Research Article

Remarks on Pre- $\mathcal{F}$ -Regular Pre- $\mathcal{F}$ -Open Sets

R. Sajuntha

Department of Mathematics, Ponjesly College of Engineering and Technology, Nagercoil, Tamil Nadu 629 003, India

Correspondence should be addressed to R. Sajuntha; r.sajuntha@gmail.com

Received 16 April 2014; Accepted 27 June 2014; Published 13 October 2014

Academic Editor: Wenwu Yu

Copyright © 2014 R. Sajuntha. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We deal with the new class of pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open sets in which the notion of pre- $\mathcal{F}$ -open set is involved. We characterize these sets and study some of their fundamental properties. We also present other notions called extremally pre- $\mathcal{F}$ -disconnectedness, locally pre- $\mathcal{F}$ -indiscreetness, and pre- $\mathcal{F}$ -regular sets by utilizing the notion of pre- $\mathcal{F}$ -open and pre- $\mathcal{F}$ -closed sets by which we obtain some equivalence relation for pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open sets.

1. Introduction

The subject of ideals in topological space has been studied by Kuratowski [1] and Vaidyanathaswamy [2]. An ideal $\mathcal{F}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{F}$ and $B \subset A$ implies $B \in \mathcal{F}$ and (ii) $A \in \mathcal{F}$ and $B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$. Given a topological space (X, τ) with an ideal $\mathcal{F}$ on X and if $\wp(X)$ is the set of all subsets of X , a set operator $(\cdot)^* : \wp(X) \rightarrow \wp(X)$, called a local function [1] of A with respect to τ and $\mathcal{F}$, is defined as follows: for $A \subset X$, $A^*(\tau, \mathcal{F}) = \{x \in X \mid U \cap A \notin \mathcal{F}, \text{ for every } \{U \in \tau(x)\} \text{ where } \tau(x) = \{U \in \tau \mid x \in U\}\}$. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(X, \tau)$, called the $\star$ -topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{F}, \tau)$ [3]; when there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{F}, \tau)$ and τ^* for $\tau^*(\mathcal{F}, \tau)$. If $\mathcal{F}$ is an ideal on X , then $(X, \tau, \mathcal{F})$ is called an ideal space. A subset A of an ideal space $(X, \tau, \mathcal{F})$ is said to be $\mathcal{F}$ -open [4] if $A \subset \text{int}(A^*)$. A subset A of an ideal space $(X, \tau, \mathcal{F})$ is said to be pre- $\mathcal{F}$ -open [5] if $A \subset \text{int}(cl^*(A))$. The complement of pre- $\mathcal{F}$ -open set is called pre- $\mathcal{F}$ -closed. The family of all pre- $\mathcal{F}$ -open sets in $(X, \tau, \mathcal{F})$ is denoted by $P\mathcal{F}O(X, \tau)$ or simply $P\mathcal{F}O(X)$. Clearly $\tau \subset P\mathcal{F}O(X)$. The largest pre- $\mathcal{F}$ -open set contained in A , denoted by $p\mathcal{F} \text{int}(A)$, is called the pre- $\mathcal{F}$ -interior of A . The smallest pre- $\mathcal{F}$ -closed set containing A , denoted by $p\mathcal{F}cl(A)$, is called the pre- $\mathcal{F}$ -closure of A . A subset A of an ideal space $(X, \tau, \mathcal{F})$ is said to be α - $\mathcal{F}$ -open [6] if $A \subset \text{int}(cl^*(\text{int}(A)))$. The family of all α - $\mathcal{F}$ -open sets

is a topology finer than τ . We will denote the α - $\mathcal{F}$ -interior subset of A of X by $\alpha\mathcal{F} \text{int}(A)$. A subset A of an ideal space $(X, \tau, \mathcal{F})$ is τ^* -dense [7] if $cl^*(A) = X$. A space $(X, \tau, \mathcal{F})$ is $\mathcal{F}$ -submaximal [8] if every τ^* -dense subset of X is open.

The following lemmas will be useful in the sequel.

Lemma 1. Let $(X, \tau, \mathcal{F})$ be an ideal space and let A be a subset of X . Then $p\mathcal{F} \text{int}(A) = A \cap \text{int}(cl^*(A))$ [9].

2. Pre- $\mathcal{F}$ -Regular Pre- $\mathcal{F}$ -Open Sets

A pre- $\mathcal{F}$ -open set A of a space $(X, \tau, \mathcal{F})$ is said to be pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open if $A = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$. The complement of a pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open set is called pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed set, equivalently $A = p\mathcal{F}cl(p\mathcal{F} \text{int}(A))$. A subset A of a space $(X, \tau, \mathcal{F})$ with an ideal $\mathcal{F}$ is said to be pre- $\mathcal{F}$ -regular if it is pre- $\mathcal{F}$ -open and pre- $\mathcal{F}$ -closed. Clearly, X and $\emptyset$ are pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

Remark 2. Also every pre- $\mathcal{F}$ -regular set is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

Proof. Assume A is pre- $\mathcal{F}$ -regular that implies A is pre- $\mathcal{F}$ -open and pre- $\mathcal{F}$ -closed. A is pre- $\mathcal{F}$ -open which implies $A = p\mathcal{F} \text{int}(A) = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$. Hence A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. $\square$

But the converse is not true as shown by Example 3.

Example 3. Consider the ideal space $(X, \tau, \mathcal{F})$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, and $\mathcal{F} = \{\emptyset, \{a\}\}$. Here $\{a\}$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open but not pre- $\mathcal{F}$ -closed.

Moreover, the intersection of two pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open sets is not pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open in general as Example 4 shows.

Example 4. Consider the ideal space $(X, \tau, \mathcal{F})$ where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b, c\}, X\}$, and $\mathcal{F} = \{\emptyset, \{a\}\}$. Here $PIO(X) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Thus, $A = \{a, b\}$, $B = \{a, c\}$ are both pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. But $A \cap B$ is not, since it is not even pre- $\mathcal{F}$ -open.

The notions of pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open and $\mathcal{F}$ -open sets are independent of each other. Consider the space (X, τ) with an ideal $\mathcal{F}$ as in Example 4. Here $\{b, c\}$ is $\mathcal{F}$ -open but not pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. Consider the space $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\mathcal{F} = \{\emptyset, \{a\}\}$. Here $\{a\}$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open but not $\mathcal{F}$ -open. Observe that every pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open set is pre- $\mathcal{F}$ -open but the converse need not be true. Here $\{b, c\}$ is pre- $\mathcal{F}$ -open but not pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

Theorem 5. Let $(X, \tau, \mathcal{F})$ be an ideal space and let A, B be any subsets of X . Then the following hold.

- (a) If $A \subset B$, then $p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(B))$.
- (b) If $A \in PIO(X)$, then $A \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$.
- (c) For every $A \in PIO(X)$, $p\mathcal{F} \text{int}(p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(A)))) = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$.
- (d) If A and B are disjoint pre- $\mathcal{F}$ -open sets, then $p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$ and $p\mathcal{F} \text{int}(p\mathcal{F}cl(B))$ are disjoint.
- (e) If A is a pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open, then $p\mathcal{F}cl(X - A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed.
- (f) If A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open, then $p\mathcal{F} \text{int}(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

Proof.

- (a) Suppose $A \subset B \Rightarrow p\mathcal{F}cl(A) \subset p\mathcal{F}cl(B)$. Therefore $p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(B))$.
- (b) Suppose that $A \in PIO(X, \tau) \Rightarrow A = p\mathcal{F} \text{int}(A) \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$.
- (c) It is obvious that $p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) \in PIO(X, \tau)$, so by (b) we have $p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(A))))$. On the other hand $p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) \subset p\mathcal{F}cl(A)$ which implies $p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(A))) \subset p\mathcal{F}cl(p\mathcal{F}cl(A)) = p\mathcal{F}cl(A)$. Therefore, $p\mathcal{F} \text{int}(p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(A)))) \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$. Hence $p\mathcal{F} \text{int}(p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(A)))) = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$.
- (d) Since A and B are disjoint pre- $\mathcal{F}$ -open sets, we have $A \cap B = \emptyset$ which implies $A \cap p\mathcal{F}cl(B) = \emptyset \Rightarrow A \cap p\mathcal{F} \text{int}(p\mathcal{F}cl(B)) = \emptyset$. Since $p\mathcal{F} \text{int}(p\mathcal{F}cl(B))$ is pre- $\mathcal{F}$ -open, $p\mathcal{F}cl(A) \cap p\mathcal{F} \text{int}(p\mathcal{F}cl(B)) = \emptyset$. Hence $p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) \cap p\mathcal{F} \text{int}(p\mathcal{F}cl(B)) = \emptyset$.

(e) Given that A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open, $A = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$ implies $X - A = X - p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) = p\mathcal{F}cl(p\mathcal{F} \text{int}(X - A))$. Therefore, $p\mathcal{F}cl(X - A) = p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(X - A)))$. Hence $p\mathcal{F}cl(X - A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed.

(f) By (e) if A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open, then $p\mathcal{F}cl(X - A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed. Hence $X - (p\mathcal{F}cl(X - A))$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open that implies $p\mathcal{F} \text{int}(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. $\square$

Lemma 6. For an ideal topological space $(X, \tau, \mathcal{F})$ the following are equivalent.

- (a) Every pre- $\mathcal{F}$ -open set is open.
- (b) Every τ^* -dense set is open.

Proof. (a) $\Rightarrow$ (b): let A be a τ^* -dense set that implies $cl^*(A) = X$ which implies $\text{int}(cl^*(A)) = X$, so that $A \subset \text{int}(cl^*(A)) = X$. By (a) every pre- $\mathcal{F}$ -open set is open and hence A is open.

(b) $\Rightarrow$ (a): let B be a pre- $\mathcal{F}$ -open subset of X , so that $B \subset \text{int}(cl^*(B)) = G$, say. Then $cl^*(B) = cl^*(G)$, so that $cl^*((X - G) \cup B) = cl^*(X - G) \cup cl^*(B) = (X - G) \cup cl^*(B) = X$, and thus $(X - G) \cup B$ is τ^* -dense in X . Thus $(X - G) \cup B$ is open. Now $B = ((X - G) \cup B) \cap G$ is the intersection of two open sets, so that B is open. $\square$

Lemma 7. If a space (X, τ) with an ideal $\mathcal{F}$ is $\mathcal{F}$ -submaximal, then any finite intersection of pre- $\mathcal{F}$ -open set is pre- $\mathcal{F}$ -open.

Proof. From Lemma 6, every pre- $\mathcal{F}$ -open set is open and hence a finite intersection of pre- $\mathcal{F}$ -open set is pre- $\mathcal{F}$ -open. $\square$

Theorem 8. If a space (X, τ) with an ideal $\mathcal{F}$ is $\mathcal{F}$ -submaximal, then any finite intersection of pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open set is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

Proof. Let $\{O_i \mid i = 1, 2, \dots, n\}$ be a finite family of pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open sets. Since the space X is $\mathcal{F}$ -submaximal, then by Lemma 6, $\cap \{O_i \mid i = 1, 2, \dots, n\}$ is pre- $\mathcal{F}$ -open. Therefore, $\cap \{O_i \mid i = 1, 2, \dots, n\} \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(\cap O_i))$. Also, for each $i = 1, 2, \dots, n$, $\cap O_i \subset O_i$ which implies $p\mathcal{F} \text{int}(p\mathcal{F}cl(\cap O_i)) \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(O_i))$. Also, each O_i is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open that implies $O_i = p\mathcal{F} \text{int}(p\mathcal{F}cl(O_i))$ which implies $p\mathcal{F} \text{int}(p\mathcal{F}cl(\cap O_i)) \subset \cap O_i$ and so $\cap O_i = p\mathcal{F} \text{int}(p\mathcal{F}cl(\cap O_i))$. Hence $\cap O_i$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. $\square$

It should be noted that an arbitrary union of pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open set is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. But the intersection of two pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed sets fails to be pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed as shown by Example 9.

Example 9. Consider the ideal space $(X, \tau, \mathcal{F})$ as in Example 3. Clearly, $\{a, c\}$, $\{a, b\}$ are pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed but their intersection is not pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed.

Theorem 10. *The following hold for a subset A of a space $(X, \tau, \mathcal{F})$.*

- (a) *If A is pre- $\mathcal{F}$ -closed, then $p\mathcal{F} \text{int}(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.*
- (b) *If $A = p\mathcal{F} \text{int}(A)$, then $p\mathcal{F}cl(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed.*
- (c) *If A and B are pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed sets, then $A \subset B$ if and only if $p\mathcal{F} \text{int}(A) \subset p\mathcal{F} \text{int}(B)$.*
- (d) *If A and B are pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open sets, then $A \subset B$ if and only if $p\mathcal{F}cl(A) \subset p\mathcal{F}cl(B)$.*

Proof. (a) Since A is pre- $\mathcal{F}$ -closed, $A = p\mathcal{F}cl(A)$.

Now, $p\mathcal{F} \text{int}(p\mathcal{F}cl(p\mathcal{F} \text{int}(A))) = p\mathcal{F} \text{int}(p\mathcal{F} \text{int}(A)) = p\mathcal{F} \text{int}(A)$. Hence $p\mathcal{F} \text{int}(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

(b) Now $p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(A))) = p\mathcal{F}cl(p\mathcal{F}cl(A)) = p\mathcal{F}cl(A)$. Hence $p\mathcal{F}cl(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed.

(c) Given that A and B are pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed sets, therefore, $A = p\mathcal{F}cl(p\mathcal{F} \text{int}(A))$ and $B = p\mathcal{F}cl(p\mathcal{F} \text{int}(B))$. Clearly, $p\mathcal{F} \text{int}(A) \subset p\mathcal{F} \text{int}(B)$ if $A \subset B$.

Conversely, $p\mathcal{F} \text{int}(A) \subset p\mathcal{F} \text{int}(B)$. Now $A = p\mathcal{F}cl(p\mathcal{F} \text{int}(A)) \subset p\mathcal{F}cl(p\mathcal{F} \text{int}(B)) \subset B$. Hence $A \subset B$.

(d) Given that A and B are pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open, therefore, $A = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$ and $B = p\mathcal{F} \text{int}(p\mathcal{F}cl(B))$. Suppose $A \subset B$, $p\mathcal{F}cl(A) = p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(A))) \subset p\mathcal{F}cl(p\mathcal{F} \text{int}(p\mathcal{F}cl(B))) \subset p\mathcal{F}cl(B)$. Therefore, $p\mathcal{F}cl(A) \subset p\mathcal{F}cl(B)$.

Conversely, $p\mathcal{F}cl(A) \subset p\mathcal{F}cl(B)$. Now $A = p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) \subset p\mathcal{F} \text{int}(p\mathcal{F}cl(B)) \subset B$. $\square$

A subset A of an ideal topological space $(X, \tau, \mathcal{F})$ is said to be $\mathcal{F}$ -rare if it has no interior points in τ^* .

Theorem 11. *Let $(X, \tau, \mathcal{F})$ be an ideal space. Then the following hold.*

- (a) *The empty set is the only subset which is nowhere dense and pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.*
- (b) *If A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed, then every $\mathcal{F}$ -rare set is pre- $\mathcal{F}$ -open.*

Proof. (a) Suppose A is nowhere dense and A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. Then $A = p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) = p\mathcal{F}cl(A) \cap \text{int}(cl^*(p\mathcal{F}cl(A)))$, by Lemma 1. Therefore, $A \subset p\mathcal{F}cl(A) \cap \text{int}(cl^*(cl(A))) \subset p\mathcal{F}cl(A) \cap \text{int}(cl(A)) = p\mathcal{F}cl(A) \cap \emptyset = \emptyset$.

(b) Suppose A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed. Then $A = p\mathcal{F}cl(p\mathcal{F} \text{int}(A)) = p\mathcal{F} \text{int}(A) \cup cl(\text{int}^*(p\mathcal{F} \text{int}(A))) \subset p\mathcal{F} \text{int}(A) \cup cl(\text{int}^*(A)) = p\mathcal{F} \text{int}(A) \cup \emptyset = p\mathcal{F} \text{int}(A)$. Therefore, $A = p\mathcal{F} \text{int}(A)$. Hence A is pre- $\mathcal{F}$ -open. $\square$

An ideal space $(X, \tau, \mathcal{F})$ is called extremally pre- $\mathcal{F}$ -disconnected if the pre- $\mathcal{F}$ -closure of every pre- $\mathcal{F}$ -open set is pre- $\mathcal{F}$ -open.

Theorem 12. *For a topological space $(X, \tau, \mathcal{F})$ the following are equivalent.*

- (a) *$(X, \tau, \mathcal{F})$ is extremally pre- $\mathcal{F}$ -disconnected.*

- (b) *Every pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open subset is pre- $\mathcal{F}$ -regular.*

Proof. (a) $\Rightarrow$ (b): assume $(X, \tau, \mathcal{F})$ is extremally pre- $\mathcal{F}$ -disconnected. Suppose A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. Then A is pre- $\mathcal{F}$ -open and so $p\mathcal{F}cl(A)$ is a pre- $\mathcal{F}$ -open set. Hence $A = p\mathcal{F} \text{int}(p\mathcal{F}cl(A)) = p\mathcal{F}cl(A)$. Hence A is pre- $\mathcal{F}$ -closed which implies A is pre- $\mathcal{F}$ -regular.

(b) $\Rightarrow$ (a): suppose A is pre- $\mathcal{F}$ -open. Then $p\mathcal{F}cl(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed which implies $X - p\mathcal{F}cl(A)$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. Hence $X - p\mathcal{F}cl(A)$ is pre- $\mathcal{F}$ -regular. Therefore, $X - p\mathcal{F}cl(A)$ is pre- $\mathcal{F}$ -closed and so $p\mathcal{F}cl(A)$ is pre- $\mathcal{F}$ -open. Hence $(X, \tau, \mathcal{F})$ is extremally pre- $\mathcal{F}$ -disconnected. $\square$

Theorem 13. *Let $(X, \tau, \mathcal{F})$ be an extremally pre- $\mathcal{F}$ -disconnected space and $A \subset X$. Then the following are equivalent:*

- (a) *A is pre- $\mathcal{F}$ -regular,*
- (b) *$A = p\mathcal{F}cl(p\mathcal{F} \text{int}(A))$,*
- (c) *$X - A$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open,*
- (d) *A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.*

Proof. (a) $\Rightarrow$ (b): suppose A is pre- $\mathcal{F}$ -regular. Then A is pre- $\mathcal{F}$ -open and pre- $\mathcal{F}$ -closed and so $A = p\mathcal{F} \text{int}(A)$ and $A = p\mathcal{F}cl(A)$. Hence $A = p\mathcal{F}cl(p\mathcal{F} \text{int}(A))$.

(b) $\Rightarrow$ (c): let $A = p\mathcal{F}cl(p\mathcal{F} \text{int}(A))$. Then $X - A = X - p\mathcal{F}cl(p\mathcal{F} \text{int}(A)) = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$ so $X - A$ is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a) follows from Theorem 12. $\square$

An ideal space $(X, \tau, \mathcal{F})$ is called locally pre- $\mathcal{F}$ -indiscrete if every pre- $\mathcal{F}$ -open subset of X is pre- $\mathcal{F}$ -closed (or) if every pre- $\mathcal{F}$ -closed subset of X is pre- $\mathcal{F}$ -open.

Theorem 14. *Let $(X, \tau, \mathcal{F})$ be an ideal space. Then the following are equivalent.*

- (a) *$(X, \tau, \mathcal{F})$ is locally pre- $\mathcal{F}$ -indiscrete.*
- (b) *Every pre- $\mathcal{F}$ -open subset is pre- $\mathcal{F}$ -regular.*
- (c) *Every pre- $\mathcal{F}$ -open subset is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.*
- (d) *$p\mathcal{F} \text{int}(p\mathcal{F}cl(\{x\})) \neq \emptyset$, for every $x \in X$.*
- (e) *The empty set is the only nowhere dense subset of X .*

Proof. (a) $\Rightarrow$ (b): assume that $(X, \tau, \mathcal{F})$ is locally pre- $\mathcal{F}$ -indiscrete. Let A be a pre- $\mathcal{F}$ -open subset of X . By hypothesis, A is pre- $\mathcal{F}$ -closed. Hence A is pre- $\mathcal{F}$ -regular.

(b) $\Rightarrow$ (c): if A is pre- $\mathcal{F}$ -open, then $A = p\mathcal{F} \text{int}(A)$. Also by hypothesis, A is pre- $\mathcal{F}$ -closed. Therefore, $A = p\mathcal{F} \text{int}(p\mathcal{F}cl(A))$. Hence A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

(c) $\Rightarrow$ (d): since $\{x\}$ is preopen, $\{x\}$ is pre- $\mathcal{F}$ -open. By (c), $\{x\}$ is a pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open set. Therefore, $\{x\} = p\mathcal{F} \text{int}(p\mathcal{F}cl(\{x\}))$.

(d) $\Rightarrow$ (e): by Theorem 11, in any space, the empty set is the only subset which is nowhere dense and pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.

(e) $\Rightarrow$ (a): suppose that A is a pre- $\mathcal{F}$ -closed set. Now $\text{int}(cl(A - p\mathcal{F}\text{int}(A))) = \text{int}(cl(A - (A \cap \text{int}(cl^*(A))))) = \text{int}(cl(A \cap (X - A) \cup (\text{int}(cl^*(X - A))))) = \text{int}(cl(A \cap (X - A) \cup (A - \text{int}(cl^*(A))))) = \text{int}(cl(A - \text{int}(cl^*(A)))) \subset \text{int}(cl(A - \text{int}(cl^*(A))) = \text{int}(cl(A) - cl(\text{int}(cl^*(A)))) \subset \text{int}(cl(A) - (\text{int}(cl^*(A)))) = \emptyset$. Therefore $A - p\mathcal{F}\text{int}(A)$ is nowhere dense which implies $A = p\mathcal{F}\text{int}(A)$, and so A is pre- $\mathcal{F}$ -open. Hence is A locally pre- $\mathcal{F}$ -indiscrete. $\square$

An ideal space $(X, \tau, \mathcal{F})$ is said to be $p\mathcal{F}R$ -door if every subset of X is either pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open or pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed.

Theorem 15. *Let $(X, \tau, \mathcal{F})$ be a $p\mathcal{F}R$ -door space; then every pre- $\mathcal{F}$ -open set in the space is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open.*

Proof. Let A be a pre- $\mathcal{F}$ -open subset of X . Since X is $p\mathcal{F}R$ -door, A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -closed and so $A = p\mathcal{F}\text{int}(p\mathcal{F}cl(A))$ which implies that $p\mathcal{F}\text{int} A = p\mathcal{F}\text{int}(p\mathcal{F}\text{int}(p\mathcal{F}cl(A)))$. Since A is pre- $\mathcal{F}$ -open, $A = p\mathcal{F}\text{int}(p\mathcal{F}cl(A))$. Hence A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open. $\square$

Theorem 16. *Let $(X, \tau, \mathcal{F})$ be an ideal space. A subset A of X is both α - $\mathcal{F}$ -open and α - $\mathcal{F}$ -closed; then A is a pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open set.*

Proof. Let A be an α - $\mathcal{F}$ -open and α - $\mathcal{F}$ -closed set. Then A is a pre- $\mathcal{F}$ -open and pre- $\mathcal{F}$ -closed set and hence A is a pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open set. $\square$

Theorem 17. *Let $(X, \tau, \mathcal{F})$ be an ideal space. A subset A of X is α - $\mathcal{F}$ -open and pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open; then $A = \text{int}(cl^*(\text{int}(A)))$.*

Proof. Suppose A is α - $\mathcal{F}$ -open. Then $A \subset \text{int}(cl^*(\text{int}(A)))$. And A is pre- $\mathcal{F}$ -regular pre- $\mathcal{F}$ -open which implies $A = p\mathcal{F}\text{int}(p\mathcal{F}cl(A)) \supset p\mathcal{F}\text{int}(cl^*(\text{int}(p\mathcal{F}cl(A)))) \supset p\mathcal{F}\text{int}(cl^*(\text{int}(A))) \supset \text{int}(cl^*(\text{int}(A)))$. Therefore $A = \text{int}(cl^*(A))$. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

- [1] K. Kuratowski, *Topology*, vol. 1, Academic Press, New York, NY, USA, 1966.
- [2] R. Vaidyanathaswamy, *Set Topology*, Chelsea Publishing, New York, NY, USA, 2nd edition, 1960.
- [3] R. Vaidyanathaswamy, "The localisation theory in set-topology," *Proceedings of the Indian Academy of Sciences*, vol. 20, pp. 51–61, 1944.
- [4] D. Jankovic and T. R. Hamlett, "Compatible extensions of ideals," *Unione Matematica Italiana: Bollettino B: Serie VII*, vol. 6, no. 3, pp. 453–465, 1992.
- [5] J. Dontchev, "On pre- $\mathcal{F}$ -open sets and a decomposition of $\mathcal{F}$ -continuity," *Banyan Mathematical Journal*, vol. 2, 1996.

- [6] E. Hatir and T. Noiri, "On decompositions of continuity via idealization," *Acta Mathematica Hungarica*, vol. 96, no. 4, pp. 341–349, 2002.
- [7] J. Dontchev, M. Ganster, and D. Rose, "Ideal resolvability," *Topology and its Applications*, vol. 93, no. 1, pp. 1–16, 1999.
- [8] A. Açıkgöz, S. Yüksel, and T. Noiri, " α -I-Preirresolute functions and β -I-Preirresolute functions," *Bulletin of the Malaysian Mathematical Sciences Society II*, vol. 28, no. 1, pp. 1–8, 2005.
- [9] V. Renuka Devi and D. Sivaraj, "A decomposition of continuity via ideals," *Acta Mathematica Hungarica*, vol. 118, no. 1–2, pp. 53–59, 2008.