Research Article

Homoclinic Orbits for a Class of Subquadratic Second Order Hamiltonian Systems

Li-Li Wan

School of Science, Southwest University of Science and Technology, Mianyang, Sichuan 621010, China

Correspondence should be addressed to Li-Li Wan; 15882872311@163.com

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The existence and multiplicity of homoclinic orbits are considered for a class of subquadratic second order Hamiltonian systems $\ddot{q}(t) - L(t)q(t) + \nabla W(t, q(t)) = 0$. Recent results from the literature are generalized and significantly improved. Examples are also given in this paper to illustrate our main results.

1. Introduction and Main Results

In this paper, we will study the existence and multiplicity of homoclinic orbits for the second order Hamiltonian systems of the type:

$$\ddot{q}(t) - L(t)q(t) + \nabla W(t, q(t)) = 0, \quad (1)$$

where $L \in C(\mathbb{R}, \mathbb{R}^{\mathbb{N}^2})$ is a symmetric matrix valued function and $W \in C^1(\mathbb{R} \times \mathbb{R}^{\mathbb{N}}, \mathbb{R})$. As usual, we say that q is a nontrivial homoclinic orbit (to 0) if $q \in C^2(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$, $q \neq 0$ and $q(t) \rightarrow 0$ as $|t| \rightarrow \infty$ (see [1]). In the following, $(\cdot, \cdot) : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^{\mathbb{N}}$ and $|\cdot|$ is the induced norm.

Hamiltonian system theory, a classical as well as a modern study area, widely consists in mathematical sciences, life sciences, and various aspects of social science. Lots of mechanical and field theory models even exist in the form of Hamiltonian system. Solutions of Hamiltonian system can be divided into periodic solution, subharmonic solution, homoclinic orbit, heteroclinic orbit, and so on. Homoclinic orbit was discovered among the models of nonlinear dynamical system and affects the nature of the whole system significantly. Starting from the Poincaré era, the study of homoclinic orbits in the nonlinear dynamical system exploits Perturbation method mainly. It is not until the recent decade that variational principle has been widely used. The existence and multiplicity of homoclinic orbits for the second order Hamiltonian systems have been extensively

investigated in many recent papers (see [1–16]). The main feature of the problem is the lack of global compactness due to unboundedness of domain. To overcome the difficulty, many authors have considered the periodic case, autonomous case, or asymptotically periodic case (see [1–4]). Some papers treat the coercive case (see [5–8]). Recently, the symmetric case has been dealt with (see [9–11]). Compared with the superquadratic case, the case that $W(t, x)$ is subquadratic as $|x| \rightarrow +\infty$ has been considered only by a few authors. As far as the author is aware, the author in [5] first discussed the subquadratic case. Later, the authors in [12] dealt with this case by use of a standard minimizing argument, which is the following theorem.

Theorem 1 (see [12]). *Assume that L and W satisfy the following conditions:*

- (A₁) $L \in C(\mathbb{R}, \mathbb{R}^{\mathbb{N}^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $(L(t)x, x) \geq \alpha(t)|x|^2$ and $\alpha(t) \rightarrow +\infty$ as $|t| \rightarrow +\infty$;
- (A₂) $W(t, x) = a(t)|x|^\gamma$, where $a : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function such that $a \in L^2(\mathbb{R}, \mathbb{R}) \cap L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R})$ and $1 < \gamma < 2$ is a constant.

Then, (1) has at least one nontrivial homoclinic orbit.

Subsequently, the condition (A₂) was generalized, respectively, in [13, 14] by the following conditions:

(A₃) $W(t, x) = a(t)|x|^\gamma$, where $a : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function such that $a \in L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R})$ and $1 < \gamma < 2$ is a constant.

(A₄) $a(t)|x|^\gamma \leq (\nabla W(t, x), x)$, $|\nabla W(t, x)| \leq b(t)|x|^{\gamma-1} + c(t)|x|^{\delta-1}$, where $a, b, c : \mathbb{R} \rightarrow \mathbb{R}^+$ are continuous functions such that $a, b \in L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R})$, $c \in L^{2/(2-\delta)}(\mathbb{R}, \mathbb{R})$, and $1 < \gamma < 2$, $1 < \delta < 2$ are constants, $W(t, 0) = 0$, $W(t, x) = W(t, -x)$ for all $(t, x) \in (\mathbb{R}, \mathbb{R}^N)$.

The authors in [13, 14] got infinitely many homoclinic orbits by using the variant fountain theorem in [17].

Motivated by these papers, we also consider the subquadratic case. Main results are the following theorems.

Theorem 2. Assume that (A₁) is satisfied and the following conditions hold:

(W₁) there exists a constant $1 \leq \gamma < 2$ such that $|W(t, x)| \leq a(t)b(|x|)|x|^\gamma$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where $a \in C(\mathbb{R}, \mathbb{R}^+) \cap L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R})$ and $b \in C(\mathbb{R}^+, \mathbb{R}^+)$;

(W₂) $|\nabla W(t, x)| \leq a(t)d(|x|)|x|^{\gamma-1}$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where $d \in C(\mathbb{R}^+, \mathbb{R}^+)$;

(W₃) $\lim_{|x| \rightarrow +\infty} W(t, x)/|x|^2 = 0$ uniformly for $t \in \mathbb{R}$;

(W₄) there exists a constant $\delta_0 > 0$ such that $W(t, x) \geq c_0 a(t)|x|^\gamma$ for all $t \in \mathbb{R}$ and $|x| \leq \delta_0$, where c_0 is a positive constant.

Then, (1) has at least one nontrivial homoclinic orbit.

Remark 3. Theorem 2 generalizes Theorem 1 (see [12]). Obviously, the condition (A₂) is stronger than the conditions (W₁)–(W₄). On the other hand, there are functions W satisfying our Theorem 2 and not satisfying the corresponding result in [12]. For example, let

$$W_0 = a(t)|x|^\gamma \left[\ln(1 + |x|^2) + 1 \right], \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^N, \quad (2)$$

where $a \in C(\mathbb{R}, \mathbb{R}^+) \cap L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R})$ and $1 \leq \gamma < 2$. Moreover, our result is different from the corresponding results in [5, 15], since W_0 does not satisfy conditions (W₃) and (W₇) in [5] and our condition (A₁) on L is different from the condition (L') in [15].

Theorem 4. Assume that (A₁), (W₁)–(W₄) are satisfied and the following condition holds:

(W₅) $W(t, x) \geq 0$ and $W(t, x) = W(t, -x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

Then, (1) has infinitely many homoclinic orbits.

Remark 5. Theorem 4 generalizes Theorem 1.2 in [13]. It is easy to see that the condition (A₃) is stronger than our conditions (W₁)–(W₅). On the other hand, there are functions W satisfying Theorem 4 and not satisfying the corresponding result in [13]. For example, let

$$W_1 = a(t)|x|^\gamma \left[\ln(1 + |x|^2) + 1 \right], \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^N, \quad (3)$$

where $a \in C(\mathbb{R}, \mathbb{R}^+) \cap L^{2/(2-\gamma)}(\mathbb{R}, \mathbb{R})$ and $1 \leq \gamma < 2$.

Theorem 6. Assume that (A₁), (W₃)–(W₅) are satisfied and the following conditions hold:

(W₆) there exist constants $1 \leq \gamma_k < 2$ and $m \in \mathbb{N}$ such that $|W(t, x)| \leq \sum_{k=1}^m a_k(t)b_k(|x|)|x|^{\gamma_k}$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where $a_k \in C(\mathbb{R}, \mathbb{R}^+) \cap L^{2/(2-\gamma_k)}(\mathbb{R}, \mathbb{R})$ and $b_k \in C(\mathbb{R}^+, \mathbb{R}^+)$;

(W₇) $|\nabla W(t, x)| \leq \sum_{k=1}^m a_k(t)d_k(|x|)|x|^{\gamma_k-1}$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where $d_k \in C(\mathbb{R}^+, \mathbb{R}^+)$.

Then, (1) has infinitely many homoclinic orbits.

Remark 7. Theorem 6 generalizes Theorem 1.2 in [14]. It is obvious that the condition (A₄) is stronger than our conditions (W₃)–(W₇). Besides, there are functions W satisfying our Theorem 6 and not satisfying the corresponding result in [14]. For example, let

$$W_2 = (a_1(t)|x|^{\gamma_1} + a_2(t)|x|^{\gamma_2}) \left[\ln(1 + |x|^2) + 1 \right], \quad (4)$$

$$\forall t \in \mathbb{R}, x \in \mathbb{R}^N,$$

where $a_k \in C(\mathbb{R}, \mathbb{R}^+) \cap L^{2/(2-\gamma_k)}(\mathbb{R}, \mathbb{R})$, $1 \leq \gamma_k < 2$ and $k = 1, 2$.

Remark 8. We should point out that there exist errors in the proofs of [13, 14]. On one hand, in Lemma 2.2, in [13], $\|u_k - u\|_2 \rightarrow 0$ can not imply that $\sum_{k=1}^\infty \|u_k - u\|_2 < +\infty$ since the convergence of the general term is only the necessary but not the sufficient condition for the convergence of the progression. On the other hand, the proof of Lemma 2.2 in [14] is not like the proof of Lemma 2.2 in [12] as the authors said. It is easy to see that $a \in L^2(\mathbb{R}, \mathbb{R})$ plays an important role in the proof of Lemma 2.2 in [12], however, b, c in condition (A₄) do not belong to $L^2(\mathbb{R}, \mathbb{R})$. In the proofs of our results, we take some methods to avoid such mistakes.

2. Proof of Main Results

Let

$$E = \left\{ q \in H^1(\mathbb{R}, \mathbb{R}^N) \mid \int_{\mathbb{R}} (|\dot{q}|^2 + (L(t)q, q)) dt < +\infty \right\}. \quad (5)$$

Then, E is a Hilbert space with the norm given by

$$\|q\| = \left(\int_{\mathbb{R}} (|\dot{q}|^2 + (L(t)q, q)) dt \right)^{1/2}. \quad (6)$$

Obviously, E is continuously embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$ for $p \in [2, \infty]$. Thus, we have

$$\|q\|_{L^p} \leq \eta_p \|q\| \quad \text{for } p \in [2, \infty], \quad (7)$$

where $\eta_p > 0$. For $q \in E$, let

$$I(q) = \frac{1}{2} \int_{\mathbb{R}} (|\dot{q}|^2 + (L(t)q, q)) dt$$

$$- \int_{\mathbb{R}} W(t, q) dt = \frac{1}{2} \|q\|^2 - \int_{\mathbb{R}} W(t, q) dt. \quad (8)$$

In the following, we always denote by c_i ($i \in \mathbb{N}$) any suitable positive constant.

Lemma 9 (see [5]). Assume that L satisfies (A_1) . Then, E is compactly embedded in $L^p(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$ for $p \in [2, \infty]$.

Lemma 10. Under the conditions of (W_1) – (W_3) , one has that $I \in C^1(E, \mathbb{R})$ and that

$$\begin{aligned} \langle I'(q), v \rangle &= \int_{\mathbb{R}} ((\dot{q}, \dot{v}) + (L(t)q, v) - (\nabla W(t, q), v)) dt, \\ &\quad \forall q, v \in E. \end{aligned} \quad (9)$$

Proof. Let $J(q) = \int_{\mathbb{R}} W(t, q) dt$ for all $q \in E$. For any $\varepsilon > 0$, it follows from (W_1) and (W_3) that

$$|W(t, x)| \leq \varepsilon |x|^2 + c_1 a(t) |x|^\gamma, \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^{\mathbb{N}}. \quad (10)$$

Then, by the Hölder inequality and (7), we have

$$\begin{aligned} \left| \int_{\mathbb{R}} W(t, q) dt \right| &\leq \varepsilon \eta_2^2 \|q\|^2 + c_1 \|a\|_{L^{2/(2-\gamma)}} \|q\|_{L^2}^\gamma \\ &\leq \varepsilon \eta_2^2 \|q\|^2 + c_2 \|a\|_{L^{2/(2-\gamma)}} \|q\|^\gamma, \end{aligned} \quad (11)$$

for all $q \in E$. Therefore, J is well defined. Moreover, $J \in C^1(E, \mathbb{R})$ and

$$\langle J'(q), v \rangle = \int_{\mathbb{R}} (\nabla W(t, q), v) dt, \quad \forall q, v \in E. \quad (12)$$

In fact, for any given $q \in E$, by (W_2) , the Hölder inequality, and (7), one has

$$\begin{aligned} \left| \int_{\mathbb{R}} (\nabla W(t, q), v) dt \right| &\leq c_3 \int_{\mathbb{R}} a(t) |q|^{p-1} |v| dt \\ &\leq c_3 \left(\int_{\mathbb{R}} a^2(t) |q|^{2p-2} dt \right)^{1/2} \|v\|_{L^2} \\ &\leq c_3 \|a\|_{L^{2/(2-\gamma)}} \|q\|_{L^2}^{p-1} \|v\|_{L^2} \\ &\leq c_4 \|a\|_{L^{2/(2-\gamma)}} \|v\|, \end{aligned} \quad (13)$$

for all $v \in E$. For any $q, v \in E$, by the mean value theorem, we have

$$\begin{aligned} &\int_{\mathbb{R}} W(t, q+v) dt - \int_{\mathbb{R}} W(t, q) dt \\ &= \int_{\mathbb{R}} (\nabla W(t, q+hv), v) dt, \end{aligned} \quad (14)$$

where $h(t) \in (0, 1)$. Besides, for any given $q \in E$, by (W_2) , the Hölder inequality, and (7), there exists a positive constant θ_0 such that

$$\begin{aligned} &\left| \int_{\mathbb{R}} (\nabla W(t, q+hv), v) dt \right| \\ &\leq c_5 \int_{\mathbb{R}} a(t) |q+hv|^{p-1} |v| dt \\ &\leq c_5 \left(\int_{\mathbb{R}} a^2(t) |q+hv|^{2p-2} dt \right)^{1/2} \|v\|_{L^2} \\ &\leq c_5 \|a\|_{L^{2/(2-\gamma)}} \|q+hv\|_{L^2}^{p-1} \|v\|_{L^2} \\ &\leq c_6 \|a\|_{L^{2/(2-\gamma)}} \|v\|, \end{aligned} \quad (15)$$

for all $\|v\| \leq \theta_0$. The combination of (13)–(15) shows

$$\begin{aligned} &\int_{\mathbb{R}} W(t, q+v) dt \\ &- \int_{\mathbb{R}} W(t, q) dt - \int_{\mathbb{R}} (\nabla W(t, q), v) dt \rightarrow 0, \end{aligned} \quad (16)$$

as $v \rightarrow 0$ in E , which gives (12) and (9) immediately. In addition, J is weakly continuous. In fact, let $q_n \rightarrow q$ in E . By Lemma 9, $q_n \rightarrow q$ in $L^2(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$ and $L^\infty(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$. Since $|q(t)| \rightarrow 0$ as $|t| \rightarrow \infty$, there exists a positive constant R_1 such that

$$|q_n| \leq 1, \quad |q| \leq 1, \quad \forall |t| \geq R_1. \quad (17)$$

It follows from (W_1) and the Hölder inequality such that

$$\begin{aligned} &\int_{|t| \geq R_1} |W(t, q_n) - W(t, q)| dt \\ &\leq c_7 \int_{|t| \geq R_1} a(t) (|q_n|^\gamma + |q|^\gamma) dt \\ &\leq c_7 \|a\|_{L^{2/(2-\gamma)}} \left[\left(\int_{|t| \geq R_1} |q_n|^2 dt \right)^{\gamma/2} + \left(\int_{|t| \geq R_1} |q|^2 dt \right)^{\gamma/2} \right]. \end{aligned} \quad (18)$$

Therefore, for any $\varepsilon > 0$, since $q_n \rightarrow q$ in $L^2(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$, one can take $R_2 \geq R_1$ such that

$$\int_{|t| \geq R_2} |W(t, q_n) - W(t, q)| dt < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}. \quad (19)$$

On the other hand, one has

$$\int_{|t| < R_2} W(t, q_n) dt \rightarrow \int_{|t| < R_2} W(t, q) dt \quad \text{as } n \rightarrow +\infty. \quad (20)$$

Hence, there exists $n_0 \in \mathbb{N}$ such that

$$|J(q_n) - J(q)| < \varepsilon, \quad (21)$$

proving the weak continuity of J . Moreover, J' is compact for J is weakly continuous (see [5]). $\square$

Lemma 11. Under the conditions of (W_3) , (W_6) , and (W_7) , one has that $I \in C^1(E, \mathbb{R})$ and that

$$\begin{aligned} \langle I'(q), v \rangle &= \int_{\mathbb{R}} ((\dot{q}, \dot{v}) + (L(t)q, v) - (\nabla W(t, q), v)) dt, \\ &\quad \forall q, v \in E. \end{aligned} \quad (22)$$

Proof. Since the proof is exactly similar to the proof of Lemma 10, we omit it here. $\square$

By Lemmas 10 and 11, it is routine to verify that any critical point of I on E is a classical solution of (1) with $q(\pm\infty) = 0$. Now, we state the critical point theorem used in [12].

Lemma 12 (see [18]). Let E be a real Banach space and let us have $I \in C^1(E, \mathbb{R})$ satisfying the (PS) condition. If I is bounded from below, then

$$c \equiv \inf_E I \quad (23)$$

is a critical value of I .

Proof of Theorem 2. We divide our proof into three steps.

Step 1. I is bounded from below. In fact, using (11), we have

$$\begin{aligned} I(q) &= \frac{1}{2} \|q\|^2 - \int_{\mathbb{R}} W(t, q) dt \\ &\geq \frac{1}{2} \|q\|^2 - \varepsilon \eta_2^2 \|q\|^2 - c_2 \|a\|_{L^{2/(2-\gamma)}} \|q\|^\gamma, \end{aligned} \quad (24)$$

for all $q \in E$. Since $1 \leq \gamma < 2$ and ε is an arbitrary positive constant, $I(q) \rightarrow +\infty$ as $\|q\| \rightarrow +\infty$ and then I is bounded from below.

Step 2. I satisfies the (PS) condition. Assume that $\{q_k\}_{k \in \mathbb{N}} \subset E$ is a sequence such that

$$I(q_k) \text{ is bounded and } I'(q_k) \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (25)$$

By (11), one gets

$$\begin{aligned} \frac{1}{2} \|q_k\|^2 - I(q_k) \\ = \int_{\mathbb{R}} W(t, q_k) dt \leq \varepsilon \eta_2^2 \|q_k\|^2 + c_2 \|a\|_{L^{2/(2-\gamma)}} \|q_k\|^\gamma, \end{aligned} \quad (26)$$

for all $k \in \mathbb{N}$. Since $1 \leq \gamma < 2$ and ε is an arbitrary positive constant, it is obvious that $\{q_k\}$ is bounded. Noting that J' is compact, a routine verification shows that $\{q_k\}$ has a convergent subsequence, proving the (PS) condition. Now, by Lemma 12, there is a $q_0 \in E$ such that

$$I(q_0) = \inf_E I(q), \quad I'(q_0) = 0. \quad (27)$$

Step 3. q_0 is nontrivial. In fact, let

$$e(t) = \begin{cases} |\sin t| e_1, & t \in [-\pi, \pi], \\ 0, & t \in \mathbb{R} \setminus [-\pi, \pi], \end{cases} \quad (28)$$

where $e_1 = (1, 0, \dots, 0)$. By (W_4) one obtains

$$\begin{aligned} I(\xi e) &= \frac{\xi^2}{2} \|e\|^2 - \int_{-\pi}^{\pi} W(t, \xi e) dt \\ &\leq \frac{\xi^2}{2} \|e\|^2 - c_0 \xi^\gamma \int_{-\pi}^{\pi} a(t) |e|^\gamma dt, \end{aligned} \quad (29)$$

for $|\xi| \leq \delta_0$. Since $1 \leq \gamma < 2$, we can choose ξ small enough such that $I(\xi e) < 0$. Hence, $q_0 \neq 0$. The proof is complete. $\square$

In order to prove Theorems 4 and 6, we need the Dual fountain theorem. For the reader's convenience, we recall it here.

Lemma 13 (see [19]). Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j \in \mathbb{N}} X_j}$ with $\dim X_j < +\infty$. For any $k \in \mathbb{N}$, define

$$Y_k = \bigoplus_{j=1}^{j=k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \quad (30)$$

Assume that

(B_1) the compact group G acts isometrically on E . The spaces X_j are invariant and there exists a finite dimensional space V such that, for every $j \in \mathbb{N}$, $X_j \simeq V$ and the action of G on V is admissible.

Besides, let $I \in \mathcal{C}^1(E, \mathbb{R})$ be an invariant functional. If for every $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that

- (B_2) $a_k = \inf_{q \in Z_k, \|q\| = \rho_k} I(q) \geq 0$;
- (B_3) $b_k = \max_{q \in Y_k, \|q\| = r_k} I(q) < 0$;
- (B_4) $d_k = \inf_{q \in Z_k, \|q\| \leq \rho_k} I(q) \rightarrow 0$ as $k \rightarrow +\infty$;
- (B_5) I satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0]$.

Then, I has a sequence of negative critical values converging to 0.

Proof of Theorem 4. Let $\{e_j\}_{j \in \mathbb{N}}$ be the completely orthogonal basis of E and $X_j = \mathbb{R}e_j$. Then, $\dim X_j < +\infty$ and $E = \overline{\bigoplus_{j \in \mathbb{N}} X_j}$. For any $k \in \mathbb{N}$, define

$$Y_k = \bigoplus_{j=1}^{j=k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \quad (31)$$

In the following, we will check that all conditions in Lemma 13 are satisfied and we divide our proof into several steps.

Step 1. Conditions (B_2) and (B_4) hold. Let

$$\mu_k = \sup_{q \in Z_k, \|q\|=1} \|q\|_{L^2}. \quad (32)$$

We claim that $\mu_k \rightarrow 0$ as $k \rightarrow +\infty$. In fact, it is obvious that $0 < \mu_{k+1} \leq \mu_k$ and then $\mu_k \rightarrow \mu_0$ for some $\mu_0 \geq 0$

as $k \rightarrow +\infty$. For every $k \in \mathbb{N}$, there is $q_k \in Z_k$ such that $\|q_k\| = 1$ and $\|q_k\|_{L^2} > \mu_k/2$. It follows from the definition of Z_k that $q_k \rightarrow 0$ in E . Then, by Lemma 9, we have $q_k \rightarrow 0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ as $k \rightarrow +\infty$ and thus $\mu_0 = 0$. Take $\varepsilon = 1/4\eta_2^2$ in (11), then for all $q \in Z_k$, one gets

$$\begin{aligned} I(q) &= \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}} W(t, q) dt \\ &\geq \frac{1}{2}\|q\|^2 - \varepsilon\eta_2^2\|q\|^2 - c_1\|a\|_{L^{2/(2-\gamma)}}\|q\|_{L^2}^\gamma \\ &= \frac{1}{2}\|q\|^2 - \frac{1}{4}\|q\|^2 - c_1\|a\|_{L^{2/(2-\gamma)}}\|q\|_{L^2}^\gamma \\ &\geq \frac{1}{4}\|q\|^2 - c_1\mu_k^\gamma\|a\|_{L^{2/(2-\gamma)}}\|q\|^\gamma. \end{aligned} \quad (33)$$

Since $1 \leq \gamma < 2$, there exists $\rho_k > 0$ such that

$$\begin{aligned} a_k &= \inf_{q \in Z_k, \|q\| = \rho_k} I(q) \geq 0, \\ d_k &= \inf_{q \in Z_k, \|q\| \leq \rho_k} I(q) \geq -c_1\mu_k^\gamma\|a\|_{L^{2/(2-\gamma)}}\|q\|^\gamma. \end{aligned} \quad (34)$$

Similar to the third step in the proof of Theorem 2, we can show that $d_k \leq 0$. Since $\mu_k \rightarrow 0$ as $k \rightarrow +\infty$, we get $d_k \rightarrow 0$ as $k \rightarrow +\infty$.

Step 2. Condition (B_3) holds. We claim that there exists $\varepsilon_1 > 0$ such that

$$\text{meas} \left\{ t \in \mathbb{R} : a(t) |q(t)|^\gamma \geq \varepsilon_1 \|q\|^\gamma \right\} \geq \varepsilon_1, \quad \forall q \in Y_k \setminus \{0\}. \quad (35)$$

If not, there exists a sequence $\{q_m\}_{m \in \mathbb{N}} \subset Y_k$ with $\|q_m\| = 1$ such that

$$\text{meas} \left\{ t \in \mathbb{R} : a(t) |q_m(t)|^\gamma \geq \frac{1}{m} \right\} < \frac{1}{m}, \quad (36)$$

for all $m \in \mathbb{N}$. Since we have $\dim Y_k < +\infty$, it follows from the compactness of the unit sphere of Y_k that, going to a subsequence if necessary, q_m converges to some q^* in Y_k and $\|q^*\| = 1$. By the equivalence of the norms on the finite-dimensional space Y_k , one has

$$\int_{\mathbb{R}} a(t) |q_m - q^*|^\gamma dt \leq \|a\|_{L^{2/(2-\gamma)}} \left(\int_{\mathbb{R}} |q_m - q^*|^2 dt \right)^{\gamma/2} \rightarrow 0, \quad (37)$$

as $m \rightarrow +\infty$. It is easy to check that there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\text{meas} \left\{ t \in \mathbb{R} : a(t) |q^*(t)|^\gamma \geq \delta_1 \right\} \geq \delta_2. \quad (38)$$

To be specific, by negation, we have

$$\text{meas} \left\{ t \in \mathbb{R} : a(t) |q^*(t)|^\gamma \geq \frac{1}{n} \right\} = 0, \quad (39)$$

for all $n \in \mathbb{N}$. Then by (7), one has

$$0 \leq \int_{\mathbb{R}} a(t) |q^*|^{y+2} dt \leq \frac{1}{n} \|q^*\|_{L^2}^2 \leq \frac{\eta_2^2}{n} \|q^*\|^2 = \frac{\eta_2^2}{n} \rightarrow 0, \quad (40)$$

as $n \rightarrow +\infty$, which contradicts that $\|q^*\| = 1$. Therefore, (37) holds. Thus, define

$$\begin{aligned} \Omega_0 &= \left\{ t \in \mathbb{R} : a(t) |q^*(t)|^\gamma \geq \delta_1 \right\}, \\ \Omega_m &= \left\{ t \in \mathbb{R} : a(t) |q_m(t)|^\gamma < \frac{1}{m} \right\}, \end{aligned} \quad (41)$$

and $\Omega_m^c = \mathbb{R} \setminus \Omega_m$. Combining (35) and (37), one has

$$\begin{aligned} \text{meas}(\Omega_m \cap \Omega_0) &= \text{meas}(\Omega_0 \setminus (\Omega_m^c \cap \Omega_0)) \\ &\geq \text{meas} \Omega_0 - \text{meas}(\Omega_m^c \cap \Omega_0) \\ &\geq \delta_2 - \frac{1}{m}, \end{aligned} \quad (42)$$

for all $m \in \mathbb{N}$. Selecting $m > \max\{2/\delta_1, 2/\delta_2\}$, we have

$$\delta_1 - \frac{1}{m} \geq \frac{1}{2}\delta_1, \quad \delta_2 - \frac{1}{m} \geq \frac{1}{2}\delta_2. \quad (43)$$

Then, one gets

$$\begin{aligned} &a(t) |q_m(t) - q^*(t)|^\gamma \\ &\geq a(t) |q^*(t)|^\gamma - a(t) |q_m(t)|^\gamma \\ &\geq \delta_1 - \frac{1}{m} \geq \frac{1}{2}\delta_1, \quad \forall t \in \Omega_m \cap \Omega_0, \end{aligned} \quad (44)$$

which implies that

$$\begin{aligned} \int_{\mathbb{R}} a(t) |q_m - q^*|^\gamma dt &\geq \int_{\Omega_m \cap \Omega_0} a(t) |q_m - q^*|^\gamma dt \\ &\geq \frac{1}{2}\delta_1 \cdot \text{meas}(\Omega_m \cap \Omega_0) \\ &\geq \frac{1}{2}\delta_1 \cdot \left(\delta_2 - \frac{1}{m} \right) \geq \frac{1}{4}\delta_1\delta_2, \end{aligned} \quad (45)$$

for $m > \max\{2/\delta_1, 2/\delta_2\}$, which is in contradiction to (36). Hence, (22) holds. Define

$$\Omega_q = \left\{ t \in \mathbb{R} : a(t) |q(t)|^\gamma \geq \varepsilon_1 \|q\|^\gamma \right\}, \quad \forall q \in Y_k. \quad (46)$$

It follows from (W_5) , (7), (W_4) , and (22) that

$$\begin{aligned} I(q) &= \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}} W(t, q) dt \\ &\leq \frac{1}{2}\|q\|^2 - \int_{\Omega_q} W(t, q) dt \\ &\leq \frac{1}{2}\|q\|^2 - c_0 \int_{\Omega_q} a(t) |q|^\gamma dt \\ &\leq \frac{1}{2}\|q\|^2 - c_0\varepsilon_1 \|q\|^\gamma \cdot \text{meas}(\Omega_q) \\ &\leq \frac{1}{2}\|q\|^2 - c_0\varepsilon_1^2 \|q\|^\gamma, \end{aligned} \quad (47)$$

for all $q \in Y_k$ with $\|q\| \leq \delta_0/\eta_\infty$. Since $1 \leq \gamma < 2$, there exists $\rho_k > r_k > 0$ such that

$$b_k = \max_{q \in Y_k, \|q\|=r_k} I(q) < 0. \quad (48)$$

Step 3. Condition (B_5) holds. Since it is similar to the second step in the proof of Theorem 2, we omit it here.

At last, it is standard to verify that condition (B_1) holds and I is invariant. So, Theorem 4 is proved by Lemma 13 immediately. $\square$

Proof of Theorem 6. Since the proof of Theorem 6 is exactly similar to the proof of Theorem 4, we omit it here. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding this work.

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