Research Article

Linear Independent Solutions and Operational Representations for Hypergeometric Functions of Four Variables

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In investigation of boundary-value problems for certain partial differential equations arising in applied mathematics, we often need to study the solution of system of partial differential equations satisfied by hypergeometric functions and find explicit linearly independent solutions for the system. Here we choose the Exton function K_2 among his 21 functions to show how to find the linearly independent solutions of partial differential equations satisfied by this function K_2 . Based upon the classical derivative and integral operators, we introduce a new operational images for hypergeometric function K_2 . By means of these operational images, a number of finite series and decomposition formulas are then found.

1. Introduction and Preliminaries

Solutions of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, and quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendent) functions [1–4]. Such kinds of functions are often referred to as special functions of mathematical physics. They mainly appear in the solution of partial differential equations which are dealt with harmonic analysis method (see [5]). In view of various applications, it is interesting in itself and seems to be very important to conduct a continuous research of multiple hypergeometric functions. For instance, in [6], a comprehensive list of hypergeometric functions of three variables as many as 205 is recorded, together with their regions of convergence. It is noted that Riemann functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions of several variables (see [7–19]). Therefore, in investigation of boundary-value problems for these partial differential equations, we need to study the solution of the system of hypergeometric functions and find explicit linearly independent

solutions (see [12–17, 19]). Exton [20, pages 78–79] introduced 21 complete hypergeometric functions $K_1, K_2, \dots, K_{21}$ of four variables. In [21] Sharma and Parihar introduced 83 complete hypergeometric functions $F_1^{(4)}, F_2^{(4)}, \dots, F_{83}^{(4)}$ of four variables. It is remarked that, out of these 83 functions, the following 19 functions had already appeared in the work of Exton [20] in the different notations:

$$\begin{aligned} F_9^{(4)} &= K_1, & F_1^{(4)} &= K_2, & F_{38}^{(4)} &= K_3, \\ F_{10}^{(4)} &= K_4, & F_2^{(4)} &= K_5, & F_{59}^{(4)} &= K_6, \\ F_{39}^{(4)} &= K_7, & F_{11}^{(4)} &= K_8, & F_{12}^{(4)} &= K_9, \\ F_3^{(4)} &= K_{10}, & F_{60}^{(4)} &= K_{11}, & F_{40}^{(4)} &= K_{12}, \\ F_{13}^{(4)} &= K_{13}, & F_{77}^{(4)} &= K_{14}, & F_{78}^{(4)} &= K_{15}, \\ F_{79}^{(4)} &= K_{16}, & F_{82}^{(4)} &= K_{19}, \\ F_{81}^{(4)} &= K_{20}, & F_{83}^{(4)} &= K_{21}. \end{aligned} \quad (1)$$

Each quadruple hypergeometric function is of the form

$$F^{(4)}(\cdot) = \sum_{m,n,p,q=0}^{\infty} \Delta(m, n, p, q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{t^q}{q!}, \quad (2)$$

where $\Delta(m, n, p, q)$ is a certain sequence of complex parameters and there are twelve parameters in each function $F^{(4)}(\cdot)$. Here, for example, we choose the Exton function K_2 among his twenty-one functions

$$\begin{aligned} K_2[a, a, a, a, b, c, b, b; e_1, e_2, e_3, e_4; x, y, z, t] \\ = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_{m+n+q} (c)_p}{(e_1)_m (e_2)_n (e_3)_p (e_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{t^q}{q!} \end{aligned} \quad (3)$$

to find the linearly independent solutions of partial differential equations satisfied by this function K_2 .

2. The System of Partial Differential Equations for K_2

According to the theory of multiple hypergeometric functions (see [22]), the system of partial differential equations for Exton hypergeometric function K_2 is readily seen to be given as follows:

$$\begin{aligned} & \left(e_1 + x \frac{\partial}{\partial x} \right) \left(x \frac{\partial}{\partial x} + 1 \right) x^{-1} u \\ & - \left(a + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} \right) \\ & \times \left(b + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + t \frac{\partial}{\partial t} \right) u = 0, \\ & \left(e_2 + y \frac{\partial}{\partial y} \right) \left(y \frac{\partial}{\partial y} + 1 \right) y^{-1} u \\ & - \left(a + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} \right) \\ & \times \left(b + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + t \frac{\partial}{\partial t} \right) u = 0, \\ & \left(e_3 + z \frac{\partial}{\partial z} \right) \left(z \frac{\partial}{\partial z} + 1 \right) z^{-1} u \\ & - \left(a + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} \right) \left(c + z \frac{\partial}{\partial z} \right) u = 0, \\ & \left(e_4 + t \frac{\partial}{\partial t} \right) \left(t \frac{\partial}{\partial t} + 1 \right) t^{-1} u \\ & - \left(a + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + t \frac{\partial}{\partial t} \right) \\ & \times \left(b + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + t \frac{\partial}{\partial t} \right) u = 0, \end{aligned} \quad (4)$$

where $u = K_2[a, b, c; e_1, e_2, e_3, e_4; x, y, z, t]$.

Now by making use of some elementary calculations, we find the following system of second-order partial differential equations:

$$\begin{aligned} & x(1-x)u_{xx} - y^2u_{yy} - t^2u_{tt} - 2xyu_{xy} \\ & - xzu_{xz} - 2xtu_{xt} - yzu_{yz} - 2ytu_{yt} - ztu_{zt} \\ & + [e_1 - (a+b+1)x]u_x - (a+b+1)yu_y \\ & - bzu_z - (a+b+1)tu_t - abu = 0, \\ & y(1-y)u_{yy} - x^2u_{xx} - t^2u_{tt} - 2xyu_{xy} - xzu_{xz} \\ & - 2xtu_{xt} - yzu_{yz} - 2ytu_{yt} - ztu_{zt} \\ & - (a+b+1)xu_x + [e_2 - (a+b+1)y]u_y \\ & - bzu_z - (a+b+1)tu_t - abu = 0, \\ & z(1-z)u_{zz} - xzu_{xz} - yzu_{yz} - ztu_{zt} - cxu_x \\ & - cyu_y + [e_3 - (a+c+1)z]u_z - ctu_t - acu = 0, \\ & t(1-t)u_{tt} - x^2u_{xx} - y^2u_{yy} - 2xyu_{xy} - xzu_{xz} \\ & - 2xtu_{xt} - yzu_{yz} - 2ytu_{yt} - ztu_{zt} \\ & - (a+b+1)xu_x - (a+b+1)yu_y - bzu_z \\ & + [e_4 - (a+b+1)t]u_t - abu = 0. \end{aligned} \quad (5)$$

It is noted that three equations of the system (5) are linearly dependent, because the hypergeometric function K_2 satisfies the system. Now, in order to find the linearly independent solutions of the system (5), we consider u as in the form $u = x^\alpha y^\beta z^\gamma t^\delta w$ an unknown function, and α, β, γ , and δ are constants which are to be determined. So, substituting $u = x^\alpha y^\beta z^\gamma t^\delta w$ into the system (5), we obtain

$$\begin{aligned} & x(1-x)w_{xx} - y^2w_{yy} - t^2w_{tt} - 2xyw_{xy} - xzw_{xz} \\ & - 2xtw_{xt} - yzw_{yz} - 2ytw_{yt} - ztw_{zt} \\ & + [E_1 - (A+B+1)x]w_x - (A+B+1)yw_y \\ & - Bzw_z - (A+B+1)tw_t \\ & + [\alpha(\alpha-1+e_1)x^{-1} - AB]w = 0, \\ & y(1-y)w_{yy} - x^2w_{xx} - t^2w_{tt} - 2xyw_{xy} \\ & - xzw_{xz} - 2xtw_{xt} - yzw_{yz} - 2ytw_{yt} - ztw_{zt} \\ & - (A+B+1)xw_x + [E_2 - (A+B+1)y]w_y \end{aligned}$$

$$\begin{aligned}
& -Bzw_z - (A+B+1)tw_t \\
& + [\beta(\beta-1+e_2)y^{-1} - AB]w = 0, \\
& z(1-z)w_{zz} - xzw_{xz} - yzw_{yz} - ztw_{zt} - Cxw_x \\
& - Cyw_y + [E_3 - (A+C+1)z]w_z - Ctw_t \\
& + [\gamma(\gamma-1+e_3)z^{-1} - AC]w = 0, \\
& t(1-t)w_{tt} - x^2w_{xx} - y^2w_{yy} - 2xyw_{xy} - xzw_{xz} \\
& - 2xtw_{xt} - yzw_{yz} - 2ytw_{yt} - ztw_{zt} \\
& - (A+B+1)xw_x - (A+B+1)yw_y \\
& - Bzw_z + [E_4 - (A+B+1)t]w_t \\
& + [\delta(\delta-1+e_4)t^{-1} - AB]w = 0,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
A &= \alpha + \beta + \gamma + \delta + a, & B &= \alpha + \beta + \delta + b, \\
C &= \gamma + c, & E_1 &= 2\alpha + e_1, & E_2 &= 2\beta + e_2, \\
E_3 &= 2\gamma + e_3, & E_4 &= 2\delta + e_4.
\end{aligned} \tag{7}$$

It is noted that the system (6) is analogical to the system (5). Therefore, it is required that the following conditions should be satisfied:

$$\begin{aligned}
\alpha(\alpha-1+e_1) &= 0, \\
\beta(\beta-1+e_2) &= 0, \\
\gamma(\gamma-1+e_3) &= 0, \\
\delta(\delta-1+e_4) &= 0.
\end{aligned} \tag{8}$$

It is not difficult to see that the system (8) satisfies the following solutions:

$$\begin{array}{c|cccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\alpha: & 0 & 1-e_1 & 0 & 0 & 0 & 1-e_1 & 1-e_1 & 1-e_1 \\
\beta: & 0 & 0 & 1-e_2 & 0 & 0 & 1-e_2 & 0 & 0 \\
\gamma: & 0 & 0 & 0 & 1-e_3 & 0 & 0 & 1-e_3 & 0 \\
\delta: & 0 & 0 & 0 & 0 & 1-e_4 & 0 & 0 & 1-e_4
\end{array}, \tag{9}$$

$$\begin{array}{c|cccccccc}
& 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
\alpha: & 0 & 0 & 0 & 1-e_1 & 1-e_1 & 1-e_1 & 0 & 1-e_1 \\
\beta: & 1-e_2 & 1-e_2 & 0 & 1-e_2 & 1-e_2 & 0 & 1-e_2 & 1-e_2 \\
\gamma: & 1-e_3 & 0 & 1-e_3 & 1-e_3 & 0 & 1-e_3 & 1-e_3 & 1-e_3 \\
\delta: & 0 & 1-e_4 & 1-e_4 & 0 & 1-e_4 & 1-e_4 & 1-e_4 & 1-e_4
\end{array}.$$

Finally, substituting all eight solutions (9) into (6), we find the following linearly independent solutions of the system (5):

$$\begin{aligned}
u_1(x, y, z, t) &= K_2(a, b, c; e_1, e_2, e_3, e_4; x, y, z, t), \\
u_2(x, y, z, t) &= x^{1-e_1} K_2[1-e_1+a, 1-e_1+b, c; \\
&\quad 2-e_1, e_2, e_3, e_4; x, y, z, t], \\
u_3(x, y, z, t) &= y^{1-e_2} K_2[1-e_2+a, 1-e_2+b, c; \\
&\quad e_1, 2-e_2, e_3, e_4; x, y, z, t], \\
u_4(x, y, z, t) &= z^{1-e_3} K_2[1-e_3+a, b, 1-e_3+c; \\
&\quad e_1, e_2, 2-e_3, e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_5(x, y, z, t) &= t^{1-e_4} K_2[1-e_4+a, 1-e_4+b, c; \\
&\quad e_1, e_2, e_3, 2-e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_6(x, y, z, t) &= x^{1-e_1} y^{1-e_2} \\
&\quad \times K_2[2-e_1-e_2+a, 2-e_1-e_2+b, c; \\
&\quad 2-e_1, 2-e_2, e_3, e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_7(x, y, z, t) &= x^{1-e_1} z^{1-e_3} \\
&\quad \times K_2[2-e_1-e_3+a, 1-e_1+b, 1-e_3+c; \\
&\quad 2-e_1, e_2, 2-e_3, e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_8(x, y, z, t) &= x^{1-e_1} t^{1-e_4} \\
&\times K_2 [2 - e_1 - e_4 + a, 2 - e_1 - e_4 + b, c; \\
&\quad 2 - e_1, e_2, e_3, 2 - e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_9(x, y, z, t) &= y^{1-e_2} z^{1-e_3} \\
&\times K_2 [2 - e_2 - e_3 + a, 1 - e_2 + b, 1 - e_3 + c; \\
&\quad e_1, 2 - e_2, 2 - e_3, e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_{10}(x, y, z, t) &= y^{1-e_2} t^{1-e_4} \\
&\times K_2 [2 - e_2 - e_4 + a, 2 - e_2 - e_4 + b, c; \\
&\quad e_1, 2 - e_2, e_3, 2 - e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_{11}(x, y, z, t) &= z^{1-e_3} t^{1-e_4} \\
&\times K_2 [2 - e_3 - e_4 + a, 1 - e_4 + b, 1 - e_3 + c; \\
&\quad e_1, e_2, 2 - e_3, 2 - e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_{12}(x, y, z, t) &= x^{1-e_1} y^{1-e_2} z^{1-e_3} \\
&\times K_2 [3 - e_1 - e_2 - e_3 + a, 2 - e_1 - e_2 + b, 1 - e_3 + c; \\
&\quad 2 - e_1, 2 - e_2, 2 - e_3, e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_{13}(x, y, z, t) &= x^{1-e_1} y^{1-e_2} t^{1-e_4} \\
&\times K_2 [3 - e_1 - e_2 - e_4 + a, 3 - e_1 - e_2 - e_4 + b, c; \\
&\quad 2 - e_1, 2 - e_2, e_3, 2 - e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_{14}(x, y, z, t) &= x^{1-e_1} z^{1-e_3} t^{1-e_4} \\
&\times K_2 [3 - e_1 - e_3 - e_4 + a, 2 - e_1 - e_4 + b, 1 - e_3 + c; \\
&\quad 2 - e_1, e_2, 2 - e_3, 2 - e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_{15}(x, y, z, t) &= y^{1-e_2} z^{1-e_3} t^{1-e_4} \\
&\times K_2 [3 - e_2 - e_3 - e_4 + a, 2 - e_2 - e_4 + b, 1 - e_3 + c; \\
&\quad e_1, 2 - e_2, 2 - e_3, 2 - e_4; x, y, z, t],
\end{aligned}$$

$$\begin{aligned}
u_{16}(x, y, z, t) &= x^{1-e_1} y^{1-e_2} z^{1-e_3} t^{1-e_4} \\
&\times K_2 [4 - e_1 - e_2 - e_3 - e_4 + a, \\
&\quad 3 - e_1 - e_2 - e_4 + b, 1 - e_3 + c; \\
&\quad 2 - e_1, 2 - e_2, 2 - e_3, 2 - e_4; x, y, z, t].
\end{aligned} \tag{10}$$

Moreover, it is seen that the global solution of the system (5) is combined to be in the form

$$u = \sum_{j=1}^{16} k_j u_j, \tag{11}$$

where k_i ($i = 1, 2, \dots, 16$) are constants.

3. Operational Representations for K_2 and Applications

In this section, we apply the concept of the right Riemann-Liouville fractional derivative to obtain operational images for hypergeometric functions K_2 . Indeed, in this section we build up the right Riemann-Liouville fractional derivative operator, which plays the role of augmenting parameters in the hypergeometric functions involved (see [23, Chapters 4 and 5]). In particular, we will deal with operational definitions ruled by the operators $\widehat{D}_x$ and $\widehat{D}_x^{-1}$ where $\widehat{D}_x$ denotes the derivative operator and $\widehat{D}_x^{-1}$ defines the inverse of the derivative and once acting on unity, it yields

$$\widehat{D}_x^{-m} 1 = \frac{x^m}{m!}, \quad m \in N \cup \{0\}. \tag{12}$$

It is evident that $\widehat{D}_x^{-1}$ is essentially an integral operator and the lower limit has been assumed to be zero. The following two formulas are well-known consequences of the derivative operator $\widehat{D}_x$ and the integral operator $\widehat{D}_x^{-1}$ (see [24]). Consider

$$\begin{aligned}
\widehat{D}_x^m x^\lambda &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - m + 1)} x^{\lambda - m}, \\
\widehat{D}_x^{-m} x^\lambda &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + m + 1)} x^{\lambda + m},
\end{aligned} \tag{13}$$

where $m \in N \cup \{0\}, \lambda \in C / \{-1, -2, \dots\}$. Based on the operational relations (13), we have

$$\widehat{D}_t^m \widehat{D}_u^{-m} \{t^{\beta+m-1} u^{\gamma-1}\} = \frac{(\beta)_m}{(\gamma)_m} \{t^{\beta-1} u^{\gamma+m-1}\}. \tag{14}$$

In view of the definition of the Gaussian hypergeometric function ${}_2F_1$ [6],

$${}_2F_1[\alpha, \beta; \gamma; x] = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n. \tag{15}$$

Bin-Saad [25] introduced the following operational representations for Gaussian hypergeometric ${}_2F_1$.

Lemma 1. Let $\operatorname{Re}(\beta) > 0$ and $\operatorname{Re}(\gamma) > 0$; then

$$\begin{aligned} (1 - x\widehat{D}_t u^{-1} \widehat{D}_u^{-1} t)^{-\alpha} \{t^{\beta-1} u^{\gamma-1}\} &= \{t^{\beta-1} u^{\gamma-1}\}_2 F_1[\alpha, \beta; \gamma; x], \\ (1 - \widehat{D}_t \widehat{D}_x^{-1} t)^{-\alpha} \{t^{\beta-1} x^{\gamma-1}\} &= \{t^{\beta-1} x^{\gamma-1}\}_2 F_1[\alpha, \beta; \gamma; x]. \end{aligned} \quad (16)$$

For the quadruple hypergeometric function K_2 , one introduces the following operational representations.

Theorem 2. Let $\operatorname{Re}(a) > 0$, $\operatorname{Re}(b) > 0$, $\operatorname{Re}(c) > 0$, $\operatorname{Re}(e_1) > 0$, $\operatorname{Re}(e_2) > 0$, $\operatorname{Re}(e_3) > 0$, and $\operatorname{Re}(e_4) > 0$; then

$$\begin{aligned} &(1 - x\widehat{D}_{t_1} t_2^{-1} \widehat{D}_{t_2}^{-1} t_1 - y\widehat{D}_{t_1} t_3^{-1} \widehat{D}_{t_3}^{-1} t_1 - u\widehat{D}_{t_1} t_5^{-1} \widehat{D}_{t_5}^{-1} t_1)^{-b} \\ &\quad \times (1 - z\widehat{D}_{t_1} t_4^{-1} \widehat{D}_{t_4}^{-1} t_1)^{-c} \{t_1^{a-1} t_2^{e_1-1} t_3^{e_2-1} t_4^{e_3-1} t_5^{e_4-1}\} \\ &= \{t_1^{a-1} t_2^{e_1-1} t_3^{e_2-1} t_4^{e_3-1} t_5^{e_4-1}\} \\ &\quad \times K_2[a, a, a, a; b, b, c, b; e_1, e_2, e_3, e_4; x, y, z, u], \\ &(1 - x\widehat{D}_{t_1} t_3^{-1} \widehat{D}_{t_3}^{-1} t_1 - y\widehat{D}_{t_1} t_4^{-1} \widehat{D}_{t_4}^{-1} t_1 \\ &\quad - z\widehat{D}_{t_2} t_5^{-1} \widehat{D}_{t_5}^{-1} t_2 - u\widehat{D}_{t_1} t_6^{-1} \widehat{D}_{t_6}^{-1} t_1)^{-a} \\ &\quad \times \{t_1^{b-1} t_2^{c-1} t_3^{e_1-1} t_4^{e_2-1} t_5^{e_3-1} t_6^{e_4-1}\} \\ &= \{t_1^{b-1} t_2^{c-1} t_3^{e_1-1} t_4^{e_2-1} t_5^{e_3-1} t_6^{e_4-1}\} \\ &\quad \times K_2[a, a, a, a; b, b, c, b; e_1, e_2, e_3, e_4; x, y, z, u]. \end{aligned} \quad (17)$$

Proof. Denote, for convenience, the left-hand side of assertion (17) by I . Then, as a consequence of the binomial theorem, it is easily seen that

$$\begin{aligned} I &= \sum_{m,n,p,q=0}^{\infty} \frac{(b)_{m+n+q}(c)_p x^m y^n z^p u^q t_2^{-m} t_3^{-n} t_4^{-p} t_5^{-q}}{m!n!p!q!} \\ &\quad \times \widehat{D}_{t_1}^{m+n+p+q} \widehat{D}_{t_2}^{-m} \widehat{D}_{t_3}^{-n} \widehat{D}_{t_4}^{-p} \widehat{D}_{t_5}^{-q} \\ &\quad \times \{t_1^{a+m+n+p+q-1} t_2^{e_1-1} t_3^{e_2-1} t_4^{e_3-1} t_5^{e_4-1}\}. \end{aligned} \quad (18)$$

Upon using (14) and considering definition (3), we are led finally to right-hand side of assertion (17). The proof of assertion (18) runs parallel to that of assertion (17); then we skip the details. $\square$

One of the advantages offered by the use of the operational images (17) and (18) is the possibility of establishing finite sums for the quadruple hypergeometric series K_2 . First, let, for convenience,

$$\begin{aligned} \widehat{M} &= x\widehat{D}_{t_1} t_3^{-1} \widehat{D}_{t_3}^{-1} t_1 - y\widehat{D}_{t_1} t_4^{-1} \widehat{D}_{t_4}^{-1} t_1 \\ &\quad - z\widehat{D}_{t_2} t_5^{-1} \widehat{D}_{t_5}^{-1} t_2 - u\widehat{D}_{t_1} t_6^{-1} \widehat{D}_{t_6}^{-1} t_1, \\ \widehat{N} &= x\widehat{D}_{t_1} t_2^{-1} \widehat{D}_{t_2}^{-1} t_1 - y\widehat{D}_{t_1} t_3^{-1} \widehat{D}_{t_3}^{-1} t_1 - u\widehat{D}_{t_1} t_5^{-1} \widehat{D}_{t_5}^{-1} t_1, \\ \widehat{R} &= z\widehat{D}_{t_1} t_4^{-1} \widehat{D}_{t_4}^{-1} t_1. \end{aligned} \quad (19)$$

By employing (18) and taking advantage from the identity

$$\begin{aligned} &[1 - (1 - \widehat{M})]^n \{t_1^{b-1} t_2^{c-1} t_3^{e_1-1} t_4^{e_2-1} t_5^{e_3-1} t_6^{e_4-1}\} \\ &= [\widehat{M}]^n \{t_1^{b-1} t_2^{c-1} t_3^{e_1-1} t_4^{e_2-1} t_5^{e_3-1} t_6^{e_4-1}\} \end{aligned} \quad (20)$$

and the definition of the general triple hypergeometric series $F^{(3)}[x, y, z]$ (see [6]), we can state that

$$\begin{aligned} &\sum_{r=0}^n (-1)^r \binom{n}{r} K_2[-r, -r, -r, -r; b, b, c, b; e_1, e_2, e_3, e_4; x, y, z, u] \\ &= \frac{(b)_n}{(e_1)_n} F^{(3)} \left[\begin{matrix} -n, 1 - e_1 - n :: --; c; --; & \frac{y}{x}, \frac{z}{x}, \frac{u}{x} \\ - :: e_2; e_3; e_4; & \end{matrix} \right]. \end{aligned} \quad (21)$$

Similarly, in view of (17), the definition of Appell hypergeometric series $F_4[x, y]$ (see [6, page 23, equation (5)]) and the use of the following identity

$$\begin{aligned} &[1 - (1 - \widehat{N})]^n [1 - (1 - \widehat{R})]^m \{t_1^{a-1} t_2^{e_1-1} t_3^{e_2-1} t_4^{e_3-1} t_5^{e_4-1}\} \\ &= [\widehat{N}]^n [\widehat{R}]^m \{t_1^{a-1} t_2^{e_1-1} t_3^{e_2-1} t_4^{e_3-1} t_5^{e_4-1}\} \end{aligned} \quad (22)$$

allow to conclude that

$$\begin{aligned} &\sum_{r=0}^n \sum_{s=0}^m (-1)^{r+s} \binom{n}{r} \binom{m}{s} \\ &\quad \times K_2[a, a, a, a; -r, -r, -s, -r; e_1, e_2, e_3, e_4; x, y, z, u] \\ &= \frac{(a)_{m+n}}{(e_1)_n (e_3)_m} F_4 \left[\begin{matrix} -n, 1 - e_1 - n; e_2, e_4; & \frac{y}{x}, \frac{u}{x} \\ - :: e_2; e_4; & \end{matrix} \right]. \end{aligned} \quad (23)$$

Also, by virtue of the identity

$$\begin{aligned} &[(1 - \widehat{N})]^{-b} [(1 - \widehat{R})]^{-b} \{t_1^{a-1} t_2^{e_1-1} t_3^{e_2-1} t_4^{e_3-1} t_5^{e_4-1}\} \\ &= [1 - \widehat{N} - \widehat{R} + \widehat{N}\widehat{R}]^{-b} \{t_1^{a-1} t_2^{e_1-1} t_3^{e_2-1} t_4^{e_3-1} t_5^{e_4-1}\}, \end{aligned} \quad (24)$$

we find the following decomposition formula:

$$\begin{aligned} &\sum_{s,k,r=0}^{\infty} \frac{(a)_{2s+2k+2r} (b)_{s+k+r}}{(e_1)_s (e_2)_k (e_3)_{s+k+r} (e_4)_r} (-xz)^s (-yz)^k (-uz)^r \\ &\quad \times F_C^{(4)}[a + 2s + 2k + 2r, b + s + k + r; \\ &\quad e_1, e_2, e_3, e_4; x, y, z, u] \end{aligned} \quad (25)$$

$$= K_2[a, a, a, a; b, b, b, b; e_1, e_2, e_3, e_4; x, y, z, u],$$

where $F_C^{(4)}$ is Lauricella function of four variables [6, page 33, equation (3)].

Moreover, in view of the composition rule

$$\begin{aligned} &[(1 - \widehat{M})]^{-2a} = [(1 - \widehat{M})]^{-a} [(1 - \widehat{M})]^{-a} \\ &= [1 - 2\widehat{M} + \widehat{M}^2]^{-a} \end{aligned} \quad (26)$$

and the operational image (18), we easily find for the function K_2 that

$$\begin{aligned}
 & K_2 [2a, 2a, 2a, 2a, b, b, c, b; e_1, e_2, e_3, e_4; x, y, z, u] \\
 &= \sum_{m,n,p,q,s,r,k,l,h,t=0}^{\infty} \frac{(a)_M (b)_N (c)_P}{(e_1)_{Q^{(1)}} (e_2)_{Q^{(2)}} (e_3)_{Q^{(3)}} (e_4)_{Q^{(4)}}} \\
 &\quad \times x^{2m} y^{2n} z^{2p} u^{2q} (2xy)^s (2xz)^k (2xu)^r \\
 &\quad \times (2yz)^l (2yu)^h (2zu)^t \\
 &\quad \times K_2 [a + M, a + M, a + M, a + M, b + N, b + N, \\
 &\quad \quad c + N, b + N; e_1 + Q^{(1)}, \\
 &\quad \quad e_2 + Q^{(2)}, e_3 + Q^{(3)}, e_4 + Q^{(4)}; x, y, z, u], \quad (28)
 \end{aligned}$$

where

$$\begin{aligned}
 M &:= m + n + p + q + s + k + r + l + h + t, \\
 P &:= 2p + k + l + t, \\
 N &:= 2(m + n + q + s + r + h) + k + l + t, \\
 Q^{(1)} &:= 2m + s + k + r, \\
 Q^{(2)} &:= 2n + s + l + h, \quad Q^{(3)} := 2p + k + l + t, \\
 Q^{(4)} &:= 2q + r + h + t. \quad (29)
 \end{aligned}$$

Finally, let us stress that the schema suggested in Sections 2 and 3 can be applied to find linear independent solutions, symbolic operational images, finite sums, and decomposition formulas for other quadruple hypergeometric functions.

Conflict of Interests

The authors declare that they have no competing interests.

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