Research Article

Almost Periodic Solution of a Discrete Schoener's Competition Model with Delays

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Received 8 June 2014; Accepted 12 July 2014; Published 24 July 2014

Academic Editor: Cengiz Çinar

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We consider an almost periodic discrete Schoener's competition model with delays. By means of an almost periodic functional hull theory and constructing a suitable Lyapunov function, sufficient conditions are obtained for the existence of a unique strictly positive almost periodic solution which is globally attractive. An example together with numerical simulation indicates the feasibility of the main result.

1. Introduction

In 2009, Wu et al. [1] had studied a discrete Schoener's competition mode with delays:

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left[\frac{a_{10}(n)}{x_1(n-\tau_{10}) + k_1(n)} - a_{11}(n)x_1(n-\tau_{11}) \right. \\ &\quad \left. - a_{12}(n)x_2(n-\tau_{12}) - c_1(n) \right], \\ x_2(n+1) &= x_2(n) \exp \left[\frac{a_{20}(n)}{x_2(n-\tau_{20}) + k_2(n)} - a_{21}(n)x_1(n-\tau_{21}) \right. \\ &\quad \left. - a_{22}(n)x_2(n-\tau_{22}) - c_2(n) \right], \end{aligned} \quad (1)$$

where $\{k_i(n)\}$, $\{a_{ij}(n)\}$, and $\{c_i(n)\}$ are real positive bounded sequences and τ_{ij} are positive integers, $i = 1, 2$; $j = 0, 1, 2$. Sufficient conditions which guarantee the permanence and the global attractivity of positive solutions for system (1) are obtained.

By the biological meaning, the system (1) is considered together with the following initial condition:

$$\begin{aligned} x_i(s) &= \varphi_i(s) \geq 0, \quad \varphi_i(0) > 0, \\ s &\in [-\tau, 0] \cap \mathbb{Z} = \{-\tau, -\tau+1, \dots, 0\}, \quad i = 1, 2, \end{aligned} \quad (2)$$

where $\tau = \max_{1 \leq i \leq 2, 0 \leq j \leq 2} \{\tau_{ij}\}$. Let $(x_1(n), x_2(n))$ be any solution of system (1) with the initial condition (2). One could easily see that $x_i(n) > 0$, $i = 1, 2$ for all $n \in \mathbb{N}$.

Schoener's competition system has been studied by many scholars. Topics such as existence, uniqueness, and global attractivity of positive periodic solutions of the system were extensively investigated, and many excellent results have been derived (see [1–6] and the references cited therein). Recently, a few papers investigate the global stability of the pure-delay model (see [7–13]).

Notice that the investigation of almost periodic solutions for difference equations is one of most important topics in the qualitative theory of difference equations due to the applications in biology, ecology, neural network, and so forth (see [14–21] and the references cited therein). But to the best of the author's knowledge, to this day, still no scholars have studied the almost periodic version which is corresponding to system (1). Therefore, with stimulation from the works of [12, 19], the main purpose of this paper is to derive a set of sufficient conditions ensuring the existence of a unique

strictly positive almost periodic solution of system (1) which is globally attractive.

Denote by Z and Z^+ the set of integers and the set of nonnegative integers, respectively. For any bounded sequence $\{f(n)\}$ defined on Z , define $f^u = \sup_{n \in Z} f(n)$, $f^l = \inf_{n \in Z} f(n)$.

Throughout this paper, we assume the following.

(H1) $k_i(n)$, $a_{ij}(n)$, and $c_i(n)$ are bounded positive almost periodic sequences such that

$$\begin{aligned} 0 < k_i^l &\leq k_i(n) \leq k_i^u, \\ 0 < a_{ij}^l &\leq a_{ij}(n) \leq a_{ij}^u, \\ 0 < c_i^l &\leq c_i(n) \leq c_i^u, \\ i = 1, 2, \quad j = 0, 1, 2. \end{aligned} \quad (3)$$

The remaining part of this paper is organized as follows. In Section 2, we will introduce some definitions and several useful lemmas. In Section 3, by applying the theory of difference inequality, we present the permanence results for system (1). In Section 4, we establish the sufficient conditions for the existence of a unique globally attractive almost periodic solution of system (1). The main result is illustrated by an example with a numerical simulation in the last section.

2. Preliminaries

In this section, we give the definitions and lemmas of the terminologies involved.

Definition 1 (see [22]). A sequence $x : Z \rightarrow R$ is called an almost periodic sequence if the ε -translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : |x(n + \tau) - x(n)| < \varepsilon, \forall n \in Z\} \quad (4)$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$|x(n + \tau) - x(n)| < \varepsilon, \quad \forall n \in Z. \quad (5)$$

τ is called an ε -translation number of $x(n)$.

Definition 2 (see [23]). Let D be an open subset of R^m , $f : Z \times D \rightarrow R^m$. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in D$ if for any $\varepsilon > 0$, and any compact set $S \subset D$, there exists a positive integer $l = l(\varepsilon, S)$ such that any interval of length l contains an integer τ for which

$$|f(n + \tau, x) - f(n, x)| < \varepsilon, \quad \forall (n, x) \in Z \times S. \quad (6)$$

τ is called an ε -translation number of $f(n, x)$.

Definition 3 (see [24]). The hull of f , denoted by $H(f)$, is defined by

$$\begin{aligned} H(f) &= \left\{ g(n, x) : \lim_{k \rightarrow \infty} f(n + \tau_k, x) \right. \\ &\quad \left. = g(n, x) \text{ uniformly on } Z \times S \right\}, \end{aligned} \quad (7)$$

for some sequence $\{\tau_k\}$, where S is any compact set in D .

Definition 4. Suppose that $X(n) = (x_1(n), x_2(n))$ is any solution of system (1). $X(n)$ is said to be a strictly positive solution in Z if

$$0 < \inf_{n \in Z} x_i(n) \leq \sup_{n \in Z} x_i(n) < \infty, \quad i = 1, 2, \quad (8)$$

for $n \in Z$.

Lemma 5 (see [25]). Assume that $\{x(n)\}$ satisfies $x(n) > 0$ and

$$x(n+1) \leq x(n) \exp\{a(n) - b(n)x(n)\} \quad (9)$$

for $n \in N$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants. Then

$$\limsup_{n \rightarrow +\infty} x(n) \leq \frac{1}{b^l} \exp\{a^u - 1\}. \quad (10)$$

Lemma 6 (see [25]). Assume that $\{x(n)\}$ satisfies

$$x(n+1) \geq x(n) \exp\{a(n) - b(n)x(n)\}, \quad n \geq N_0, \quad (11)$$

$$\limsup_{n \rightarrow +\infty} x(n) \leq x^*,$$

and $x(N_0) > 0$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants and $N_0 \in N$. Then

$$\liminf_{n \rightarrow +\infty} x(n) \geq \min \left\{ \frac{a^l}{b^u} \exp\{a^l - b^u x^*\}, \frac{a^l}{b^u} \right\}. \quad (12)$$

3. Permanence and Global Attractivity

Now we state several lemmas which will be useful in proving our main result.

Proposition 7 (see [1]). Any solution $(x_1(n), x_2(n))$ of system (1) with the initial condition (2) is positive and ultimately bounded; that is,

$$\limsup_{n \rightarrow +\infty} x_i(n) \leq M_i, \quad (13)$$

where

$$M_i = \frac{1}{a_{ii}^l} \exp \left\{ \frac{a_{i0}^u}{k_i^l} (\tau_{ii} + 1) - 1 \right\}, \quad i = 1, 2. \quad (14)$$

Proposition 8. Assume that

$$\begin{aligned} (H2) \quad & \frac{a_{10}^l}{M_1 + k_1^u} - a_{12}^u M_2 - c_1^u > 0, \\ & \frac{a_{20}^l}{M_2 + k_2^u} - a_{21}^u M_1 - c_2^u > 0 \end{aligned} \quad (15)$$

holds; then for any solution $(x_1(n), x_2(n))$ of system (1) with the initial condition (2), one has

$$\liminf_{n \rightarrow +\infty} x_i(n) \geq m_i, \quad i = 1, 2, \quad (16)$$

where m_1 and m_2 are defined by (24) and (26), respectively.

Proof. For any small positive constant $\varepsilon > 0$, according to Proposition 7, there exists a $N_1 > 0$ such that, for all $n > N_1$ and $i = 1, 2$,

$$x_i(n) \leq M_i + \varepsilon. \quad (17)$$

It follows from (17) and the first equation of system (1), for $n \geq N_1 + \tau$,

$$\begin{aligned} & x_1(n+1) \\ & \geq x_1(n) \exp \left[\frac{a_{10}^l}{(M_1 + \varepsilon) + k_1^u} - a_{11}^u (M_1 + \varepsilon) \right. \\ & \quad \left. - a_{12}^u (M_2 + \varepsilon) - c_1^u \right] \triangleq x_1(n) \exp \{A_1^\varepsilon\}, \end{aligned} \quad (18)$$

where $A_1^\varepsilon = (a_{10}^l / ((M_1 + \varepsilon) + k_1^u)) - a_{11}^u (M_1 + \varepsilon) - a_{12}^u (M_2 + \varepsilon) - c_1^u$. Thus, by using (18) we obtain

$$x_1(n - \tau_{11}) \leq x_1(n) \exp \{-\tau_{11} A_1^\varepsilon\}. \quad (19)$$

Substituting (19) into the first equation of system (1), for $n \geq N_1 + \tau_{11}$, it follows that

$$\begin{aligned} & x_1(n+1) \\ & \geq x_1(n) \exp \left[\frac{a_{10}^l}{(M_1 + \varepsilon) + k_1^u} - a_{12}^u (M_2 + \varepsilon) \right. \\ & \quad \left. - c_1^u - a_{11}(n) \exp \{-\tau_{11} A_1^\varepsilon\} x_1(n) \right]. \end{aligned} \quad (20)$$

When ε is an arbitrary small positive constant, it follows from condition (H2)

$$\frac{a_{10}^l}{(M_1 + \varepsilon) + k_1^u} - a_{12}^u (M_2 + \varepsilon) - c_1^u > 0. \quad (21)$$

Thus, as a direct corollary of Lemma 6, according to (14) and (20), one has

$$\liminf_{n \rightarrow +\infty} x_1(n) \geq \min \{B_{1\varepsilon}, B_{2\varepsilon}\}, \quad (22)$$

where

$$\begin{aligned} B_{1\varepsilon} &= \frac{1}{a_{11}^u} \left[\frac{a_{10}^l}{(M_1 + \varepsilon) + k_1^u} - a_{12}^u (M_2 + \varepsilon) - c_1^u \right] \exp \{\tau_{11} A_1^\varepsilon\}, \\ B_{2\varepsilon} &= B_{1\varepsilon} \exp \left\{ \frac{a_{10}^l}{(M_1 + \varepsilon) + k_1^u} - a_{12}^u (M_2 + \varepsilon) \right. \\ & \quad \left. - c_1^u - M_1 a_{11}^u \exp \{-\tau_{11} A_1^\varepsilon\} \right\}. \end{aligned} \quad (23)$$

Letting $\varepsilon \rightarrow 0$, it follows that

$$\liminf_{n \rightarrow +\infty} x_1(n) \geq \frac{1}{2} \min \{B_1, B_2\} \triangleq m_1 > 0, \quad (24)$$

where

$$\begin{aligned} B_1 &= \frac{1}{a_{11}^u} \left[\frac{a_{10}^l}{M_1 + k_1^u} - a_{12}^u M_2 - c_1^u \right] \exp \{\tau_{11} A_1\}, \\ B_2 &= B_1 \exp \left\{ \frac{a_{10}^l}{M_1 + k_1^u} - a_{12}^u M_2 \right. \\ & \quad \left. - c_1^u - M_1 a_{11}^u \exp \{-\tau_{11} A_1\} \right\}, \\ A_1 &= \frac{a_{10}^l}{M_1 + k_1^u} - a_{11}^u M_1 - a_{12}^u M_2 - c_1^u. \end{aligned} \quad (25)$$

Similar to the analysis of (18)–(24), by applying (17), from the second equation of system (1), we also have that

$$\liminf_{n \rightarrow +\infty} x_2(n) \geq \frac{1}{2} \min \{D_1, D_2\} \triangleq m_2 > 0, \quad (26)$$

where

$$\begin{aligned} D_1 &= \frac{1}{a_{22}^u} \left[\frac{a_{20}^l}{M_2 + k_2^u} - a_{21}^u M_1 - c_2^u \right] \exp \{\tau_{22} C_1\}, \\ D_2 &= D_1 \exp \left\{ \frac{a_{20}^l}{M_2 + k_2^u} - a_{21}^u M_1 - c_2^u \right. \\ & \quad \left. - M_2 a_{22}^u \exp \{-\tau_{22} C_1\} \right\}, \\ C_1 &= \frac{a_{20}^l}{M_2 + k_2^u} - a_{22}^u M_2 - a_{21}^u M_1 - c_2^u. \end{aligned} \quad (27)$$

The proof is completed. $\square$

Note that condition (H2) of Proposition 8 is weakened compared to condition (H) in [1].

Theorem 9 (see [1]). Assume that (H2) holds; system (1) is permanent.

Proposition 10 (see [1]). Assume that (H2) holds and further that there exist positive constants α_1, α_2 , and $\eta > 0$ such that

$$(H3) \quad \alpha_i E_{ii} - (\alpha_i F_{ij} + \alpha_j G_{ji}) \geq \eta, \quad i, j = 1, 2, \quad i \neq j, \quad (28)$$

where

$$\begin{aligned} E_{ii} &= \min \left(a_{ii}^l, \frac{2}{M_i} - a_{ii}^u \right) - \frac{a_{i0}^u}{(k_i^l)^2} (1 + \tau_{ii} a_{ii}^u M_i B_i^u) \\ & \quad - \tau_{ii} (a_{ii}^u)^2 M_i B_i^u, \\ F_{ij} &= \tau_{ii} a_{ii}^u A_i^u \left(\frac{a_{i0}^u}{k_i^l} + a_{ii}^u M_i + a_{ij}^u M_j + c_i^u \right), \\ G_{ji} &= a_{ji}^u + \tau_{jj} a_{jj}^u M_j B_j^u a_{ji}^u, \quad i, j = 1, 2; \quad i \neq j, \end{aligned}$$

$$\begin{aligned}
A_i(s) &= \exp \left\{ \theta_i(s) \left(\frac{a_{i0}(s)}{x_i^*(s - \tau_{i0}) + k_i(s)} - a_{ii}x_i^*(s - \tau_{ii}) \right. \right. \\
&\quad \left. \left. - a_{ij}x_j^*(s - \tau_{ij}) - c_i(s) \right) \right\}, \\
B_i(s) &= \exp \left\{ \varphi_i(s) \left(\frac{a_{i0}(s)}{x_i(s - \tau_{i0}) + k_i(s)} - a_{ii}x_i(s - \tau_{ii}) \right. \right. \\
&\quad \left. \left. - a_{ij}x_j(s - \tau_{ij}) - c_i(s) \right) \right. \\
&\quad \left. + (1 - \varphi_i(s)) \left(\frac{a_{i0}(s)}{x_i^*(s - \tau_{i0}) + k_i(s)} \right. \right. \\
&\quad \left. \left. - a_{ii}x_i^*(s - \tau_{ii}) \right. \right. \\
&\quad \left. \left. - a_{ij}x_j^*(s - \tau_{ij}) - c_i(s) \right) \right\}, \quad (29)
\end{aligned}$$

$\theta_i(s), \varphi_i(s) \in (0, 1)$, $i, j = 1, 2$, $i \neq j$. Then system (1) is globally attractive; that is, for any two positive solutions $(x_1(n), x_2(n))$ and $(x_1^*(n), x_2^*(n))$ of system (1), we have

$$\lim_{n \rightarrow +\infty} (x_i^*(n) - x_i(n)) = 0, \quad i = 1, 2. \quad (30)$$

4. Almost Periodic Solution

In this section, by means of an almost periodic functional hull theory and constructing a suitable Lyapunov function, we will study the existence of a globally attractive almost periodic solution of system (1) with initial condition (2) and obtain the sufficient conditions.

Let $\{\delta_k\}$ be any integer valued sequence such that $\delta_k \rightarrow \infty$ as $k \rightarrow \infty$. According to Lemma 5, taking a subsequence if necessary, we have $k_i(n + \delta_k) \rightarrow k_i^*(n)$, $a_{ij}(n + \delta_k) \rightarrow a_{ij}^*(n)$, $c_i(n + \delta_k) \rightarrow c_i^*(n)$, $i = 1, 2$, $j = 0, 1, 2$, as $k \rightarrow \infty$ for $n \in Z$. Then we get a hull equation of system (1) as follows:

$$\begin{aligned}
&x_1(n+1) \\
&= x_1(n) \exp \left[\frac{a_{10}^*(n)}{x_1(n - \tau_{10}) + k_1^*(n)} - a_{11}^*(n)x_1(n - \tau_{11}) \right. \\
&\quad \left. - a_{12}^*(n)x_2(n - \tau_{12}) - c_1^*(n) \right], \\
&x_2(n+1) \\
&= x_2(n) \exp \left[\frac{a_{20}^*(n)}{x_2(n - \tau_{20}) + k_2^*(n)} - a_{21}^*(n)x_1(n - \tau_{21}) \right. \\
&\quad \left. - a_{22}^*(n)x_2(n - \tau_{22}) - c_2^*(n) \right]. \quad (31)
\end{aligned}$$

By the almost periodic theory, we can conclude that if system (1) satisfies (H2) and (H3), then the hull equation (31) of system (1) also satisfies (H2) and (H3).

By Theorem 3.4 in [26], we can easily obtain the lemma as follows.

Lemma 11. *If each hull equation of system (1) has a unique strictly positive solution, then the almost periodic difference system (1) has a unique strictly positive almost periodic solution.*

Now we investigate a globally attractive almost periodic solution of system (1).

Theorem 12. *If the almost periodic difference system (1) satisfies (H1), (H2), and (H3), then the almost periodic difference system (1) admits a unique strictly positive almost periodic solution, which is globally attractive.*

Proof. By Lemma 11, we only need to prove that each hull equation of system (1) has a unique globally attractive almost periodic sequence solution; hence we firstly prove that each hull equation of system (1) has at least one strictly positive solution (the existence) and then we prove that each hull equation of system (1) has a unique strictly positive solution (the uniqueness).

Now we prove the existence of a strictly positive solution of any hull equation (31). By the almost periodicity of $\{k_i(n)\}$, $\{a_{ij}(n)\}$, and $\{c_i(n)\}$, $i = 1, 2$, $j = 0, 1, 2$, there exists an integer valued sequence $\{\tau_k\}$ with $\tau_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $k_i^*(n + \tau_k) \rightarrow k_i^*(n)$, $a_{ij}^*(n + \tau_k) \rightarrow a_{ij}^*(n)$, $c_i^*(n + \tau_k) \rightarrow c_i^*(n)$, $i = 1, 2$, $j = 0, 1, 2$, as $k \rightarrow \infty$ for $n \in Z$. Suppose that $X = (x_1(n), x_2(n))$ is any solution of hull equation (31). By the proof of Propositions 7 and 8, we have

$$m_i \leq \liminf_{n \rightarrow +\infty} x_i(n) \leq \limsup_{n \rightarrow +\infty} x_i(n) \leq M_i, \quad i = 1, 2. \quad (32)$$

And also

$$0 < \inf_{n \in Z^+} x_i(n) \leq \sup_{n \in Z^+} x_i(n) < \infty, \quad i = 1, 2. \quad (33)$$

Let ε be an arbitrary small positive number. It follows from (32) that there exists a positive integer n_0 such that $m_i - \varepsilon \leq x_i(n) \leq M_i + \varepsilon$, $n \geq n_0$, $i = 1, 2$. Write $X_k(n) = X(n + \tau_k) = (x_{1k}(n), x_{2k}(n))$, for all $n \geq n_0 + \tau - \tau_k$, $k \in Z^+$. We claim that there exists a sequence $\{y_i(n)\}$, and a subsequence of $\{\tau_k\}$, which we still denote by $\{\tau_k\}$ such that $x_{ik}(n) \rightarrow y_i(n)$ uniformly in n on any finite subset B of Z as $k \rightarrow \infty$, where $B = \{a_1, a_2, \dots, a_m\}$, $a_h \in Z$ ($h = 1, 2, \dots, m$), and m is a finite number.

In fact, for any finite subset $B \subset Z$, when k is large enough, $\tau_k + a_h - \tau > n_0$, $h = 1, 2, \dots, m$. So

$$m_i - \varepsilon \leq x_i(n + \tau_k) \leq M_i + \varepsilon, \quad i = 1, 2; \quad (34)$$

that is, $\{x_i(n + \tau_k)\}$ are uniformly bounded for k large enough.

Now, for $a_1 \in B$, we can choose a subsequence $\{\tau_k^{(1)}\}$ of $\{\tau_k\}$ such that $\{x_i(a_1 + \tau_k^{(1)})\}$ uniformly converges on Z^+ for k large enough.

Similarly, for $a_2 \in B$, we can choose a subsequence $\{\tau_k^{(2)}\}$ of $\{\tau_k^{(1)}\}$ such that $\{x_i(a_2 + \tau_k^{(2)})\}$ uniformly converges on Z^+ for k large enough.

Repeating this procedure, for $a_m \in B$, we can choose a subsequence $\{\tau_k^{(m)}\}$ of $\{\tau_k^{(m-1)}\}$ such that $\{x_i(a_m + \tau_k^{(m)})\}$ uniformly converges on Z^+ for k large enough.

Now pick the sequence $\{\tau_k^{(m)}\}$ which is a subsequence of $\{\tau_k\}$, which we still denote by $\{\tau_k\}$, then for all $n \in B$, we have $x_i(n + \tau_k) \rightarrow y_i(n)$ uniformly in $n \in B$, as $k \rightarrow \infty$.

By the arbitrary of B , the conclusion is valid.

Combining with

$$\begin{aligned} & x_{1k}(n+1) \\ &= x_{1k}(n) \exp \left[\frac{a_{10}^*(n + \tau_k)}{x_{1k}(n - \tau_{10}) + k_1^*(n + \tau_k)} \right. \\ &\quad \left. - a_{11}^*(n + \tau_k) x_{1k}(n - \tau_{11}) \right. \\ &\quad \left. - a_{12}^*(n + \tau_k) x_{2k}(n - \tau_{12}) - c_1^*(n + \tau_k) \right], \end{aligned}$$

$$\begin{aligned} & x_{2k}(n+1) \\ &= x_{2k}(n) \exp \left[\frac{a_{20}^*(n + \tau_k)}{x_{2k}(n - \tau_{20}) + k_2^*(n + \tau_k)} \right. \\ &\quad \left. - a_{21}^*(n + \tau_k) x_{1k}(n - \tau_{21}) \right. \\ &\quad \left. - a_{22}^*(n + \tau_k) x_{2k}(n - \tau_{22}) - c_2^*(n + \tau_k) \right], \end{aligned} \quad (35)$$

gives

$$\begin{aligned} & y_1(n+1) \\ &= y_1(n) \exp \left[\frac{a_{10}^*(n)}{y_1(n - \tau_{10}) + k_1^*(n)} - a_{11}^*(n) y_1(n - \tau_{11}) \right. \\ &\quad \left. - a_{12}^*(n) y_2(n - \tau_{12}) - c_1^*(n) \right], \\ & y_2(n+1) \\ &= y_2(n) \exp \left[\frac{a_{20}^*(n)}{y_2(n - \tau_{20}) + k_2^*(n)} - a_{21}^*(n) y_1(n - \tau_{21}) \right. \\ &\quad \left. - a_{22}^*(n) y_2(n - \tau_{22}) - c_2^*(n) \right]. \end{aligned} \quad (36)$$

We can easily see that $Y(n) = (y_1(n), y_2(n))$ is a solution of hull equation (31) and $m_i - \varepsilon \leq y_i(n) \leq M_i + \varepsilon$, $i = 1, 2$, for $n \in Z$. Since ε is an arbitrary small positive number, it follows that $m_i \leq y_i(n) \leq M_i$, $i = 1, 2$, for $n \in Z$; that is,

$$0 < \inf_{n \in Z} y_i(n) \leq \sup_{n \in Z} y_i(n) < \infty, \quad i = 1, 2. \quad (37)$$

Hence each hull equation of almost periodic difference system (1) has at least one strictly positive solution.

Now we prove the uniqueness of the strictly positive solution of each hull equation (31). Suppose that the hull equation (31) has two arbitrary strictly positive solutions

$(x_1^*(n), x_2^*(n))$ and $(y_1^*(n), y_2^*(n))$. We construct a Lyapunov functional

$$V^*(n) = \sum_{i=1}^2 \beta_i (V_{i1}^*(n) + V_{i2}^*(n) + V_{i3}^*(n)), \quad n \in Z, \quad (38)$$

where

$$\begin{aligned} & V_{i1}^*(n) = |\ln x_i^*(n) - \ln y_i^*(n)| \\ & V_{i2}^*(n) \\ &= \sum_{s=n-\tau_{i0}}^{n-1} \frac{a_{i0}(s + \tau_{i0})}{k_i^2(s + \tau_{i0})} |x_i^*(s) - y_i^*(s)| \\ &\quad + \sum_{s=n-\tau_{ij}}^{n-1} a_{ij}(s + \tau_{ij}) |x_j^*(s) - y_j^*(s)| \\ &\quad + \sum_{s=n}^{n-1+\tau_{ii}} a_{ii}(s) \\ &\quad \times \sum_{u=s-\tau_{ii}}^{n-1} \left\{ A_i(u) \left[\frac{a_{i0}(u)}{k_i(u)} + a_{ii}(u) M_i \right. \right. \\ &\quad \left. \left. + a_{ij}(u) M_j + c_i(u) \right] \right. \\ &\quad \times |x_i^*(u) - y_i^*(u)| \\ &\quad + M_i B_i(u) \frac{a_{i0}(u)}{k_i^2(u)} \\ &\quad \times |x_i^*(u - \tau_{i0}) - y_i^*(u - \tau_{i0})| \\ &\quad + M_i B_i(u) a_{ii}(u) \\ &\quad \times |x_i^*(u - \tau_{ii}) - y_i^*(u - \tau_{ii})| \\ &\quad + M_i B_i(u) a_{ij}(u) \\ &\quad \times |x_j^*(u - \tau_{ij}) - y_j^*(u - \tau_{ij})| \Big\}, \end{aligned}$$

$$\begin{aligned} & V_{i3}^*(n) = M_i \sum_{l=n-\tau_{i0}}^{n-1} \frac{a_{i0}(l + \tau_{i0})}{k_i^2(l + \tau_{i0})} B_i(l + \tau_{i0}) \\ &\quad \times |x_j^*(l) - y_j^*(l)| \sum_{s=l+\tau_{i0}+1}^{l+\tau_{i0}+\tau_{ii}} a_{ii}(s) \\ &\quad + M_i \sum_{l=n-\tau_{ii}}^{n-1} B_i(l + \tau_{ii}) a_{ii}(l + \tau_{ii}) \\ &\quad \times |x_i^*(l) - y_i^*(l)| \sum_{s=l+\tau_{ii}+1}^{l+2\tau_{ii}} a_{ii}(s) \end{aligned}$$

$$\begin{aligned}
& + M_i \sum_{l=n-\tau_{ij}}^{n-1} B_i(l+\tau_{ij}) a_{ij}(l+\tau_{ij}) \\
& \quad \times |x_j^*(l) - y_j^*(l)| \sum_{s=l+\tau_{ij}+1}^{l+\tau_{ij}+\tau_{ii}} a_{ii}(s), \\
& \quad i, j = 1, 2, \quad i \neq j.
\end{aligned} \tag{39}$$

Calculating the difference of $V^*(n)$ along the solution of the hull equation (31), one has

$$\Delta V^*(n) \leq -\eta \sum_{i=1}^2 |x_i^*(n) - y_i^*(n)|, \quad n \in \mathbb{Z}. \tag{40}$$

From (40), we can see that $V^*(n)$ is a nonincreasing function on $\mathbb{Z}$. Summing both sides of the above inequalities from n to 0, we have

$$\begin{aligned}
\eta \sum_{k=n}^0 \sum_{i=1}^2 |x_i^*(k) - y_i^*(k)| & \leq V^*(0) - V^*(n+1), \\
n & < 0.
\end{aligned} \tag{41}$$

Note that $V^*(n)$ is bounded. Hence we have

$$\sum_{k=-\infty}^0 \sum_{i=1}^2 |x_i^*(k) - y_i^*(k)| < +\infty, \tag{42}$$

which implies that

$$\lim_{n \rightarrow -\infty} |x_i^*(n) - y_i^*(n)| = 0, \quad i = 1, 2. \tag{43}$$

Define $Q = \sum_{i=1}^2 \beta_i Q_i$, where

$$\begin{aligned}
Q_i & = \frac{1}{m_i} + \frac{\tau_{i0} a_{i0}^u}{(k_i^l)^2} \left(1 + M_i B_i^u a_{ii}^u \tau_{ij} \right) + \tau_{ij} a_{ij}^2 \\
& \quad + \left(\tau_{ii} a_{ii}^u + \tau_{ij} a_{ij}^u \right) M_i B_i^u \tau_{ii} a_{ii}^u \\
& \quad + \tau_{ii}^2 a_{ii}^2 \left[A_i^u \left(\frac{a_{i0}^u}{K_i^l} + A_{ii}^u M_i + A_{ij}^u M_j + c_i^u \right) \right. \\
& \quad \left. + B_i^u M_i \left[\frac{a_{i0}^u}{(K_i^l)^2} + a_{ii}^u + a_{ij}^u \right] \right], \\
& \quad i, j = 1, 2, \quad i \neq j.
\end{aligned} \tag{44}$$

Let ε be an arbitrary small positive number. It follows from (43) that there exists a positive integer $n_1 > 0$ such

that $|x_i^*(n) - y_i^*(n)| < \varepsilon/Q$, $n < -n_1$, $i = 1, 2$. Therefore, for $n < -n_1$, $i, j = 1, 2$, $i \neq j$,

$$\begin{aligned}
V_{i1}^*(n) & \leq \frac{1}{m_i} |x_i^*(n) - y_i^*(n)| \leq \frac{1}{m_i} \frac{\varepsilon}{Q}, \\
V_{i2}^*(n) & \leq \frac{\tau_{i0} a_{i0}^u}{(k_i^l)^2} \max_{p \leq n} |x_i^*(p) - y_i^*(p)| \\
& \quad + \tau_{ij} a_{ij}^u \max_{p \leq n} |x_j^*(p) - y_j^*(p)| \\
& \quad + \tau_{ii}^2 a_{ii}^u \left[A_i^u \left(\frac{a_{i0}^u}{K_i^l} + a_{ii}^u M_i + a_{ij}^u M_j + c_i^u \right) \right. \\
& \quad \times \max_{p \leq n} |x_i^*(p) - y_i^*(p)| \\
& \quad + M_i B_i^u \frac{a_{i0}^u}{(k_i^l)^2} \max_{p \leq n} |x_i^*(p) - y_i^*(p)| \\
& \quad + M_i B_i^u a_{ii}^u \max_{p \leq n} |x_i^*(p) - y_i^*(p)| \\
& \quad \left. + M_i B_i^u a_{ij}^u \max_{p \leq n} |x_j^*(p) - y_j^*(p)| \right] \\
& \leq \left\{ \frac{\tau_{i0} a_{i0}^u}{(k_i^l)^2} + \tau_{ij} a_{ij}^u \right. \\
& \quad + \tau_{ii}^2 a_{ii}^u \left[A_i^u \left(\frac{a_{i0}^u}{K_i^l} + a_{ii}^u M_i + a_{ij}^u M_j + c_i^u \right) \right. \\
& \quad + M_i B_i^u \frac{a_{i0}^u}{(k_i^l)^2} \\
& \quad \left. \left. + M_i B_i^u a_{ii}^u + M_i B_i^u a_{ij}^u \right] \right\} \frac{\varepsilon}{Q}, \\
V_{i3}^*(n) & \leq M_i \tau_{i0} \tau_{ii} \frac{a_{i0}^u B_i^u}{(k_i^l)^2} a_{ii}^u \max_{p \leq n} |x_j^*(p) - y_j^*(p)| \\
& \quad + M_i \tau_{ii}^2 B_i^u (a_{ii}^u)^2 \max_{p \leq n} |x_i^*(p) - y_i^*(p)| \\
& \quad + M_i \tau_{ij} \tau_{ii} B_i^u a_{ij}^u a_{ii}^u \max_{p \leq n} |x_j^*(p) - y_j^*(p)| \\
& \leq \left\{ M_i \tau_{i0} \tau_{ii} \frac{a_{i0}^u B_i^u}{(k_i^l)^2} a_{ii}^u + M_i \tau_{ii}^2 B_i^u (a_{ii}^u)^2 \right. \\
& \quad \left. + M_i \tau_{ij} \tau_{ii} B_i^u a_{ij}^u a_{ii}^u \right\} \frac{\varepsilon}{Q}, \\
& \quad i, j = 1, 2, \quad i \neq j.
\end{aligned} \tag{45}$$

It follows from (38) and above inequalities that

$$V^*(n) \leq \sum_{i=1}^2 \beta_i Q_i \frac{\varepsilon}{Q} = \varepsilon, \quad n < -n_1, \quad (46)$$

so $\lim_{n \rightarrow -\infty} V^*(n) = 0$. Note that $V^*(n)$ is a nonincreasing function on Z , and then $V^*(n) \equiv 0$; that is $x_i^*(n) = y_i^*(n)$, $i = 1, 2$, for all $n \in Z$. Therefore, each hull equation of system (1) has a unique strictly positive solution.

In view of the above discussion, any hull equation of system (1) has a unique strictly positive solution. By Propositions 7–10 and Lemma 11, the almost periodic difference system (1) has a unique strictly positive almost periodic solution which is globally attractive. The proof is completed. $\square$

Let $\tau_{ij} = 0$, $i = 1, 2$, $j = 0, 1, 2$. Like in the proof of Theorem 12, we have the following corollary.

Corollary 13. Let $\tau_{ij} = 0$, $i = 1, 2$, $j = 0, 1, 2$. Assume that

$$\begin{aligned} \frac{a_{10}^l}{M_1 + k_1^u} - a_{12}^u M_2 - c_1^u &> 0, \\ \frac{a_{20}^l}{M_2 + k_2^u} - a_{21}^u M_1 - c_2^u &> 0 \end{aligned} \quad (47)$$

hold and further that there exist two positive constants α_1 and α_2 , such that

$$\begin{aligned} \alpha_i \left[\min \left\{ a_{ii}^l, \frac{2}{M_i} - a_{ii}^u \right\} - \frac{a_{i0}^u}{(k_i^l)^2} \right] - \alpha_j a_{ji}^u &> 0, \\ i, j = 1, 2, \quad i \neq j, \end{aligned} \quad (48)$$

where $M_i = (1/a_{ii}^l) \exp\{(a_{i0}^u/k_i^l) - 1\}$. Then the almost periodic difference system (1) admits a unique strictly positive almost periodic solution, which is globally attractive.

5. Example and Numerical Simulation

In this section, we give the following example to check the feasibility of our result.

Example 1. Consider the following almost periodic discrete Schoener's competition model with delays

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left[\frac{0.4 + 0.1 \sin(\sqrt{2}n)}{x_1(n-3) + 10.2 + 0.2 \cos(\sqrt{2}n)} \right. \\ &\quad - (2.02 + 0.02 \sin(\sqrt{3}n)) x_1(n-1) \\ &\quad - (0.015 + 0.005 \sin(\sqrt{2}n)) x_2(n-4) \\ &\quad \left. - (0.002 + 0.0005 \sin(\sqrt{5}n)) \right], \end{aligned}$$

FIGURE 1: Dynamic behavior of the first component $x_1(n)$ of the solution $(x_1(n), x_2(n))$ to system (49) with the initial conditions $(x_1(n), x_2(n)) = (0.013, 0.018)$ and $(0.032, 0.025)$, $n = 1, 2, 3, 4, 5$ for $n \in [1, 100]$, respectively.

$$\begin{aligned} x_2(n+1) &= x_2(n) \exp \left[\frac{0.3 + 0.2 \sin(\sqrt{3}n)}{x_2(n-4) + 10.5 + 0.5 \cos(\sqrt{3}n)} \right. \\ &\quad - (0.009 + 0.001 \cos(\sqrt{5}n)) x_1(n-2) \\ &\quad - (2.53 + 0.03 \cos(\sqrt{3}n)) x_2(n-1) \\ &\quad \left. - (0.005 + 0.001 \sin(\sqrt{2}n)) \right]. \end{aligned} \quad (49)$$

By simple computation, we derive

$$\begin{aligned} M_1 &\approx 0.2033, & M_2 &\approx 0.1626, \\ \frac{a_{10}^l}{M_1 + k_1^u} - a_{12}^u M_2 - c_1^u &\approx 0.0237, \\ \frac{a_{20}^l}{M_2 + k_2^u} - a_{21}^u M_1 - c_2^u &\approx 0.0018, \\ E_{11} &\approx 1.1047, & E_{22} &\approx 1.377, \\ F_{12} &\approx 1.0075, & F_{21} &\approx 1.3366, \\ G_{12} &\approx 0.0548, & G_{21} &\approx 0.0217. \end{aligned} \quad (50)$$

Then

$$\begin{aligned} \frac{2}{3} E_{11} - \left(\frac{2}{3} F_{12} + G_{21} \right) &\approx 0.01, \\ E_{22} - \left(F_{21} + \frac{2}{3} G_{12} \right) &\approx 0.026. \end{aligned} \quad (51)$$

FIGURE 2: Dynamic behavior of the second component $x_2(n)$ of the solution $(x_1(n), x_2(n))$ to system (49) with the initial conditions $(x_1(n), x_2(n)) = (0.013, 0.018)$ and $(0.032, 0.025)$, $n = 1, 2, 3, 4, 5$ for $n \in [1, 100]$, respectively.

FIGURE 3: Dynamic behavior of the first component $x_1(n)$ of the solution $(x_1(n), x_2(n))$ to system (49) with the initial conditions $(x_1(n), x_2(n)) = (0.013, 0.018)$ and $(0.032, 0.025)$, $n = 1, 2, 3, 4, 5$ for $n \in [500, 550]$, respectively.

Also it is easy to see that conditions (H2) and (H3) are verified. Therefore, system (49) has a unique strictly positive almost periodic solution which is globally attractive. Our numerical simulations support our results (see Figures 1, 2, 3, and 4).

Conflict of Interests

There are no financial interest conflicts between the authors and the commercial identity.

FIGURE 4: Dynamic behavior of the second component $x_2(n)$ of the solution $(x_1(n), x_2(n))$ to system (49) with the initial conditions $(x_1(n), x_2(n)) = (0.013, 0.018)$ and $(0.032, 0.025)$, $n = 1, 2, 3, 4, 5$ for $n \in [500, 550]$, respectively.

Acknowledgments

This work is supported by National Natural Science Foundation of China (no. 61132008) and Scientific Research Program Funded by Shaanxi Provincial Education Department of China (no. 2013JK1098).

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