Research Article

Analysis of Free Edge Stresses in Composite Laminates Using Higher Order Theories

Hamidreza Yazdani Sarvestani and Ali Naghashpour

Concordia Center for Composites (CONCOM), Department of Mechanical and Industrial Engineering, Concordia University, 1455 De Maisonneuve Boulevard West, Montreal, QC, Canada H3G 1M8

Correspondence should be addressed to Hamidreza Yazdani Sarvestani; h_yazd@encs.concordia.ca

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This paper presents the determination of the interlaminar stresses close to the free edges of general cross-ply composite laminates based on higher order equivalent single-layer theory (HESL). The laminates with finite dimensions were subjected to a bending moment, an axial force, and/or a torque for investigation. Full three-dimensional stresses in the interior and the boundary-layer regions were determined. The computed results were compared with those obtained from Reddy’s layerwise theory. It was found that HESL theory predicts precisely the interlaminar stresses near the free edges of laminates. Besides, high efficiency in terms of computational time is obtainable when HESL theory is used as compared with layerwise theory. Finally, various numerical results were presented for the cross-ply laminates. Also design guidelines were proposed to minimize the edge-effect problems in composite laminates.

1. Introduction

Laminated composite materials are being used in several industries due to their high strength-to-weight ratio and stiffness-to-weight ratio. However, they are susceptible to different types of damage such as delamination which occurs due to high stress concentration near the edge of composite laminates. These stresses are induced by mismatch in elastic properties between adjacent plies of composite laminates [1]. It has been shown that the state of stresses in the edge zone of the laminate is three-dimensional (3D) in nature. Many attempts have been made to compute these stresses next to laminate’s free edges [1–12]. However, because of intrinsic complexities involved in the problem, no exact solution is known for elasticity equations. Therefore, many approximate methods used to determine the interlaminar stresses are recorded in the survey paper by Kant and Swaminathan [3]. Based on a laminated model containing anisotropic layers, the first approximate solution of interlaminar shear stresses was presented by Puppo and Evensen [4]. Other approximate analytical methods used to examine the problem are the employment of the higher order plate theory proposed by Pagano [2], the perturbation technique by Hsu and Herakovich [5], the boundary-layer theory by Tang and Levy [6], and the approximate elasticity solutions by Pipes and Pagano [7]. An approximate theory is also utilized by Pagano [8, 9] based on assumed in-plane stresses and the use of Reissner’s variational principle. The principle of minimum complementary energy and the force balance method are used by Kassapoglou and Lagace [10] to study the symmetric laminates under uniaxial loading. A variational method involving Lekhnitskii’s stress function is utilized by Yin [11, 12] to determine the interlaminar stresses in a multilayer strip of a laminate subjected to combinations of mechanical loads. Lin et al. [13] improved the technique developed by Kassapoglou and Lagace [10] for symmetric laminates under uniaxial tension to evaluate the interlaminar stress distribution near the straight free edges of symmetric and unsymmetric laminates under different types of loading conditions. The first numerical method to solve the 2D governing elasticity equations is given by Pipes and Pagano [1]. They utilized a finite-difference technique to establish the interlaminar stresses in a long symmetric laminate under uniform axial strain. A layer reduction technique and a layerwise
2. Higher Order Equivalent Single-Layer Theory (HESL)

General cross-ply laminates are subjected to the bending moment, the axial force, and/or the torque in order to accurately determine the interlaminar stresses. The geometry of the laminate is illustrated in Figure 1. The formulation is limited to linear elastic material behavior, small strain, and displacements. The coordinate system \((x, y, z)\) is located at the middle plane of the laminate. Thickness, width, and length of the laminate are \(h, 2b\), and \(2a\), respectively.

2.1. Displacement Field and Strains. The integrations of the three-dimensional elasticity strain-displacement relations [22] within the \(k\)th layer of the laminate producing the most general form of displacement field are given by

\[
\begin{align*}
    u_1^{(k)}(x, y, z) &= B_2 x + B_6 x z + u_1^{(k)}(y, z), \\
    u_2^{(k)}(x, y, z) &= -B_1 x z + v^{(k)}(y, z), \\
    u_3^{(k)}(x, y, z) &= B_1 x y - \frac{1}{2} B_8 x^2 + w^{(k)}(y, z),
\end{align*}
\]

where \(u_1, u_2, \text{ and } u_3\) represent the displacement components in the \(x-, y-, \text{ and } z\)-directions, respectively, of a material point initially located at \((x, y, z)\) in the undeformed laminate. The displacement field in (1a), (1b), and (1c) may be used, in principle, for obtaining the stress field in any composite laminate subjected to arbitrary combinations of self-equilibrating mechanical and uniform hygrothermal loads. In the present work, however, our attention is focused on symmetric and unsymmetric cross-ply laminates under the bending moment, the axial force, and/or the torque. General cross-ply laminates based on physical grounds can be established as (see Figure 1)

\[
u_1^{(k)}(x, y, z) = -u_1^{(k)}(-x, y, z).
\]

Upon imposing this condition on (1a), (1b), and (1c), it is readily seen that \(u^{(k)}(y, z) = 0\). Thus, for cross-ply laminates the most general form of the displacement field is given as

\[
\begin{align*}
    u_1^{(k)}(x, y, z) &= B_6 x z + B_2 x, \\
    u_2^{(k)}(x, y, z) &= -B_1 x z + v^{(k)}(y, z), \\
    u_3^{(k)}(x, y, z) &= B_1 x y - \frac{1}{2} B_8 x^2 + w^{(k)}(y, z).
\end{align*}
\]

The unknown constants, namely, \(B_1, B_2, \text{ and } B_6\) appearing in (3), are global response of the laminate. On the other hand, unknown functions \(v^{(k)}(y, z)\) and \(w^{(k)}(y, z)\) are local response of the laminate.

The purpose of the present section is to show the HESL theory with infinity number of terms which provides sufficiently accurate results for the interlaminar stresses in composite laminate. In HESL, the components of the displacement vector at a material point in cross-ply laminated composite are expressed as [23]

\[
\begin{align*}
    u_1(x, y, z) &= u(x, y) + z \eta_1(x, y) + z^2 \eta_2(x, y) + \cdots, \\
    u_2(x, y, z) &= v(x, y) + z \psi_1(x, y) + z^2 \psi_2(x, y) + \cdots, \\
    u_3(x, y, z) &= w(x, y) + z \phi_1(x, y) + z^2 \phi_2(x, y) + \cdots.
\end{align*}
\]

Relations (4) may more conveniently be presented as

\[
\begin{align*}
    u_k(x, y, z) &= u(x, y) + z^k \eta_k(x, y) & k = 1, 2, \ldots, l, \\
    v_i(x, y, z) &= v(x, y) + z^i \psi_i(x, y) & i = 1, 2, \ldots, n, \\
    w_j(x, y, z) &= w(x, y) + z^j \phi_j(x, y) & j = 1, 2, \ldots, m,
\end{align*}
\]
where \( k, i, \) and \( j \) are dummy indexes indicating the summation of terms from \( k = i = j = 1 \) to desirable number (i.e., \( l \) or \( n \) or \( m \)). It is expected that the accuracy of such theories can, generally, be increased by taking more terms in (5). Also, \( u_1, u_2, \) and \( u_3 \) are the displacement components of any point within a laminate in the \( x, y, \) and \( z \)-directions, respectively. \( k + i + j + 3 \) unknown displacement functions should be found for a single layer or more layers in the laminate. The displacement field in (3) within HESL is simplified by

\[
\begin{align*}
\delta v : N'_y &= 0, \\
\delta w : N'_{yz} &= 0, \tag{7a}
\end{align*}
\]

\[
\begin{align*}
\delta \psi_i : B_{yz} - \frac{dA_{iy}}{dy} &= 0 \quad i = 1, 2, \ldots, n, \tag{7c}
\end{align*}
\]

\[
\begin{align*}
\delta \phi_j : B_{jz} - \frac{dA_{jy}}{dy} &= 0 \quad j = 1, 2, \ldots, m, \tag{7d}
\end{align*}
\]

\[
\begin{align*}
\delta B_1 : \int_{-b}^{b} (Q_x y - M_{xy}) dy &= T_0, \tag{8a}
\end{align*}
\]

\[
\begin{align*}
\delta B_2 : \int_{-b}^{b} N_x dy &= F_0, \tag{8b}
\end{align*}
\]

\[
\begin{align*}
\delta B_6 : \int_{-b}^{b} M_x dy &= M_0, \tag{8c}
\end{align*}
\]

where a prime indicates an ordinary differentiation with respect to variable \( y \) and the stress and moment resultants appearing in (7a), (7b), (7c), and (7d) and (8a), (8b), and (8c) which are defined as

\[
\begin{align*}
(M_x, M_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_{xy}) z \, dz, \tag{9}
\end{align*}
\]

\[
\begin{align*}
(N_x, N_y, N_{yz}, Q_x) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{yz}, \sigma_{xy}) \, dz, \tag{10a}
\end{align*}
\]

\[
\begin{align*}
(A_{iy}, A_{jy}) &= \int_{-h/2}^{h/2} (\sigma_j \psi_i, \sigma_j \phi_i) \, dz, \tag{10b}
\end{align*}
\]

\[
\begin{align*}
(Q_x) &= A_{jy} B_1 y, \tag{10c}
\end{align*}
\]
where

\[
(A_{pq}, B_{pq}, D_{pq}) = \sum_{i=1}^{N} t_{z_i} \bar{C}_{pq}(1, z, z^2) dz \quad p, q = 1, 2, 6,
\]

\[
F_{pq} = \sum_{i=1}^{N} t_{z_i} \bar{C}_{pq} z_{i} dz,
\]

\[
H_{pq} = \sum_{i=1}^{N} t_{z_i} \bar{C}_{pq} z_{i} dz,
\]

\[
L_{ijpq} = \sum_{i=1}^{N} t_{z_i} \bar{C}_{pq} z_{i} j_{i} dz,
\]

(11)

where \(\bar{C}_{pq}\) are the transformed stiffness of an orthotropic lamina. For free edges of the laminate at \(y = \pm b\) according to the principle of minimum total potential energy the traction-free boundary conditions must be imposed as

\[
M_{xy} = N_{y} = N_{yz} = A_{xy} = A_{yze} = 0 \quad \text{at} \quad y = \pm b. \quad (12)
\]

Substituting (10a), (10b), (10c) into (7a), (7b), (7c), the equilibrium equations are obtained in terms of the displacement components which can be given as

\[
\delta v : B_{13}B_{6} + A_{12}B_{2} - B_{26}B_{1} + A_{22} v' + F_{122}ψ_{1} = 0, \quad (13a)
\]

\[
\delta ω : A_{44} w' + iF_{i-144}ψ_{1} + H_{j} 44ψ_{j} = 0, \quad (13b)
\]

\[
\delta ψ_{i} : i^2 F_{2i-2} 44 ψ_{i} + iF_{n-144} w' + iL_{i-134}ψ_{j} = 0, \quad (13c)
\]

\[
\delta φ_{j} : jH_{j} 13 B_{6} + jH_{j-113} B_{2} + jH_{j-123} v' = 0, \quad (13d)
\]

To determine the parameters \(B_{1}, B_{2}, \) and \(B_{6}\) as well as the interlaminar stresses in (13a)–(13d), the general solution of the ordinary differential equations in (13a)–(13d) is first obtained in terms of \(B_{1}, B_{2}, \) and \(B_{6}\). Then, by applying the boundary conditions in (12) and using the global equilibrium equation in (8a), (8b), and (8c), the constant parameters \(B_{1}, B_{2}, \) and \(B_{6}\) will be found in terms of the bending moment \(M_{0}\), axial force \(F_{0}\), and torque \(T_{0}\). For completeness, the details of the steps involved are displayed in Appendix A.

3. Layerwise Theory

3.1. Displacement Field and Strains. It should be emphasized that the solution proposed by Tahani and Nosier [25] was used to verify HESL theory for predicting stress in the edge of composite laminates. The proposed solution was used while different boundary conditions are considered. This offers to develop new analytical solution which is expressed in the present work. The displacement field in this theory may be represented as [25]

\[
u_{1}(x, y, z) = U_{k}(x, y) \Phi_{k}(z), \quad u_{2}(x, y, z) = V_{k}(x, y) \Phi_{k}(z), \quad u_{3}(x, y, z) = W_{k}(x, y) \Phi_{k}(z),
\]

\[(k = 1, 2, \ldots, N + 1), \quad (14)\]

where \(u_{1}, u_{2}, \) and \(u_{3}\) represent the displacement components in the \(x-, y-\), and \(z-\)directions, respectively [26]. Also \(U_{k}(x, y), V_{k}(x, y), \) and \(W_{k}(x, y)\) show the displacement components of all points located on the \(k\)th plane in the undeformed laminate, and \(\Phi_{k}(z)\) is continuous function of the thickness coordinate \(z\). Moreover, \(N\) indicates the total number of numerical layers considered in a laminate. It should be noted that a repeated index indicates summation over all values of that index. Substituting (14) into the linear strain-displacement relations of elasticity [27], the results are obtained as

\[
e_{x} = \frac{\partial U_{k}}{\partial x} \Phi_{k}, \quad y_{xy} = \frac{\partial V_{k}}{\partial y} \Phi_{k} + \frac{\partial W_{k}}{\partial x} \Phi_{k},
\]

\[
e_{y} = \frac{\partial V_{k}}{\partial y} \Phi_{k}, \quad y_{xz} = \frac{\partial U_{k}}{\partial x} \Phi_{k} + \frac{\partial W_{k}}{\partial y} \Phi_{k},
\]

\[
e_{z} = \frac{\partial W_{k}}{\partial z} \Phi_{k}, \quad y_{yz} = \frac{\partial U_{k}}{\partial y} \Phi_{k} + \frac{\partial V_{k}}{\partial x} \Phi_{k}.
\]

3.2. Equilibrium Equations. Using the principle of minimum total potential energy [23], \(3(N + 1)\) equations of equilibrium corresponding to \(3(N + 1)\) unknowns \(U_{k}, V_{k}, \) and \(W_{k}\) can be shown by

\[
\frac{\partial M_{k}}{\partial x} + \frac{\partial M_{k}}{\partial y} - Q_{k}' = 0, \quad \frac{\partial M_{k}}{\partial y} + \frac{\partial M_{k}}{\partial x} - Q_{k}' = 0,
\]

\[
\frac{\partial R_{k}^x}{\partial x} + \frac{\partial R_{k}^y}{\partial y} - N_{k} = 0,
\]

(16)

where \(k = 1, 2, \ldots, N + 1\). In (16), the generalized stress and moment resultants are defined as

\[
N_{k} = \int_{-h/2}^{h/2} \frac{\partial \Phi_{k}}{\partial z} dz,
\]

\[
(R_{x}^{k}, R_{y}^{k}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}) \Phi_{k} dz,
\]

\[
(Q_{xx}'^{k}, Q_{yy}'^{k}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}) \frac{\partial \Phi_{k}}{\partial z} dz,
\]

\[
(M_{k}^{x}, M_{k}^{y}, M_{k}^{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) \Phi_{k} dz.
\]

(17)
The boundary conditions for a laminated plate with a rectangular platform in the layerwise theory at an edge parallel to y-axis involve the specification of $U_k$, $M_{xy}^k$, $V_k$, and $M_{yy}^k$, and $W_k$ or $R_x^k$. Similarly, at an edge parallel to x-axis, the boundary conditions involve the specification of $U_k$ or $M_{xy}^k$, $V_k$, or $M_{yy}^k$, and $W_k$ or $R_x^k$. The present results are directly compared with those of Appendix B. Finally, substituting (19) into (16) yields the governing equations of equilibrium as:

$$D_{ij}^{kk}U_{j,xx} + D_{ij}^{kk}U_{j,yy} + D_{ij}^{kk}V_{j,xx} + D_{ij}^{kk}V_{j,yy} - A_{kk}^{ij}V_{j}$$

$$+ (B_{ij}^{kk} - B_{kk}^{kk}) W_{j,xx} = 0,$$

$$D_{ij}^{kk}U_{j,xx} + D_{ij}^{kk}V_{j,yy} + D_{ij}^{kk}V_{j,xx} - A_{kk}^{ij}V_{j}$$

$$+ (B_{ij}^{kk} - B_{kk}^{kk}) W_{j,yy} = 0,$$

$$D_{ij}^{kk}W_{j,yy} - A_{kk}^{ij}W_{j} = 0,$$

where the coefficients $A^{ij}$, $B^{ij}$, and $D^{ij}$ appearing in (19) are defined in Appendix B. Finally, substituting (19) into (16) yields the governing equations of equilibrium as:

$$D_{ij}^{kk}U_{j,xx} + D_{ij}^{kk}U_{j,yy} - A_{kk}^{ij}U_{j}$$

$$+ (B_{ij}^{kk} - B_{kk}^{kk}) W_{j,xx} = 0,$$

$$D_{ij}^{kk}U_{j,xx} + D_{ij}^{kk}V_{j,yy} + D_{ij}^{kk}V_{j,xx} - A_{kk}^{ij}V_{j}$$

$$+ (B_{ij}^{kk} - B_{kk}^{kk}) W_{j,yy} = 0,$$

$$D_{ij}^{kk}W_{j,yy} - A_{kk}^{ij}W_{j} = 0,$$

where a comma followed by a variable indicates differentiation with respect to that variable.

For completeness, the details of the steps involved in the analytical solutions are displayed in Appendix C.

### 4. Results

#### 4.1. Numerical Results and Discussion

To verify the accuracy and efficiency of the present method, several numerical examples are presented for general cross-ply laminates subjected to the bending moment, axial force, and/or torque. The analyses were performed for 4-layer and 6-layer laminates made of graphite/epoxy. The used mechanical and physical properties of the layers are presented in Table 1 [24].

In addition, the thickness of each physical ply is assumed to be 0.5 mm (i.e., $h_k = 0.5$ mm). HESL $(i)$ represents that $i$ number of $(j = i)$ term is taken in (6). Clearly, as the number $i$ is increased, the accuracy of the results is also increased. In the numerical examples that follow the interlaminar stresses are determined by integrating the local equilibrium equations of elasticity. Also, the width-to-thickness ratio (i.e., $2b/h$) is assumed to be, unless otherwise mentioned, equal to 10. Furthermore, the stress components are normalized as:

$$\sigma_j = \frac{\sigma_{ij}}{\sigma_0},$$

where $\sigma_0 = (1/\text{b}h)(F_0 + M_0/h + T_0/h)$.

To study the convergence of the stresses near free edges, two symmetric laminates $[90°/0°]_s$ and $[0°/90°]_s$, subjected to the bending moment, the axial force, and the torque are considered. On account of the nature of the cross-ply laminates, both $\sigma_y$ and $\sigma_{yy}$ are zero here as expected. Figures 2 and 3 demonstrate the convergence of the solution for $\sigma_z$ versus $y$ at the $x = h/4$ for the $[90°/0°]_s$ and $[0°/90°]_s$ laminates. It is seen that, for both $[90°/0°]_s$ and $[0°/90°]_s$, $\sigma_z$ is seen to rise or fall suddenly near the free edge, while being zero in the inner region of the laminate also the peak stress steadily increases as the number of terms $(i)$ is taken is increased. This is often attributed to a possible singularity at the $0°/90°$ interfaces.

Numerical results are obtained from equilibrium equations. The present results are directly compared with those
Table 1: Engineering properties of graphite/epoxy.

| Engineering properties | $E_1$ (GPa) | $E_2 = E_3$ (GPa) | $G_{12} = G_{13}$ (GPa) | $G_{23}$ (GPa) | $v_{12} = v_{13}$ | $v_{23}$ |
|------------------------|-------------|------------------|-------------------------|----------------|------------------|
| Graphite/epoxy         | 132         | 10.8             | 5.65                    | 3.38           | 0.24             | 0.59 |

obtained from LWT. Figure 4 shows the $\sigma_z$ distribution at the $0^\circ/90^\circ$ interfaces of $[0^\circ/90^\circ/0^\circ/90^\circ]$ and $[0^\circ/90^\circ/0^\circ/90^\circ]$ laminates under the bending moment and the axial force. An excellent agreement is found between the present solutions and those of LWT.

Figures 5 and 6 show the $\sigma_z$ and $\sigma_{yz}$ distribution at the $0^\circ/90^\circ$ interfaces of $[0^\circ/90^\circ/90^\circ/0^\circ]$ and $[0^\circ/0^\circ/90^\circ/0^\circ]$ laminates under the bending moment and the torque. Good accordance is seen between the results of the two theories. It is noted that the accuracy of HESL theory can be improved by taking more terms. Also the layerwise theory needs to take many terms to approach accurately the results of the present theory and these terms cause that LWT is more complex and computationally more time consuming than the HESL theory, so, it is better that HESL theory is used to compute the local phenomena such as free-edge-effect problems and the distribution of interlaminar stresses more precisely and less computationally than the layerwise theory in laminate composites.

The effect of the laminate width-to-thickness ratio on the interlaminar stresses for the $[90^\circ/0^\circ/90^\circ/90^\circ]$ laminate under the bending moment is investigated in Figure 7. By decreasing the width-to-thickness ratio, the boundary-layer region is expanded towards the internal region of the laminate with its width being almost equal to the thickness of the laminate. It is seen that the magnitude of the interlaminar stress at the free edge does not change while the width-to-thickness ratio of the laminate changes. On the other hand, this is evident that the highly localized nature of interlaminar stresses occurs near and exactly at the free edges of the laminate.

The variation of interlaminar shear stress $\sigma_{yz}$ through the thickness and near the free edge of the $[90^\circ/0^\circ/90^\circ/0^\circ]$ laminate under the bending moment and the torque is revealed.
in Figure 8. It is observed that the maximum negative and the maximum positive values of $\sigma_{yz}$ occur within the bottom 90° layer and the top 0° layer at the 90°/0° interfaces of the laminate, respectively. The variations of interlaminar normal stress $\sigma_z$ through the thickness of the unsymmetrical cross-ply laminate [90°/0°/90°/0°] under the bending moment and the axial force are portrayed in Figure 9. The maximum negative and the maximum positive values of $\sigma_z$ occur within the bottom 90° layer and the top 0° layer both near the 90°/0° interfaces at the free edge (i.e., $y = b$), respectively. From Figure 9, it is seen that $\sigma_z$ decreases away from the free edge as the inner region of the laminate is approached.

The distributions of the interlaminar stresses $\sigma_z$ and $\sigma_{yz}$ along the upper and lower interfaces of the unsymmetrical cross-ply [90°/0°/90°/0°] laminate subjected to the bending moment, the axial force, and the torque are demonstrated in Figure 10. It is observed that the interlaminar stresses demonstrate high stress gradient near the free edge. Both stresses are seen to grow suddenly near the free edge, while being zero in the interior region of the laminate. It is also observed that the interlaminar shear stress $\sigma_{yz}$ rises toward the free edge and decreases (or increases) rather abruptly to zero at the free edge.

4.2. Preliminary Design Guidelines. A comprehensive structural analysis program for designing composite laminates is very extensive and complex. This generally involves several analysis phases such as laminate stress and strength analysis. Hence, there is still a need to provide some preliminary knowledge of the lay-up sequence of composite laminates. Usually, structural properties of composite laminates such as stiffness, strength, and dimensional stability are affected by
the laminate stacking sequences. Because each property has
different relations with a particular stacking sequence, the
choice of stacking sequence suited for a particular application
may require a compromise. The obtained preliminary design
guidelines consist of the following.

1. The laminates stacking sequence (LSS) should be
symmetric about the midplane to avoid extension-bending
coupling. If this is not possible due to other
requirements, locate the asymmetry or imbalance as
near to the laminate midplane as possible. Avoid
symmetric LSS that create high interlaminar tension
stresses ($s$) at free edges.

2. Avoid grouping of 90° plies and separate 90° plies
by a 0° ply to minimize interlaminar shear
and normal stress. Minimize groupings of plies with
the same orientations to create a more homogeneous
laminate and to minimize interlaminar stress. If plies
must be grouped, avoid grouping more than four
plies of the same orientation. Minimizing grouping
helps to increase strength and minimize interlaminar
shear and normal stresses and therefore minimize the
tendency to delaminate.

3. Shield primary load carrying plies by positioning
them inside of laminate to increase tensile strength
and buckling resistance.

4. An LSS should have at least both distinct ply angles
(e.g., 0°, 90°) with a minimum of 10% of the plies
oriented at each angle. Ply angles should be selected
such that fibers are oriented with principal load axes.

5. Locating 90° ply toward the exterior surfaces
improves the buckling allowable in many cases.

5. Conclusions

In this research, analytical solutions were established within
HESL theory for the edge-effect problem of general cross-
ply composite laminates with finite dimensions under the
bending moment, the axial force, and/or the torque. The
edges of the laminates at $y = \pm b$ were assumed to
have traction-free boundary conditions. The accuracy and
effectiveness of HESL theory in describing the localized
three-dimensional effects were demonstrated by comparing
the results of HESL theory with those calculated from the
layerwise theory. Good agreement was observed between
the results of HESL and LWT theories. Furthermore, the analysis
using HESL was found to be more cost effective and accurate,
so HESL was employed to assess the local phenomena instead
of LWT theory. Here, several numerical results were shown
for the different loading problems. The design guidelines were
developed to provide cross-ply laminate stacking sequences
with minimized the free edges effects.

Appendices

A.

According to the principle of minimum total potential energy
[22] at the equilibrium configuration of a body the variation of
the total potential energy $\Pi$ of the body must vanish as

$$\delta \Pi = \delta U + \delta V = 0,$$

where $\delta U$ is the variation of total strain energy of the body;
that is,

$$\delta U = \iiint_V \left[ \sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \sigma_z \delta \varepsilon_z + \sigma_{yz} \delta \gamma_{yz}
+ \sigma_{xz} \delta \gamma_{xz} + \sigma_{xy} \delta \gamma_{xy} \right] dx \, dy \, dz,$$

and $V$ is negative of the work done on the body by the
specified external forces. Here, $V = -2M_0 a B_0 - 2T_0 a B_1 - 2F_0 a B_2$ and therefore,
$\delta V = -2M_0 a \delta B_0 - 2T_0 a \delta B_1 - 2F_0 a \delta B_2$. Also, the variations of strains in (A.2) are found as

$$\delta \varepsilon_x = z \delta B_0 + \delta B_2, \quad \delta \varepsilon_{yz} = i z^{-1} \delta \psi_i + \delta w + z^l \delta \phi_j,$$
$$\delta \varepsilon_y = \delta v + z^l \delta \psi_j, \quad \delta \varepsilon_{xz} = B_1 y,$$
$$\delta \varepsilon_z = j z^{-1} \delta \phi_j, \quad \delta \gamma_{xy} = -B_1 z.$$

Upon substituting (A.3) and $\delta V = -2M_0 a \delta B_0 - 2T_0 a \delta B_1 - 2F_0 a \delta B_2$ into (A.1), carrying out the necessary integrations,
and employing the fundamental lemma of calculus of variations
the equations and the associated boundary conditions of a laminate under bending are obtained to be
as in (7a), (7b), (7c), and (7d), (8a), (8b), and (8c), and (12),
respectively. To solve the linear equations in (13a), (13b),
(13c), and (13d), first we obtained $v'$ and $w'$ from (13a) and
(13b); then substituting $v'$ and $w'$ into (13c) and (13d) and integrating (13c) and (13d) yield

$$\delta \psi_i : S_1(i, j) \psi''_{1} + S_2(i, j) \psi + S_3(i, j) \phi'_{1} = 0$$

$$i = 1, 2, \ldots, n,$$

$$\delta \phi_j : S_4(i, j) \psi_i + S_5(i, j) \phi''_{j} + S_6(i, j) \phi_j = 0$$

$$j = 1, 2, \ldots, m,$$

where

$$\phi_j = \int \phi_j \, d \gamma,$$

and

$$S_1 (i, j) = \begin{cases} F_{i22}F_{j22} - F_{0.5(i+j)22} & \text{if } i + j \text{ even} \\ F_{i22}F_{j22} & \text{if } i > j \\ F_{i22}F_{j22} & \text{if } j > i, \end{cases}$$

$$S_2 (i, j) = \begin{cases} i^2 F_{i+j-244} - \frac{iF_{i-144}F_{j-144}}{A_{44}} & \text{if } i + j \text{ even} \\ j^2 F_{i-244} - \frac{iF_{i-144}F_{j-144}}{A_{44}} & \text{if } i > j \\ j^2 F_{i-244} & \text{if } j > i, \end{cases}$$

$$S_3 (i, j) = jL_{i-1j44} - \frac{iF_{i-144}H_{j44}}{A_{44}} - jL_{i-1j23} + \frac{jF_{i22}H_{j-123}}{A_{22}}$$

$$\delta \phi_j : S_4 (i, j) \psi_i + S_5 (i, j) \phi''_{j} + S_6 (i, j) \phi_j = 0$$

$$j = 1, 2, \ldots, m,$$

where $[S_p] (P = 1, 2, \ldots, 6)$ and the vectors $[S_Q] (Q = 7, 8, 9)$ are $n \times m$ matrices containing $S_p(i, j)$ and $m \times 1$ column matrices containing $S_Q(j)$, respectively. We introduced (A.4) in a matrix form as follows:

$$[M] \{\mu''\} + [K] \{\mu\} = [L] \{B\} \cdot y,$$

where

$$[\mu] = [\psi_1, \phi_1, \psi_2, \phi_2, \ldots, \psi_n, \phi_n]^T,$$

$$[B] = [B_1, B_2, B_6]^T,$$

with

$$[M] = \begin{bmatrix}
S_1(1, 1) & S_1(1, 1) & S_1(1, 2) & S_1(1, 2) & \cdots & S_1(1, m) & S_1(1, m) \\
0 & S_2(1, 1) & 0 & S_2(1, 2) & \cdots & 0 & S_2(1, m) \\
S_1(2, 1) & S_1(2, 1) & S_1(2, 2) & S_1(2, 2) & \cdots & S_1(2, m) & S_1(2, m) \\
0 & S_2(2, 1) & 0 & S_2(2, 2) & \cdots & 0 & S_2(2, m) \\
& & & & \cdots & & \\
S_1(n, 1) & S_1(n, 1) & S_1(n, 2) & S_1(n, 2) & \cdots & S_1(n, m) & S_1(n, m) \\
0 & S_2(n, 1) & 0 & S_2(n, 2) & \cdots & 0 & S_2(n, m)
\end{bmatrix},$$

$$[K] = \begin{bmatrix}
\frac{H_{i44}H_{j44}}{A_{44}} - H_{0.5(i+j)44} & i + j = \text{even} \\
\frac{H_{i44}H_{j44}}{A_{44}} - H_{j44} & i > j \\
\frac{H_{i44}H_{j44}}{A_{44}} - H_{j44} & j > i,
\end{bmatrix}$$

$$[L] = \begin{bmatrix}
\frac{j^2 H_{i+j-233}}{A_{22}} & i + j = \text{even} \\
\frac{j^2 H_{j-123}}{A_{22}} & i > j \\
\frac{j^2 H_{j-123}}{A_{22}} & j > i,
\end{bmatrix}$$

$$\{\mu\} = [\psi_1, \phi_1, \psi_2, \phi_2, \ldots, \psi_n, \phi_n]^T,$$
The general solution of (A.6) can be readily shown by

\[ \{\mu\} = [\psi] \cdot \sinh(\lambda y) \cdot \{H\} + [K]^{-1} \cdot [L] \cdot \{B\} \cdot y, \quad (A.9) \]

where \([\sinh(\lambda y)]\) is a 2n × 2m diagonal matrix. Also in (A.9), \([\psi]\) and \(\{\lambda_1, \lambda_2, \ldots, \lambda_{2m}\}\) represent the modal matrix and eigenvalues of \((-[M]^{-1} \cdot [K])\), respectively. It is clear that \([M]\) and \([K]\) are \((2n) \times (2m)\) matrices containing \(S_p(i, j)\) (\(P = 1, 2, \ldots, 6\)) and \([L]\) is a 2m × 3 matrix containing \(S_p(i, j)\) (\(Q = 7, 8, 9\)). Also \([H]\) is an unknown vector containing \(i + j\) constants of integration. Next, upon imposing the remaining boundary conditions (i.e., \(A_y = A_y = 0\) at either \(y = b\) or \(y = -b\)) in (12), the unknown vector \([H]\) is determined in terms of vector\([B]\).

B.

The coefficients \(A^{kj}, B^{kj}\), and \(D^{kj}\) appearing in (19) are given as

\[
A_{pq}^{kj} = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \frac{\partial \Phi_k}{\partial z} \cdot \frac{\partial \Phi_j}{\partial z} \, dz, \\
B_{pq}^{kj} = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \frac{\partial \Phi_k}{\partial z} \cdot \frac{\partial \Phi_j}{\partial z} \, dz \quad \text{for} \quad pq = 44, 55, \\
D_{pq}^{kj} = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \frac{\partial \Phi_k}{\partial z} \cdot \Phi_j \, dz \quad \text{for} \quad pq = 11, 12, 22, 66, \\
D_{pq}^{kj} = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \frac{\partial \Phi_k}{\partial z} \cdot \Phi_j \, dz \quad \text{for} \quad pq = 44, 55. \quad (B.1)
\]

The linear global interpolation function is defined as

\[
\Phi_k(z) = \begin{cases} 
0 & z \leq z_{k-1} \\
\psi_{k-1}^j(z) & z_{k-1} \leq z \leq z_k \\
\psi_k^j(z) & z_k \leq z \leq z_{k+1} \\
0 & z \geq z_{k+1} \end{cases} \quad (k = 1, 2, \ldots, N + 1), \quad (B.2)
\]

where \(\psi_k^j(j = 1, 2)\) are the local Lagrangian linear interpolation functions within the \(k\)th layer which are defined as

\[
\psi_k^1(z) = \frac{1}{h_k} (z - z_k), \quad \psi_k^2(z) = \frac{1}{h_k} (z - z_{k+1}), \quad (B.3)
\]

with \(h_k\) being the thickness of the \(k\)th layer.

C.

C.1. Analytical Solutions. The investigation is performed when analytically a rectangular composite laminate is subjected to the bending moment, the axial force, and the torque at its two opposite ends \((x = -a \text{ and } x = a)\) (see Figure 1). The linear Lagrangian interpolation functions through the thickness are used as in [26]. It is assumed that the laminates have invariably the traction-free boundary conditions at \(y = b\) and \(y = -b\). The boundary conditions at these edges are considered as

\[
M_{xy}^k = M_y^k = R_y^k = 0 \quad \text{at} \quad y = \pm b. \quad (C.1)
\]

Here the edges of the laminate are subjected to the bending moment \(M_0\), the axial force \(F_0\), and the torque \(T_0\) at \(x = \pm a: \)

\[
R_x^k = -2T_0 \int_{-h/2}^{h/2} \frac{\partial \Phi_k}{\partial z} \, dz, \\
M_{xy}^k = -2T_0 \int_{-h/2}^{h/2} \Phi_k(z) \left( \frac{1}{z} - 1 \right) \, dz,
\]

\[
(\text{A.8})
\]
\[ M^k_x = -2M_0 \int_{-h/2}^{h/2} \frac{\Phi_k(z)}{z} \, dz \]
\[ -2 \frac{F_0}{A} \int_{-h/2}^{h/2} \Phi_k(z) \, dz \quad \text{at} \ x = \pm a. \]  

(C.2)

To find analytical solutions for (C.2), it is assumed that

\[ U_j(x, y) = u_j(x, y) + U(x), \]
\[ V_j(x, y) = v_j(x, y) + V_j(x), \]
\[ W_j(x, y) = w_j(x, y) + W_j(x), \]  
\( j = 1, 2, \ldots, N + 1 \).

(C.3)

Upon substitution of these expressions into (20), two sets of equations are obtained. The first set contains \( U_j(x), V_j(x), \) and \( W_j(x) \) which are expressed by

\[ D_{11}^{kj} u_j'' + D_{44}^{kj} u_j' + A_{55}^{kj} u_j + (D_{12}^{kj} + D_{55}^{kj}) v_j' = 0, \]
\[ D_{66}^{kj} v_j'' - A_{44}^{kj} v_j = 0, \]
\[ (B_{55}^{kj} - B_{11}^{kj}) U_j + D_{55}^{kj} W_j'' - A_{33}^{kj} W_j = 0. \]  

(C.4)

The second set of equations contains \( u_j(x, y), v_j(x, y), \) and \( w_j(x, y) \) which are given by

\[ D_{11}^{kj} u_{j,xx} + D_{44}^{kj} u_{j,yy} - A_{55}^{kj} u_j + (D_{12}^{kj} + D_{55}^{kj}) v_j = 0, \]
\[ (B_{11}^{kj} - B_{55}^{kj}) u_{j,xx} + (D_{44}^{kj} + B_{55}^{kj}) w_j = 0, \]
\[ (B_{55}^{kj} - B_{11}^{kj}) u_{j,xx} + (D_{44}^{kj} + B_{55}^{kj}) v_j = 0, \]
\[ (B_{55}^{kj} - B_{11}^{kj}) u_{j,xx} + (D_{44}^{kj} + B_{55}^{kj}) w_j = 0. \]  

(C.5)

It should be emphasized that a repeated index in (C.4) and (C.5) is a summation index from 1 to \( N + 1 \). When the boundary conditions (C.4) are considered, substituting (C.3) into (C.2) yields

\[ B_{55}^{kj} U_j + D_{55}^{kj} W_j = -2T_0 n_j^k, \]  

(C.6a)

\[ D_{66}^{kj} V_j' = -2T_0 (n_j^k - n_j^k), \]  

(C.6b)

\[ D_{11}^{kj} U_j' + B_{11}^{kj} W_j = -2M_0 n_j^k - 2 \frac{F_0}{A} n_j^k \quad \text{at} \ x = \pm a, \]  

(C.6c)

where

\[ n_j^k = \int_{-h/2}^{h/2} \frac{\Phi_k(z)}{z} \, dz, \quad n_j^k = \int_{-h/2}^{h/2} \Phi_k(z) \, dz, \]  

(C.7)

Equations (C.4) subjected to the boundary conditions (C.6a), (C.6b), and (C.6c) can be solved analytically. It is to be noted that there exist repeated zero roots (or eigenvalues) in the characteristic equation of the set of equations in (C.4). To improve the solution scheme of these equations, some small artificial terms are added to these equations so that the characteristic roots become all distinct. So, (C.4) are rewritten as

\[ D_{11}^{kj} U_j'' - A_{55}^{kj} U_j + (B_{11}^{kj} - B_{55}^{kj}) W_j' = \alpha^{kj} U_j, \]
\[ D_{66}^{kj} V_j'' - A_{44}^{kj} V_j = \alpha^{kj} V_j, \]
\[ (B_{55}^{kj} - B_{11}^{kj}) U_j + D_{55}^{kj} W_j'' - A_{33}^{kj} W_j = \alpha^{kj} W_j, \]  

(C.9)

where, for convenience, \( \alpha^{kj} \) is assumed to be

\[ \alpha^{kj} = \alpha \int_{-h/2}^{h/2} \Phi_k \Phi_j \, dz, \]  

(C.10)

where \( \alpha \) is a prescribed number such that \( \alpha^{kj} \)'s in (C.10) are comparatively small compared to the numerical values of stiffnesses \( A_{pq}^{kj} \) \( (pq = 33, 44, 55) \). The system of equations appearing in (C.9) is \( 3(N + 1) \) coupled second-order ordinary differential equations with constant coefficients which may be introduced in a matrix form as

\[ [M] \{\eta''\} + [K] \{\eta\} = \{0\}, \]  

(C.11)

where

\[ \{\eta\} = \{U\}^T, \{V\}^T, \{W\}^T \}, \]
\[ \{U\} = \{U_1, U_2, \ldots, U_{N+1}\}^T, \]
\[ \{V\} = \{V_1, V_2, \ldots, V_{N+1}\}^T, \]
\[ \{W\} = \{W_1, W_2, \ldots, W_{N+1}\}^T, \]  

(C.12a)

\[ \overline{W} = \int_0^7 W_j \, dx. \]  

(C.12b)
The coefficient matrices $[M]$ and $[K]$ appearing in (C.11) are defined as

$$[M] = \begin{bmatrix} D_{11} & 0 & B_{13} - [B_{55}]^T \\ 0 & D_{66} & 0 \\ 0 & 0 & D_{55} \end{bmatrix},$$

$$[K] = \begin{bmatrix} -[A_{55}] - [\alpha] & 0 & 0 \\ 0 & -[A_{55}] - [\alpha] & 0 \\ [B_{55}] - [B_{13}]^T & 0 & -[A_{55}] - [\alpha] \end{bmatrix},$$

$$[H] = [Z_1]^{-1} [Z_2] \cdot a,$$

$$[Z_2] = \begin{bmatrix} -2T_0 [n_5] \\ -2T_0 ([n_z] - [n]) \\ -2M_0 [n_z] - 2 \frac{f_0}{A} [n] \end{bmatrix},$$

$$[Z_1] = [H_2] [\psi] [\lambda^2 \sinh (\lambda a)] + [H_2] [\psi] [\sin (\lambda a)],$$

$$[H_1] = \begin{bmatrix} [0] & [0] & [D_{55}] \\ [0] & [0] & [0] \\ [0] & [0] & [0] \end{bmatrix},$$

$$[H_2] = \begin{bmatrix} [B_{55}] & [0] & [D_{66}] \\ [0] & [D_{11}] & [0] \\ [0] & [B_{13}] \end{bmatrix}.\tag{C.12c}$$

The general solution of (C.11) can be written as

$$[\eta] = [\psi] [\sinh (\lambda x)] \{H\}, \tag{C.13}$$

where $[\sinh (\lambda x)]$, $[\psi]$, and $(\lambda_1^2, \lambda_2^2, \ldots, \lambda_{3(N+1)}^2)$ are a $3(N+1) \times 3(N+1)$ diagonal matrix, the modal matrix, and eigenvalues of $(-[M]^{-1} [K])$, respectively. In addition, $[H]$ is an unknown vector containing 3($N+1$) integration constants. It should be noted that by satisfying the boundary condition in (C.6a), (C.6b), and (C.6c) the integration constants in $[H]$ are found; then the problem is solved entirely.

It remains to solve (C.5) subject to the boundary conditions in (C.1) and (C.8). To this end, it is emphasized that the boundary conditions in (C.8) will identically be satisfied if the following expressions for the displacement components in (C.5) are assumed:

$$u_k (x, y) = \sum_{m=1}^{\infty} f_{1m}^k (y) \sin (\alpha_{1m} x) + f_{2m}^k (y) \cos (\alpha_{2m} x) + f_{10}^k (y),$$

$$v_k (x, y) = \sum_{m=1}^{\infty} f_{3m}^k (y) \sin (\alpha_{2m} x) + f_{4m}^k (y) \cos (\alpha_{1m} x) + f_{20}^k (y),$$

$$w_k (x, y) = \sum_{m=1}^{\infty} f_{5m}^k (y) \sin (\alpha_{2m} x) + f_{6m}^k (y) \cos (\alpha_{1m} x) + f_{30}^k (y), \tag{C.14}$$

$$f_{30}^k (y) + f_{20}^k (y) + (-1)^m \times \left[ D_{11}^k f_{1m}^{k \ell} + D_{12}^k f_{2m}^{k \ell} - D_{12}^k f_{3m}^{k \ell} + D_{12}^k f_{4m}^{k \ell} \right] = 0, \tag{C.15}$$

$$f_{10}^k (y) = 0,$$

where $\alpha_{m} = m \pi / a$, $\alpha_{2m} = m \pi / 2a$ with $m$ are the Fourier integer. Upon substitution of (C.14) into (C.5), the following ordinary differential equations are obtained:

$$D_{66} f_{1m}^{k \ell}' - (D_{11}^k f_{1m}^{k \ell} + A_{55}^k f_{1m} - \alpha_{1m} (D_{12}^k + D_{66}^k) f_{3m}^{k \ell} - \alpha_{1m} (B_{13}^k - B_{55}^k) f_{6m} = 0, \tag{C.16a}$$

$$\alpha_{1m} (D_{12}^k + D_{66}^k) f_{1m}^{k \ell} + (D_{66}^k f_{2m}^{k \ell} - (D_{66}^k \alpha_{1m}^2 + A_{44}^k) f_{4m}^{k \ell} - (B_{23}^k - B_{44}^k) f_{6m} = 0, \tag{C.16b}$$

$$\alpha_{1m} (B_{55}^k - B_{13}^k) f_{1m}^{k \ell} + (B_{44}^k - B_{23}^k) f_{2m}^{k \ell} + D_{44}^k f_{6m} = 0, \tag{C.16c}$$

$$D_{22}^k f_{20}^{k \ell} + (B_{23}^k - B_{44}^k) f_{20}^{k \ell} = 0, \tag{C.17}$$

Considering (C.14), the boundary conditions in (C.1) at $y = b$ and $y = -b$ are given as

$$M_{xy}^k = \sum_{m=1}^{\infty} \left[ D_{66}^k f_{1m}^{k \ell} - D_{66}^k f_{4m}^{k \ell} \alpha_{1m} \right] \sin (\alpha_{1m} x) + \left[ D_{66}^k f_{1m}^{k \ell} + D_{66}^k f_{4m}^{k \ell} \alpha_{2m} \right] \cos (\alpha_{2m} x) + D_{66}^k f_{30}^{k \ell} = 0,$$
The coefficient matrices \([S]\) and \([G]\) appearing in (C.21a) are given as

\[
I_1^k = \frac{1}{a} \int_a^a f^k(x) \sin(\alpha_{1m}x) \, dx,
\]
\[
K_1^k = \frac{1}{a} \int_a^a K(x) \cos(\alpha_{1m}x) \, dx,
\]
\[
I_2^m = \frac{1}{a} \int_a^a f^k(x) \cos(\alpha_{2m}x) \, dx,
\]
\[
K_2^m = \frac{1}{a} \int_a^a K(x) \sin(\alpha_{2m}x) \, dx.
\]

By introducing Fourier sine and cosine expansions for the underlined terms in (C.18), the following boundary conditions at \(y = b\) and \(y = -b\) can readily be obtained as

\[
\begin{align*}
D_{22}^{kj} f''_{20} + B_{23}^{kj} f''_{30} &= -\frac{J_0}{2}, \\
D_{44}^{kj} f''_{30} + B_{44}^{kj} f''_{40} &= -K_0.
\end{align*}
\]

where

\[
D_{66}^{kj} f''_{lm} - D_{66}^{kj} f''_{4m} \alpha_{1m} = -I_{1m},
\]

\[
D_{12}^{kj} f''_{1m} \alpha_{1m} + D_{12}^{kj} f''_{4m} + B_{23}^{kj} f''_{6m} = -J_{1m},
\]

\[
B_{44}^{kj} f''_{4m} + D_{44}^{kj} f''_{6m} = -K_{1m},
\]

\[
D_{66}^{kj} f''_{2m} + D_{66}^{kj} f''_{3m} \alpha_{2m} = -I_{2m},
\]

\[
-D_{12}^{kj} f''_{2m} \alpha_{2m} + D_{22}^{kj} f''_{3m} + B_{23}^{kj} f''_{5m} = -J_{2m},
\]

\[
B_{44}^{kj} f''_{5m} + D_{44}^{kj} f''_{6m} = -K_{2m},
\]

where

\[
\begin{align*}
D_{66}^{kj} f''_{j} &= I^k(x) \\
&= \sum_{m=1}^\infty I_{1m}^k \sin(\alpha_{1m}x) + I_{2m}^k \cos(\alpha_{2m}x) + \frac{J_0}{2}, \\
D_{12}^{kj} U''_{j} + B_{23}^{kj} W_j &= J^k(x) \\
&= \sum_{m=1}^\infty J_{1m}^k \sin(\alpha_{2m}x) + J_{2m}^k \cos(\alpha_{1m}x) + \frac{J_0}{2}, \\
B_{44}^{kj} V_j &= K^k(x) \\
&= \sum_{m=1}^\infty K_{1m}^k \sin(\alpha_{2m}x) + K_{2m}^k \cos(\alpha_{1m}x) + \frac{K_0}{2}.
\end{align*}
\]
\[
[S] = \begin{bmatrix}
[D_{66}] - \alpha_{1m} ([D_{12}] + [D_{66}]) & [0] \\
[0] & [D_{22}] \\
[0] & [B_{44}] - [B_{23}]^T \\
[0] & [D_{44}]
\end{bmatrix}, \tag{C.23c}
\]

\[
[G] = \begin{bmatrix}
- ([A_{55}] + \alpha_{1m} [D_{11}]) & \alpha_{1m} ([D_{12}] + [D_{66}]) - ([A_{44}] + \alpha_{1m} [D_{66}]) & 0 & -\alpha_{1m} ([B_{13}] - [B_{55}]^T) \\
0 & 0 & 0 & 0
\end{bmatrix}, \tag{C.23d}
\]

The coefficient matrices \([S]\) and \([G]\) appearing in (C.22) for solving (C.16b) are given as

\[
[S] = \begin{bmatrix}
[D_{66}] - \alpha_{2m} ([D_{12}] + [D_{66}]) & [0] \\
[0] & [D_{22}] \\
[0] & [B_{44}] - [B_{23}]^T \\
[0] & [D_{44}]
\end{bmatrix}, \tag{C.23e}
\]

\[
[G] = \begin{bmatrix}
- ([A_{55}] + \alpha_{2m} [D_{11}]) & \alpha_{2m} ([D_{12}] + [D_{66}]) - ([A_{44}] + \alpha_{2m} [D_{66}]) & 0 & -\alpha_{2m} ([B_{13}] - [B_{55}]^T) \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

The general solution of (C.22) can be written as

\[
(\mu) = \left[ \xi \right] \sinh (\lambda y) \left[ B \right], \tag{C.24}
\]

where \(\sinh(\lambda y), \left[ \xi \right], \) and \((\lambda_1^2, \lambda_2^2, \ldots, \lambda_{N+1}^2)\) are a \(3(N+1) \times 3(N+1)\) diagonal matrix, the modal matrix, and eigenvalues of \((- [S]^{-1} [G])\), respectively. Moreover, \([B]\) is an unknown vector containing \(3(N+1)\) integration constants. By satisfying the boundary condition in (C.20a) only at one edge, say, at \(y = b\), the integration constants in \([B]\) are obtained; then the problem is solved completely.

The solution procedure for (C.16b) with the boundary conditions in (C.20b) and (C.17) and with the boundary conditions in (C.19) is similar to the one discussed in (C.16b), and therefore, for the sake of brevity, it will not be taken up here. The boundary conditions in (C.15) will definitely be satisfied.

\section*{Conflict of Interests}

The authors declare that there is no conflict of interests regarding the publication of this paper.

\section*{References}


