Research Article

A Formula for the Numerical Range of Elementary Operators

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Let $B(H)$ be the algebra of bounded linear operators on a complex Hilbert space H . For k -tuples of elements of $B(H)$, $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$, let $R_{A,B}$ denote the elementary operator on $B(H)$ defined by $R_{A,B}(X) = \sum_{i=1}^k A_i X B_i$. In this paper, we prove the following formula for the numerical range of $R_{A,B}$: $V(R_{A,B}, B(H)) = [\cup_{U \in U(H)} W(\sum_{i=1}^k U A_i U^* B_i)]^-$, where $U(H)$ is the set of unitary operators.

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with unit. For k -tuples of elements of $\mathcal{A}$, $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_k)$, let $R_{a,b}$ denote the elementary operator on $\mathcal{A}$ defined by

$$R_{a,b}(x) = \sum_{i=1}^k a_i x b_i. \quad (1)$$

This is a bounded linear operator on $\mathcal{A}$. Some interesting cases are the generalized derivation $\delta_{a,b}(x) = ax - xb$ and the multiplication $M_{a,b}(x) = axb$ for $a, b, x \in \mathcal{A}$.

The numerical range of $a \in \mathcal{A}$ is defined by

$$V(a, \mathcal{A}) = \{f(a) : f \in S\}, \quad (2)$$

where S is the set of normalized states in $\mathcal{A}$:

$$S = \{f \in \mathcal{A}^*, \|f\| = 1, f(e) = 1\}. \quad (3)$$

See [1–3]. It is well known that $V(a, \mathcal{A})$ is convex and closed and contains the spectrum $\sigma(a)$. For $\mathcal{A} = B(X)$, the algebra of bounded linear operators on a normed space X , and $T \in B(X)$, in addition to $V(T, B(X))$, we have the spatial numerical range of T , given by

$$\begin{aligned} W(T) \\ = \{f(Tx) : x \in X, \|x\| = 1, f \in X^*, \|f\| = f(x) = 1\}, \end{aligned} \quad (4)$$

and it is known that $V(T, B(X)) = \overline{\text{co}}W(T)$, the closed convex hull of $W(T)$. In the case of a Hilbert space $X = H$, then

$$W(T) = \{\langle Tx, x \rangle : x \in H, \|x\| = 1\} \quad (5)$$

is convex but not closed in general and $V(T, B(H)) = W(T)^-$.

Many facts about the relation between the spectrum of $R_{a,b}$ and the spectrums of the coefficients a_i and b_i are known. This is not the case with the relation between the numerical range of $R_{a,b}$ and the numerical ranges of a_i and b_i . Apparently, the only elementary operator on a Hilbert space for which the numerical range is computed is the generalized derivations [4–8]. It is Fong [4] who first gives the following formula:

$$W(\delta_T) = W(T)^- - W(T)^-, \quad (6)$$

where δ_T is the inner derivation defined by $\delta_T(X) = TX - XT$. Shaw [7] (see also [5, 6]) extended this formula to generalized derivations in Banach spaces. For a good survey of the numerical range of elementary operators, you can see [9], where the following problem is posed.

Problem. Determine the numerical range of the elementary operator $R_{A,B}$.

In this paper, we give a formula that answers this problem.

2. Main Result

The following theorem is the main result in this paper.

Theorem 1. *Let H be a complex Hilbert space. Let $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$ be two k -tuples of elements in $B(H)$. Then, one has*

$$V(R_{A,B}, B(B(H))) = \left[\bigcup_{U \in U(H)} W \left(\sum_{i=1}^k U A_i U^* B_i \right)^- \right]^- \quad (7)$$

In particular for multiplication and generalized derivation, one has $(A, B \in B(H))$:

$$V(M_{A,B}, B(B(H))) = \left[\bigcup_{U \in U(H)} W(AUBU^*)^- \right]^- \quad (8)$$

$$V(\delta_{A,B}, B(B(H))) = \left[\bigcup_{U \in U(H)} W(U^*AU - B)^- \right]^-$$

From Fong's formula (see [4, 6, 10]), we can deduce the following.

Corollary 2. *For $A, B \in B(H)$, one has*

$$\begin{aligned} W(A)^- - W(B)^- &= \left[\bigcup_{U \in U(H)} W(UAU^* - B)^- \right]^- \\ &= \left[\bigcup_{U \in U(H)} W(A - UB U^*)^- \right]^- \end{aligned} \quad (9)$$

3. Proof of the Main Result

One of the keys to the proof of our main result is the following lemma.

Lemma 3. *Let $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$ be two k -tuples of elements in $B(H)$. Then, one has*

$$W \left(\sum_{i=1}^k A_i B_i \right) \subset V(R_{A,B}, B(B(H))). \quad (10)$$

In particular, for $A, B \in B(H)$, one has

$$\begin{aligned} W(AB) &\subset V(M_{A,B}, B(B(H))), \\ W(A - B) &\subset V(\delta_{A,B}, B(B(H))). \end{aligned} \quad (11)$$

Proof. Let $\lambda \in W(\sum_{i=1}^k A_i B_i)$; by definition, there exists $x \in H$ with $\|x\| = 1$ such that

$$\lambda = \left\langle \sum_{i=1}^k A_i B_i x, x \right\rangle = \text{tr} \left(\sum_{i=1}^k A_i B_i (x \otimes x) \right). \quad (12)$$

Here, $\text{tr}(\cdot)$ is the linear form trace. Let $\phi_{x \otimes x}$ be the linear form defined by

$$\phi_{x \otimes x}(X) = \text{tr}(X(x \otimes x)) = \langle Xx, x \rangle. \quad (13)$$

This is a bounded linear form on $B(H)$, with norm being equal to 1, because

$$\|\phi_{x \otimes x}\| = \|x \otimes x\| = 1. \quad (14)$$

Since

$$\phi_{x \otimes x}(I_H) = \text{tr}(x \otimes x) = \langle x, x \rangle = 1, \quad (15)$$

the form $\phi_{x \otimes x}$ is a state on $B(H)$. So,

$$\begin{aligned} \phi_{x \otimes x}(R_{A,B}(I_H)) &\in V(M_{A,B}, B(B(H))), \\ \phi_{x \otimes x}(R_{A,B}(I_H)) &= \phi_{x \otimes x} \left(\sum_{i=1}^k A_i B_i \right) \\ &= \text{tr} \left(\sum_{i=1}^k A_i B_i (x \otimes x) \right) = \left\langle \sum_{i=1}^k A_i B_i x, x \right\rangle \\ &= \lambda. \end{aligned} \quad (16)$$

Hence, $W(\sum_{i=1}^k A_i B_i) \subset W(R_{A,B}) \subset V(R_{A,B}, B(B(H)))$. $\square$

Let E be a Banach space. We say that $S \in B(E)$ is an isometry if $\|Sx\| = \|x\|$ for all $x \in E$. If S is an invertible isometry, then its inverse S^{-1} is also an isometry, and we have

$$V(STS^{-1}, B(E)) = V(T, B(E)), \quad T \in B(E). \quad (17)$$

In the case of a Hilbert space, an invertible isometry is unitary and its inverse is the adjoint.

Let U and V be two unitaries operators on H ; then

$$R_{UAU^*, VB V^*} = M_{U,V^*} R_{A,B} M_{U^*,V}, \quad (18)$$

with M_{U,V^*} being an invertible isometry and its inverse being $M_{U^*,V}$.

From this result, we deduce that

$$\begin{aligned} V(R_{UAU^*, VB V^*}, B(B(H))) &= V(M_{U,V^*} R_{A,B} M_{U^*,V}, B(B(H))) \\ &= V(R_{A,B}, B(B(H))). \end{aligned} \quad (19)$$

Now, using Lemma 3, we get

$$\begin{aligned} W \left(\sum_{i=1}^k U A_i U^* V B_i V^* \right) &\subset V(R_{A,B}, B(B(H))), \\ \bigcup_{U,V \in U(H)} W \left(\sum_{i=1}^k U A_i U^* V B_i V^* \right) &\subset V(R_{A,B}, B(B(H))). \end{aligned} \quad (20)$$

But, the numerical range is closed and the product of two unitaries is also an unitary, hence:

$$\begin{aligned} \left[\bigcup_{U \in U(H)} W \left(\sum_{i=1}^k U A_i U^* B_i \right)^- \right]^- \\ = \left[\bigcup_{V \in U(H)} W \left(\sum_{i=1}^k A_i V B_i V^* \right)^- \right]^- \subset V(R_{A,B}, B(B(H))). \end{aligned} \quad (21)$$

So, we have proved the second inclusion of Theorem 1.

For the other inclusion, we will use the two following theorems.

Theorem 4 (See [11]). *Let $\mathcal{A}$ be Banach algebra. For $a \in \mathcal{A}$, one has*

$$V(a, \mathcal{A}) = \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|a - z\|\}. \quad (22)$$

The norm of an elementary operator is defined by

$$\|R_{A,B}\| = \sup \{\|R_{A,B}(X)\| : \|X\| \leq 1\}. \quad (23)$$

Let $\mathcal{A}$ be C^* -algebra. An element $U \in \mathcal{A}$ is said to be unitary if $U^*U = UU^* = I$. In the following, $U(\mathcal{A})$ denote the set of unitaries in $\mathcal{A}$.

Theorem 5 (Russo-Dye [12]). *Let $\mathcal{A}$ be C^* -algebra. Let $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ be two k -tuples of elements in $\mathcal{A}$. Then, one has*

$$\|R_{A,B}\| = \sup \{\|R_{A,B}(U)\| : U \in U(\mathcal{A})\}. \quad (24)$$

We return now to the proof of the main theorem.

Proof. We need only to show the inclusion “ $\subset$.” By Theorem 4, we have

$$V(R_{A,B}, B(B(H))) = \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|R_{A,B} - z\|\}. \quad (25)$$

And, by Theorem 5, we get

$$\|R_{A,B} - z\| = \sup \left\{ \left\| \sum_{i=1}^k A_i U B_i - zU \right\| : U \in U(H) \right\}. \quad (26)$$

But $\|UT\| = \|T\|$ for all $T \in B(H)$ and $U \in U(H)$. Hence,

$$\|R_{A,B} - z\| = \sup \left\{ \left\| \sum_{i=1}^k U^* A_i U B_i - zI_H \right\| : U \in U(H) \right\}. \quad (27)$$

Hence, if $\lambda \in V(R_{A,B}, B(B(H)))$, then, for all $z \in \mathbb{C}$,

$$\lambda \in \{|\mu - z| \leq \|R_{A,B} - zI_{B(H)}\|\}. \quad (28)$$

Let $\epsilon > 0$ fixed, there exists a unitary U_ϵ such that

$$\|R_{A,B} - zI_{B(H)}\| \leq \left\| \sum_{i=1}^k U_\epsilon^* A_i U_\epsilon B_i - zI_H \right\| + \epsilon. \quad (29)$$

Now, using Theorem 4, we have

$$\begin{aligned} W\left(\sum_{i=1}^k U_\epsilon^* A_i U_\epsilon B_i\right)^- \\ = \bigcap_{z \in \mathbb{C}} \left\{ \lambda : |\lambda - z| \leq \left\| \sum_{i=1}^k U_\epsilon^* A_i U_\epsilon B_i - zI_H \right\| \right\}. \end{aligned} \quad (30)$$

So, there exists $\mu \in W(\sum_{i=1}^k U_\epsilon^* A_i U_\epsilon B_i)$ such that $|\lambda - \mu| \leq \epsilon$. But ϵ is arbitrary, $\lambda \in [\cup_{U \in U(H)} W(\sum_{i=1}^k U^* A_i U B_i)^-]$. This finishes the proof of the main theorem. $\square$

4. Some Applications

It is well known that, for the spectrum, if $A, B \in B(H)$, then we have

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}. \quad (31)$$

For the numerical range, this not true, but we can deduce the following corollary from the proof of Theorem 1.

Corollary 6. *For all $A, B \in B(H)$, one has*

$$\left[\bigcup_{U \in U(H)} W(AUBU^*) \right]^- = \left[\bigcup_{U \in U(H)} W(UBU^*A) \right]^- \quad (32)$$

The numerical radius of an operator $T \in B(E)$ is denoted by $v(T)$ and defined by

$$v(T) = \sup \{|\lambda| : \lambda \in V(T, B(E))\}. \quad (33)$$

Corollary 7. *Let $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_k)$ be two k -tuples of elements in $B(H)$. Then, one has*

$$v(R_{A,B}) = \sup \left\{ v\left(\sum_{i=1}^k U^* A_i U B_i\right) : U \in U(H) \right\}. \quad (34)$$

In particular, for $A, B \in B(H)$,

$$\begin{aligned} v(\delta_{A,B}) &= \sup_{U \in U(H)} v(UAU^* - B) \\ &= \sup_{U \in U(H)} v(A - UBU^*) \\ &= \sup_{U, V \in U(H)} v(UAU^* - VB V^*). \end{aligned} \quad (35)$$

Let K be a nonempty subset of the plane and let

$$\text{diam}(K) = \sup_{\alpha, \beta \in K} |\alpha - \beta|. \quad (36)$$

From Corollary 7 ($B = A$), one has

$$\text{diam}(W(A)) = \sup \{v(UAU^* - A) : U \in U(H)\}. \quad (37)$$

So, the diameter of the numerical range $W(A)$ is equal to the diameter of the v -unitary orbit of the operator A .

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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