Research Article

Finitely Generated Modules over Group Rings of a Direct Product of Two Cyclic Groups

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Let K be a commutative field of characteristic $p > 0$ and let $G = G_1 \times G_2$, where G_1 and G_2 are two finite cyclic groups. We give some structure results of finitely generated $K[G]$ -modules in the case where the order of G is divisible by p . Extensions of modules are also investigated. Based on these extensions and in the same previous case, we show that $K[G]$ -modules satisfying some conditions have a fairly simple form.

1. Introduction

Let K be a field of characteristic $p > 0$ and let G be a finite group. The study of $K[G]$ -modules in the case where the order of G is divisible by p is a very difficult task. When G is a finite abelian p -group, we find in [1] the following statement: a complete classification of finitely generated $K[G]$ -modules is available only when G is cyclic or equal to $C_2 \times C_2$, where C_2 is the cyclic group of order 2. In [2] we find this classification in these two cases. Still more, in the case where the Sylow p -subgroup P of G is not cyclic, the groups G such that $p = 2$ and P is dihedral, semidihedral, or generalized quaternion are the only groups for which we can (in principle) classify the indecomposable $K[G]$ -modules (see [2]). These reasons just cited show the importance of the study of $K[G]$ -modules when G is of order divisible by p and equal to a direct product of two cyclic groups.

Now, let K be a commutative field of characteristic $p > 0$ and let $G = G_1 \times G_2$, where G_1 and G_2 are two finite cyclic groups. Let M be a finitely generated $K[G]$ -module. When M is considered as a module over a subalgebra $K[H]$ of $K[G]$ for a subgroup H of the group G , we write $M \downarrow_H$.

In Section 2, we show that if G_1 is a cyclic p -group and the characteristic of K does not divide the order of G_2 , then we can have a complete system of indecomposable pairwise nonisomorphic $K[G]$ -modules. In the rest, we assume that

$G_1 = \langle \sigma_1 \rangle$ and $G_2 = \langle \sigma_2 \rangle$ are cyclic p -groups. Under conditions that $M \downarrow_{G_1}$ is a free $K[G_1]$ -module and that $M/(\sigma_1 - 1)M$ is a free $K[G_2]$ -module, we show that M is a free $K[G]$ -module. We also show that if σ_2 is of order p^n , $n \neq 0$, and H_2 is the subgroup of G_2 generated by $\sigma_2^{p^{n-r}}$ with $0 < r \leq n$, then under certain conditions M is a free $K[G_1 \times H_2]$ -module. The fact that $M \downarrow_{G_1}$ must be a free $K[G_1]$ -module is one of these conditions, and exactly in the end of this section we give a result that shows when this condition is satisfied. In Section 3 and always in the case where G_1 and G_2 are cyclic p -groups, we show that under some conditions $K[G]$ -modules have a fairly simple form. But in case $p = 2$, G_1 and G_2 are two cyclic groups of respective orders 2 and 2^n , $n \neq 0$; these modules have this simple form without any other assumptions other than that they must be finitely generated over $K[G]$.

2. Free $K[C_{p^m} \times C_{p^n}]$ -Modules of Finite Rank

Throughout this paper, rings are assumed to be commutative with unity. We begin this section by giving a weak version of Nakayama's lemma with an elementary proof.

Lemma 1 (Nakayama). *Let G be a p -group with p odd, R a ring of characteristic p^k where k is a natural number,*

M an $R[G]$ -module (not necessarily finitely generated), and N a submodule of M and $\sigma \in G$. Then, one has the following:

- (1) if $(\sigma - 1)M = M$, then $M = 0$;
- (2) if $M = (\sigma - 1)M + N$, then $M = N$;
- (3) if $x_i, i \in I$, are representatives in M of a generating family of $M/(\sigma - 1)M$, then $(x_i)_{i \in I}$ generate M .

Proof. (1) Let p^r be the order of G . We have

$$\begin{aligned}
 (\sigma - 1)^{p^r} &= \sum_{i=0}^{p^r} C_{p^r}^i (-1)^i \sigma^{p^r-i} \\
 &= \sigma^{p^r} + (-1)^{p^r} + \sum_{i=1}^{p^r-1} C_{p^r}^i (-1)^i \sigma^{p^r-i} \\
 &= \sum_{i=1}^{p^r-1} C_{p^r}^i (-1)^i \sigma^{p^r-i} \\
 &= p \sum_{i=1}^{p^r-1} \frac{1}{p} C_{p^r}^i (-1)^i \sigma^{p^r-i}.
 \end{aligned} \tag{1}$$

For $1 \leq i \leq p^r - 1$, $p \mid C_{p^r}^i$, so $(1/p)C_{p^r}^i$ is a natural number. So

$$\begin{aligned}
 (\sigma - 1)^{kp^r} &= ((\sigma - 1)^{p^r})^k \\
 &= \left(p \sum_{i=1}^{p^r-1} \frac{1}{p} C_{p^r}^i (-1)^i \sigma^{p^r-i} \right)^k \\
 &= p^k \left(\sum_{i=1}^{p^r-1} \frac{1}{p} C_{p^r}^i (-1)^i \sigma^{p^r-i} \right)^k \\
 &= 0,
 \end{aligned} \tag{2}$$

since R has characteristic p^k . Now $M = (\sigma - 1)M = (\sigma - 1)^2 M = \dots = (\sigma - 1)^{kp^r} M = 0$.

(2) If $M = (\sigma - 1)M + N$, then $M/N = ((\sigma - 1)M + N)/N = (\sigma - 1)(M/N)$, and then, by (1), $M/N = 0$ and then $M = N$.

(3) If N is the submodule generated by x_i , then $M = (\sigma - 1)M + N$, and then by (2) we have $M = N$. $\square$

Remark 2. Lemma 1 remains true if $p = 2$ and R is of characteristic 2.

For a ring R of prime characteristic p and for a cyclic group G of order p^r generated by an element σ , we have the following lemma.

Lemma 3. Let $k = p^{r'}$ with $0 < r' \leq r$ and let H be a subgroup of G generated by $\sigma^{p^{r-k}}$. Then, one has $R[G]/(\sigma - 1)^k R[G] \cong R[H]$ (as R -algebras).

Proof. Define

$$\begin{aligned}
 \psi : \frac{R[G]}{(\sigma - 1)^k R[G]} &\longrightarrow R[H], \\
 \bar{\sigma} &\longmapsto \sigma^{p^{r-k}},
 \end{aligned} \tag{3}$$

where ψ is a well-defined R -algebra homomorphism. It is easy to see that ψ is surjective. As $R[G]/(\sigma - 1)^k R[G]$ and $R[H]$ are finite free modules of the same rank k over R , ψ is an isomorphism. $\square$

Remark 4. With the notation of Lemma 3, H is simply the subgroup of G generated by $\sigma^{p^{r-r'}}$.

Let K be a commutative field of characteristic $p > 0$ and let G be a direct product of two finite groups G_1 and G_2 . We have $K[G] = K[G_1 \times G_2] \cong R[G_2]$, where $R = K[G_1]$.

Assume that G_1 is a cyclic group of order p^m generated by σ_1 and p does not divide the order of G_2 . $R = K[G_1]$ is a principal Artinian local ring. Indeed $K[G_1] \cong K[X]/(X - 1)^{p^m}$; this isomorphism is induced by the homomorphism $\Psi : K[X] \rightarrow K[G]$ defined by $\Psi(X) = \sigma_1$. $K[X]/(X - 1)^{p^m}$ is a principal Artinian local ring with residue field K (up to isomorphism) whose maximal ideal is generated by $\bar{X} - 1$. So R is a principal Artinian local ring with residue field K (up to isomorphism) whose maximal ideal is generated by $\sigma_1 - 1$. We have $K[G] \cong R[G_2]$, where R is a principal Artinian local ring of residue field K . The characteristic of K does not divide the order of G_2 . Under these conditions, we can apply [3, Theorem 3.6] to have a complete system of indecomposable pairwise nonisomorphic $K[G]$ -modules.

In the remainder of this section, we assume that $G_1 = C_{p^m}$ and $G_2 = C_{p^n}$ are two cyclic groups of respective orders p^m and p^n and are generated, respectively, by σ_1 and σ_2 . We have $K[G] \cong R[G_2]$. As R is a commutative ring and local and G_2 is a p -group, by [4, Proposition 10, page 239], $R[G_2]$ is a local ring. Therefore $K[G]$ is a local ring. As K is commutative ring and local and G is a p -group, $K[G]$ is a local ring by [4, Proposition 10, page 239]. So the $K[G]$ -projective modules are free $K[G]$ -modules.

Lemma 5. Let M be a $K[G]$ -module. Then, $M/(\sigma_1 - 1)M$ is a $K[G_2]$ -module (also $M/(\sigma_2 - 1)M$ is a $K[G_1]$ -module).

Proof. This lemma is a particular case of a more general result (see [5, page 386]). But for this particular case, we can give the following direct proof: $M/(\sigma_1 - 1)M$ is a $(K[G_1]/(\sigma_1 - 1)K[G_1])[G_2]$ -module, and we have already seen that $(\sigma_1 - 1)K[G_1]$ is the unique maximal ideal of $K[G_1]$ and $K[G_1]/(\sigma_1 - 1)K[G_1] \cong K$. So $M/(\sigma_1 - 1)M$ is a $K[G_2]$ -module.

Similarly we show that $M/(\sigma_2 - 1)M$ is a $K[G_1]$ -module. $\square$

Proposition 6. Let M be a free $K[G]$ -module of rank l . Then, $M/(\sigma_1 - 1)M$ is a free $K[G_2]$ -module and $M/(\sigma_2 - 1)M$ is a free $K[G_1]$ -module of the same rank l .

Proof. As $K[G]$ is a local ring, $K[G]$ -projective modules are free $K[G]$ -modules, and therefore this proposition is only a particular case of a more general result (see [5, Lemma 2.2.]). But for this particular case, we can give the following specific proof: we have $M \cong (K[G])^l = (K[G_1 \times G_2])^l$. So $M \cong (K[G_1][G_2])^l$. Then, we have

$$\begin{aligned} (\sigma_1 - 1)M &\cong (\sigma_1 - 1)(K[G_1][G_2])^l \\ &\cong ((\sigma_1 - 1)K[G_1][G_2])^l. \end{aligned} \quad (4)$$

Hence,

$$\begin{aligned} \frac{M}{(\sigma_1 - 1)M} &\cong \frac{(K[G_1][G_2])^l}{((\sigma_1 - 1)K[G_1][G_2])^l} \\ &\cong \left(\frac{K[G_1][G_2]}{(\sigma_1 - 1)K[G_1][G_2]} \right)^l \\ &\cong \left(\frac{K[G_1]}{(\sigma_1 - 1)K[G_1]} [G_2] \right)^l. \end{aligned} \quad (5)$$

As $K[G_1]/(\sigma_1 - 1)K[G_1] \cong K$ (as we have already seen), $M/(\sigma_1 - 1)M \cong (K[G_2])^l$. So $M/(\sigma_1 - 1)M$ is a free $K[G_2]$ -module of rank l .

Similarly we show that $M/(\sigma_2 - 1)M$ is a free $K[G_1]$ -module of rank l . $\square$

Proposition 7. *Let M be a $K[G]$ -module. If $M \downarrow_{G_1}$ is a free $K[G_1]$ -module and $M/(\sigma_1 - 1)M$ is a free $K[G_2]$ -module, then M is a free $K[G]$ -module.*

Proof. $R = K[G_1]$ is a principal Artinian local ring with residue field K and $\sigma_1 - 1$ is a generator of its maximal ideal. $M \downarrow_{G_1}$ is a free R -module and $Q = M/(\sigma_1 - 1)M$ is a projective $K[G_2]$ -submodule of $M/(\sigma_1 - 1)M$. Then, $M = P \oplus M'$, where P is a projective $R[G_2]$ -module and $P/(\sigma_1 - 1)P \cong Q = M/(\sigma_1 - 1)M$ (according to [3, Proposition 4.13]). We have

$$\begin{aligned} \frac{M}{(\sigma_1 - 1)M} &\cong \frac{P \oplus M'}{(\sigma_1 - 1)(P \oplus M')} \\ &\cong \frac{P \oplus M'}{(\sigma_1 - 1)P \oplus (\sigma_1 - 1)M'} \\ &\cong \frac{P}{(\sigma_1 - 1)P} \oplus \frac{M'}{(\sigma_1 - 1)M'} \\ &\cong \frac{M}{(\sigma_1 - 1)M} \oplus \frac{M'}{(\sigma_1 - 1)M'}. \end{aligned} \quad (6)$$

So $M'/(\sigma_1 - 1)M' = 0$. By Nakayama's lemma and the remark following it, $M' = 0$. Therefore, $M = P$ which is projective $R[G_2]$ -module. As $R[G_2] \cong K[G]$ is a local ring, M is a free $K[G]$ -module. $\square$

Let J_i be the Jacobson radical of $K[G_i]$ for $i \in \{1, 2\}$. Note that if K is of characteristic p (as here) and G' is a cyclic p -group, then the Jacobson radical of $K[G']$ is none other than $(\sigma - 1)K[G']$, where σ is a generator of G' (see [5, page 122]).

Let M be a finitely generated $K[G]$ -module and k a natural number such that $1 \leq k \leq p^n$. As $K[G] \cong R[G_2]$, M is a $R[G_2]$ -module. So M/J_1M is a $K[X]/(X - 1)^{p^n}$ -module. M is called of type k if M/J_1M is a free $K[X]/(X - 1)^k$ -module (terminology of [6]).

Lemma 8. *If M is a $K[G]$ -module of type k with $k = p^r$ and $0 < r \leq n$ and H_2 is the subgroup of G_2 generated by $\sigma_2^{p^{n-r}}$, then M/J_1M is a free $K[H_2]$ -module.*

Proof. As M is of type k , M/J_1M is a free $K[X]/(X - 1)^k$ -module. Define

$$\begin{aligned} \psi : \frac{K[X]}{(X - 1)^k} &\longrightarrow K[H_2], \\ \bar{X} &\longmapsto \sigma_2^{p^{n-r}}, \end{aligned} \quad (7)$$

where ψ is a well-defined K -algebra homomorphism. It is not difficult to show that ψ is an isomorphism (using an argument similar to that done in the proof of Lemma 3). So M/J_1M is a free $K[H_2]$ -module. $\square$

Theorem 9. *Let M be a $K[G]$ -module of type k , with $J_2^k M = 0$, and let H_2 be the subgroup of G_2 generated by $\sigma_2^{p^{n-r}}$ with $0 < r \leq n$. If $M \downarrow_{G_1}$ is R -free and $k = p^r$, then M is a free $K[G_1 \times H_2]$ -module.*

Proof. M is an $R[G_2]$ -module R -free. We have $J_2^k M = 0$, so $(\sigma_2 - 1)^k M = 0$, and therefore $((\sigma_2 - 1)^k R[G_2])M = 0$. So M is an $R[G_2]/(\sigma_2 - 1)^k R[G_2]$ -module R -free. By Lemma 3, $R[G_2]/(\sigma_2 - 1)^k R[G_2] \cong R[H_2]$; then M is an $R[H_2]$ -module R -free. M/J_1M is a free $K[X]/(X - 1)^k$ -module, so by Lemma 8 this is a free $K[H_2]$ -module. In conclusion M is a $K[G_1 \times H_2]$ -module such that

$$\begin{aligned} M \downarrow_{G_1} &\text{ is a free } k[G_1]\text{-module,} \\ \frac{M}{J_1M} &\text{ is a free } K[H_2]\text{-module.} \end{aligned} \quad (8)$$

So by Proposition 7 M is a free $K[G_1 \times H_2]$ -module. $\square$

In Theorem 9 we assumed that the $k[G]$ -module M satisfies the following condition: $M \downarrow_{G_1}$ is R -free. So it is useful to know when this condition is satisfied. This is the subject of the following result.

Theorem 10. *Let M be a $k[G]$ -module and σ an element of G of order p^r . The following conditions are equivalent:*

- (1) $M \downarrow_{\langle \sigma \rangle}$ is free;
- (2) $\dim_k(M) = (p^r/(p^r - 1))\dim_k((\sigma - 1)M)$;
- (3) $\dim_k(M) = \dim_k((\sigma - 1)M) + \dim_k((\sigma - 1)^{p^r-1}M)$.

Proof. (1) $\Rightarrow$ (2) Assume that $M \downarrow_{\langle \sigma \rangle}$ is free. There exists a nonzero natural number n such that $M \downarrow_{\langle \sigma \rangle} \cong (k[\langle \sigma \rangle])^n$. The endomorphism φ of M defined by $\varphi(m) = (\sigma - 1)m$ for all

$m \in M$ is nilpotent of nilpotency index p^r , and $k[\langle \sigma \rangle]$ is an indecomposable $k[\langle \sigma \rangle]$ -module. Therefore M has a basis in which the matrix of φ is a Jordan matrix. This matrix is formed of n blocks of order p^r all equal to

$$\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}. \quad (9)$$

So $\dim_k(M) = p^r \times n$. We can easily see that $\dim_k((\sigma-1)M) = n \times (p^r-1)$. Therefore $\dim_k(M) = (p^r/(p^r-1))\dim_k((\sigma-1)M)$.

(2) $\Rightarrow$ (3) Now, assume that $\dim_k(M) = (p^r/(p^r-1))\dim_k((\sigma-1)M)$. So $\dim_k(M) = p^r \times (\dim_k(M) - \dim_k((\sigma-1)M))$. As $\dim_k(M) - \dim_k((\sigma-1)M)$ is the number of blocks of the Jordan matrix of φ , the order of each block is less than or equal to p^r , and $\dim_k(M)$ is equal to the sum of the orders of these blocks, then the order of each block is p^r . Therefore $\dim_k((\sigma-1)^{p^r-1}M)$ is equal to the number of Jordan blocks of φ . So $\dim_k(M) - \dim_k((\sigma-1)M) = \dim_k((\sigma-1)^{p^r-1}M)$; that is, $\dim_k(M) = \dim_k((\sigma-1)M) + \dim_k((\sigma-1)^{p^r-1}M)$.

(3) $\Rightarrow$ (1) Assume that $\dim_k(M) = \dim_k((\sigma-1)M) + \dim_k((\sigma-1)^{p^r-1}M)$. So $\dim_k((\sigma-1)^{p^r-1}M)$ is equal to the number of Jordan blocks of φ . Therefore the order of each Jordan block is equal to p^r . So the modules contained in a decomposition of $M_{\downarrow \langle \sigma \rangle}$ as a direct sum of indecomposable modules are of the form $k[\langle \sigma \rangle]$; that is, $M_{\downarrow \langle \sigma \rangle}$ is free. $\square$

3. Classification of Finitely Generated $K[C_{p^m} \times C_{p^n}]$ -Modules: Use of Module Extensions

Let G be a finite group and let R be a ring. Let M_1 and M_2 be two $R[G]$ -modules. We put $T = \text{Hom}_R(M_2, M_1)$. T has a natural structure as a $(R[G], R[G])$ -bimodule centralized by R (see [7, section 25]). Explicitly, we have

$$\begin{aligned} (\lambda f)m_2 &= \lambda \cdot f(m_2), & (f\lambda)m_2 &= f(\lambda m_2) \\ \forall \lambda &\in R[G], \quad m_2 \in M_2, \quad f \in T. \end{aligned} \quad (10)$$

A derivation $F : R[G] \rightarrow T$ is an R -homomorphism satisfying

$$F_{\lambda'\lambda} = \lambda'F_\lambda + F_{\lambda'}\lambda \quad \forall \lambda, \lambda' \in R[G]. \quad (11)$$

Derivations from $R[G]$ into T form an R -module $\text{Der}(R[G], T)$. For $F \in \text{Der}(R[G], T)$ we equip $M_1 \times M_2$ with an $R[G]$ -module structure by

$$\begin{aligned} \lambda(m_1, m_2) &= (\lambda m_1 + F_\lambda(m_2), \lambda m_2) \\ \forall (m_1, m_2) &\in M_1 \times M_2, \quad \lambda \in R[G]. \end{aligned} \quad (12)$$

This $R[G]$ -module is denoted by $M_1 \times_F M_2$ as in [8].

An extension of M_2 by M_1 is an $R[G]$ -exact sequence $O \rightarrow M_1 \rightarrow X \rightarrow M_2 \rightarrow O$. Let $O \rightarrow M_1 \xrightarrow{u} X \xrightarrow{v} M_2 \rightarrow O$ and $O \rightarrow M_1 \xrightarrow{u'} X' \xrightarrow{v'} M_2 \rightarrow O$

be a pair of extensions of M_2 by M_1 . These two extensions are equivalent if there exists an isomorphism of $R[G]$ -modules $\Phi : X \rightarrow X'$ such that $\Phi u = u'$ and $v'o\Phi = v$. These equivalence classes of extensions form an R -module $\text{Ext}_{R[G]}^1(M_2, M_1)$. The $R[G]$ -modules sequence $O \rightarrow M_1 \xrightarrow{i} M_1 \times_F M_2 \xrightarrow{j} M_2 \rightarrow O$, where i and j denote, respectively, the canonical injection from M_1 to $M_1 \times_F M_2$ and the second projection from $M_1 \times_F M_2$ to M_2 , is exact. The equivalence class of this sequence is denoted by $[M_1 \times_F M_2]$.

Remark 11. With the previous notations, derivations F and modules $M_1 \times_F M_2$ play the same role as the cocycles α and modules $M_1 \times_\alpha M_2$ defined in [8].

From Proposition 25.10 of [7] we have the following result.

Proposition 12. *The correspondence $\theta : \text{Der}(R[G], T) \rightarrow \text{Ext}_{R[G]}^1(M_2, M_1)$ defined by $\theta(F) = [M_1 \times_F M_2]$ is surjective whenever M_2 is finitely generated and projective as R -module.*

From Theorems 5.2 and 5.3 of [9] we have the following result.

Proposition 13. *Let G be a cyclic group of order p^r generated by an element σ , K a field of characteristic p , and M an indecomposable $K[G]$ -module. Then, M is isomorphic to $(\sigma-1)^s K[G]$, where s is a natural number strictly less than p^r .*

Lemma 14. *Let R be a ring and $G = G_1 \times G_2$ a direct product of two finite groups. Let M be an $R[G]$ -module such that the action of G_1 on M is trivial and let M' be an $R[G_2]$ -module. If M is isomorphic to M' as $R[G_2]$ -modules and if we extend the action of G_2 on M' to G by $\sigma \cdot m' = m'$, $\forall (\sigma, m') \in G_1 \times M'$, then M is isomorphic to M' as $R[G]$ -modules.*

Proof. Let $\psi : M \rightarrow M'$ be an isomorphism of $R[G_2]$ -modules. We extend the action of G_2 on M' to G by $\sigma \cdot m' = m'$, $\forall (\sigma, m') \in G_1 \times M'$. We easily see that the application $\psi : M \rightarrow M'$ is an isomorphism of $R[G]$ -modules. $\square$

Let K be a commutative field of characteristic $p > 0$. Let $G = G_1 \times G_2$, where $G_1 = C_{p^m}$ and $G_2 = C_{p^n}$ are two cyclic groups of respective orders p^m and p^n and are generated, respectively, by σ_1 and σ_2 , and let J_1 be the Jacobson radical of $K[G_1]$.

Proposition 15. *Let M be a finitely generated $K[G]$ -module. If $J_1 M = 0$, then there exists a nonzero natural number n' such that $M \cong \oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$, $0 \leq k_i < p^n$, as $K[G]$ -modules, where the action of G_1 on $\oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$ is trivial.*

Proof. If $J_1 M = 0$, then the action of G_1 on M is trivial since $J_1 = (\sigma_1 - 1)K[G_1]$. By Proposition 13, there exists a nonzero natural number n' such that $M \cong \oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$, $0 \leq k_i < p^n$, as $K[G_2]$ -modules. Then, Lemma 14 allows concluding the following. $\square$

Theorem 16. Let M be a finitely generated $K[G]$ -module. If $J_1^2 M = 0$, then there exist two nonzero natural numbers n' and n'' and two $K[G]$ -modules $M_1 = \oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$, $0 \leq k_i < p^n$, and $M_2 = \oplus_{i=1}^{n''} (\sigma_2 - 1)^{k'_i} K[G_2]$, $0 \leq k'_i < p^n$, where the action of G_1 on M_1 and M_2 is trivial, and there is a derivation F from $K[G]$ in $\text{Hom}_K(M_2, M_1)$ such that $M \cong M_1 \times_F M_2$.

Proof. We have the exact sequence of $K[G]$ -modules $0 \rightarrow J_1 M \hookrightarrow M \rightarrow M/J_1 M \rightarrow 0$. As $J_1^2 M = 0$, $J_1(J_1 M) = 0$. So by Proposition 15 there exists a nonzero natural number n' such that $J_1 M \cong \oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$, $0 \leq k_i < p^n$, as $K[G]$ -modules, where the action of G_1 on $\oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$ is trivial. We set $M_1 = \oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$. We have $J_1(M/J_1 M) = 0$. So by Proposition 15 there exists a nonzero natural number n'' such that $M/J_1 M \cong \oplus_{i=1}^{n''} (\sigma_2 - 1)^{k'_i} K[G_2]$, $0 \leq k'_i < p^n$, as $K[G]$ -modules, where the action of G_1 on $\oplus_{i=1}^{n''} (\sigma_2 - 1)^{k'_i} K[G_2]$ is trivial. We set $M_2 = \oplus_{i=1}^{n''} (\sigma_2 - 1)^{k'_i} K[G_2]$. Then, Proposition 12 shows that $M \cong M_1 \times_F M_2$ for a derivation F from $K[G]$ in $\text{Hom}_K(M_2, M_1)$. $\square$

If $p = 2$, $G_1 = C_2$, and $G_2 = C_{2^n}$, then we have the following corollary.

Corollary 17. For all finitely generated $K[G]$ -modules there exist two nonzero natural numbers n' and n'' and two $K[G]$ -modules $M_1 = \oplus_{i=1}^{n'} (\sigma_2 - 1)^{k_i} K[G_2]$, $0 \leq k_i < 2^n$, and $M_2 = \oplus_{i=1}^{n''} (\sigma_2 - 1)^{k'_i} K[G_2]$, $0 \leq k'_i < 2^n$, where the action of G_1 on M_1 and M_2 is trivial, and there is a derivation F from $K[G]$ in $\text{Hom}_K(M_2, M_1)$ such that $M \cong M_1 \times_F M_2$.

Proof. We have $J_1 = (\sigma_1 - 1)K[G_1]$ and as $(\sigma_1 - 1)^2 = 0$ since the field K is of characteristic $p = 2$, $J_1^2 = 0$. So $J_1^2 M = 0$ for $K[G]$ -module of finite type M . The rest is a simple application of Theorem 16. $\square$

Now we return to cases $G_1 = C_{p^m}$ and $G_2 = C_{p^n}$, where G_2 is generated by an element σ_2 , and let J_2 be the Jacobson radical of $K[G_2]$. For an integer $k = p^r$ with $0 < r \leq n$ and for the subgroup H_2 of G_2 generated by $\sigma_2^{p^{n-r}}$, we have the following result.

Theorem 18. Let M be a finitely generated $K[G]$ -module, with $J_2^{k+1} M = 0$. If $M/J_2^k M \downarrow_{G_1}$ is R -free and of type k , then there exist two nonzero natural numbers n' and n'' and two $K[G]$ -modules $M_1 = \oplus_{i=1}^{n'} (\sigma_1 - 1)^{k_i} K[G_1]$, $0 \leq k_i < p^n$, and $M_2 = (K[G_1 \times H_2])^{n''}$, where the action of G_2 on M_1 is trivial, and there is a derivation F from $K[G]$ in $\text{Hom}_K(M_2, M_1)$ such that $M \cong M_1 \times_F M_2$.

Proof. We have the following exact sequence:

$$0 \rightarrow J_2^k M \hookrightarrow M \rightarrow \frac{M}{J_2^k M} \rightarrow 0. \quad (13)$$

As $J_2(J_2^k M) = 0$, by Proposition 15, there exists a nonzero natural number n' such that $J_2^k M \cong \oplus_{i=1}^{n'} (\sigma_1 - 1)^{k_i} K[G_1]$,

$0 \leq k_i < p^n$, as $K[G]$ -modules, where the action of G_2 on $\oplus_{i=1}^{n'} (\sigma_1 - 1)^{k_i} K[G_1]$ is trivial. $M/J_2^k M$ is a $K[G]$ -module of type k with $J_2^k(M/J_2^k M) = 0$, more $M/J_2^k M \downarrow_{G_1}$ is R -free, and $k = p^r$ with $0 < r \leq n$. Then, Theorem 9 shows that $M/J_2^k M$ is a free $K[G_1 \times H_2]$ -module. Therefore there exists a nonzero natural number n'' such that $M/J_2^k M \cong (K[G_1 \times H_2])^{n''}$. The rest is a simple application of Proposition 12. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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