
- [28] M. M. Kabir and R. Bagherzadeh, "Application of (G'/G) -expansion method to nonlinear variants of the $(2+1)$ -dimensional camassa-holm-KP equation," *Middle-East Journal of Scientific Research*, vol. 9, pp. 602-610, 2011.
- [29] K. Bulent and B. Erdal, "Complex solutions for the fisher equation and the benjamin-bona-mahony equation," *Cankaya University Journal of Science and Engineering*, vol. 7, no. 2, pp. 87-93, 2010.
- [30] Q. Yanhong and T. Baodan, "Generalized G'/G expansion method and its applications," *International Mathematical Forum*, vol. 6, no. 3, pp. 147-157, 2011.