Research Article

Algebraic Numbers Satisfying Polynomials with Positive Rational Coefficients

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A theorem of Dubickas, affirming a conjecture of Kuba, states that a nonzero algebraic number is a root of a polynomial f with positive rational coefficients if and only if none of its conjugates is a positive real number. A certain quantitative version of this result, yielding a growth factor for the coefficients of f similar to the condition of the classical Eneström-Kakeya theorem of such polynomial, is derived. The bound for the growth factor so obtained is shown to be sharp for some particular classes of algebraic numbers.

1. Introduction

A nonzero complex number γ is called *positively algebraic* if it is a root of a polynomial all of whose coefficients are positive rational numbers. In 2005, Kuba [1] conjectured that a necessary condition for an algebraic number γ to be positively algebraic is that none of its conjugates is a positive real number. This conjecture was confirmed affirmatively by Dubickas [2], in 2007, through the following result.

Theorem 1 (see [2]). *A nonzero complex number γ is a root of a polynomial with positive rational coefficients if and only if γ is an algebraic number such that none of its conjugates is a positive real number.*

In 2009, Brunotte [3] gave an elementary proof of Dubickas-Kuba theorem based on the following lemma ([3, Lemma 2]), which is originated from [4] (see also [5]): if $f(x) \in \mathbb{R}[x]$ is a polynomial having no nonnegative roots, then there exists $m \in \mathbb{N}$ such that $(1+x)^m f(x)$ has only positive coefficients. See also [6], where a bound for the degree of the polynomial with rational positive coefficients was given.

We fix the following notation and terminology throughout. Denote by $\mathbb{R}^+$ the set of positive real numbers. For $r \in \mathbb{R}^+$, let

$$\begin{aligned} P &:= \left\{ f(x) = \sum_{j=0}^J c_j x^j \in \mathbb{R}[x]; \right. \\ &\quad \left. c_J > 0, c_j \geq 0 \ (0 \leq j \leq J-1), J \in \mathbb{N} \cup \{0\} \right\}, \\ Q &:= \left\{ f(x) = \sum_{j=0}^J c_j x^j \in \mathbb{R}[x]; \right. \\ &\quad \left. c_J > 0, c_j \leq 0 \ (0 \leq j \leq J-1), J \in \mathbb{N} \cup \{0\} \right\}, \\ Q(r) &:= \left\{ f(x) = \sum_{j=0}^J c_j x^j \in \mathbb{R}[x]; \right. \\ &\quad \left. (x-r)f(x) \in Q, J \in \mathbb{N} \cup \{0\} \right\}, \\ Q(0) &:= Q, \\ Q^+(r) &:= Q(r) \cap \mathbb{R}^+[x], \\ Q^+(>r) &:= \cup_{s>r} Q^+(s). \end{aligned} \tag{1}$$

Some basic properties, which are needed in our work here, modified with the same proofs for $Q^+(r)$ from [7], are in the following lemma.

Lemma 2. Let $r > 0$ and $f(x) = \sum_{j=0}^J c_j x^j \in \mathbb{R}[x] \setminus \{0\}$. Then

- (1) $f(x) \in Q(r) \Leftrightarrow c_0 \geq 0, c_j \leq r c_{j+1} \ (0 \leq j < J)$;
 $f(x) \in Q^+(r) \Leftrightarrow c_0 > 0, c_j \leq r c_{j+1} \ (0 \leq j < J)$;
- (2) $0 < a < b \Rightarrow X(a) \subsetneq X(b)$, where $X \in \{Q, Q^+\}$;
- (3) $Q^+(r) \subsetneq Q(r) \subsetneq P$;
- (4) $f(x) \in Q$ and $c_j \neq 0$ for some $j \in \{0, \dots, J-1\} \Rightarrow f(x)$ has a unique positive zero z_0 ; this zero is simple and all other roots have absolute values $\leq z_0$;
- (5) let $Y \in \{P, Q, Q(r), Q^+(r)\}$. Then $f(x) \in Y \Rightarrow f(x^k) \in Y$ for all $k \in \mathbb{N}$;
- (6) ([7, Lemma 2]) $a_1, a_2, \dots, a_s$ are positive real numbers, and $f_k(x) \in X(a_k) \ (1 \leq k \leq s) \Rightarrow \prod_{k=1}^s f_k(x) \in X(\sum_{k=1}^s a_k)$, where $X \in \{Q, Q^+\}$;
- (7) ([7, Lemma 3]) γ is a complex number which is not real positive $\Rightarrow \gamma$ is a root of a polynomial in $Q(r)$ for any $r > |\gamma|$.

Our first main result is a certain quantitative improvement of Dubickas-Kuba theorem (Theorem 1 above).

Theorem 3. Let γ be a nonzero algebraic number of degree n (over $\mathbb{Q}$), let $\gamma_1(=\gamma), \dots, \gamma_n$ be all its conjugates, and let

$$r > \overline{|\gamma|} := \max \{|\gamma_t|; 1 \leq t \leq n\}. \quad (2)$$

Then all conjugates of γ are in $\mathbb{C} \setminus \mathbb{R}^+$ if and only if there exists $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$ such that $p(\gamma) = 0$.

The nontrivial half of Theorem 3 is its necessity part, and its main difficulty is to show the existence of a polynomial in $Q^+(r)$ all of whose coefficients are rational numbers. Should we need only a polynomial with nonnegative real coefficients, this necessity part follows from Roitman-Rubinstein's result [7, Lemma 4] but with a bound for r which depends not only on the conjugates of α but also on other roots. Theorem 3 is proved in the next section.

Combining Theorem 3 with part (2) of Lemma 2, it is natural to ask for the least possible value of r for which Theorem 3 holds. In the last section, we show that the bound $|\gamma|$ is optimal for a class of algebraic numbers without nonnegative conjugates. The investigation in this second part leads us to some interesting connections with the classical result of Eneström-Kakeya [8], which gives upper and lower bounds for the absolute values of the roots of polynomials with positive coefficients.

2. Proof of Theorem 3

The proof of Theorem 3 is done by analyzing the nature of the roots which are complex numbers, and we simplify by treating them separately in the next two lemmas.

Lemma 4. Let $\lambda = |\lambda|e^{2\pi i\theta} \neq 0$ with $\theta \in (0, 1) \setminus \mathbb{Q}$. Then, for any real $r > |\lambda|$, there exists $F(x) \in Q^+(r)$ such that $F(\lambda) = 0$.

Proof. Let $u = r/|\lambda| > 1$. Since $\theta \in \mathbb{R} \setminus \mathbb{Q}$, by Kronecker's theorem ([9, Theorem 439, page 376]), the set $\{e^{2\pi i n\theta} : n \in \mathbb{N}\}$ is dense in the unit circle and so there exists $m(=m_\theta) \in \mathbb{N}$ such that

$$-\frac{1}{2} < \cos(2\pi m\theta) < -\frac{1}{2u}. \quad (3)$$

This yields

$$1 < -2u \cos(2\pi m\theta) < u, \quad \frac{1}{u} < -2 \cos(2\pi m\theta) < 1 < u. \quad (4)$$

Let

$$\begin{aligned} f_1(x) &= \left(x - \left(\frac{\lambda}{|\lambda|}\right)^m\right) \left(x - \left(\frac{\bar{\lambda}}{|\lambda|}\right)^m\right) \\ &= x^2 - 2x \cos(2\pi m\theta) + 1. \end{aligned} \quad (5)$$

Thus, (4) shows that all the coefficients of f_1 are $> 1/u > 0$. Again, using (4) with Lemma 2 part (2), we get

$$f_1(x) \in Q^+(u) \subseteq Q^+(u^m). \quad (6)$$

Thus, $(x - u^m)f_1(x) \in Q$ and so, by Lemma 2 part (5), $(x^m - u^m)f_1(x^m) \in Q$. Let

$$f_2(x) = \left(\frac{x^m - u^m}{x - u}\right) f_1(x^m), \quad (7)$$

so that $(x - u)f_2(x) \in Q$. Consequently,

$$\left(\frac{x}{|\lambda|} - \frac{r}{|\lambda|}\right) f_2\left(\frac{x}{|\lambda|}\right) \in Q. \quad (8)$$

Putting

$$F(x) = \frac{1}{|\lambda|} f_2\left(\frac{x}{|\lambda|}\right), \quad (9)$$

we see that

$$F(x) = \frac{1}{|\lambda|^{3m}} \left(\frac{x^m - r^m}{x - r}\right) (x^m - \lambda^m) (x^m - \bar{\lambda}^m) \in Q^+(r), \quad (10)$$

and λ and $\bar{\lambda}$ are roots of $F(x)$. $\square$

Lemma 5. Let $\lambda = |\lambda|e^{2\pi i\theta} \neq 0$ with $\theta \in (0, 1) \cap \mathbb{Q}$. Then there exists $G(x) \in Q^+(|\lambda|)$ such that $G(\lambda) = 0$.

Proof. Since $\theta \in \mathbb{Q}$, there is an $M(=M_\theta) \in \mathbb{N}$ such that $(\lambda/|\lambda|)^M = 1$. Putting

$$g(x) = \frac{x^M - (\lambda/|\lambda|)^M}{x - 1}, \quad (11)$$

we clearly see that $(x - 1)g(x) \in Q$, and so

$$\left(\frac{x}{|\lambda|} - 1\right) g\left(\frac{x}{|\lambda|}\right) \in Q. \quad (12)$$

The result follows by taking

$$\begin{aligned} G(x) &= \frac{1}{|\lambda|} g\left(\frac{x}{|\lambda|}\right) = \frac{1}{|\lambda|^M} \left(\frac{x^M - \lambda^M}{x - |\lambda|} \right) \\ &= \frac{1}{|\lambda|} \left\{ \left(\frac{x}{|\lambda|} \right)^{M-1} + \left(\frac{x}{|\lambda|} \right)^{M-2} \right. \\ &\quad \left. + \cdots + \left(\frac{x}{|\lambda|} \right)^2 + \frac{x}{|\lambda|} + 1 \right\} \in Q^+(|\lambda|), \end{aligned} \quad (13)$$

noting that λ and $\bar{\lambda}$ are two roots of $G(x)$. $\square$

Observe that the proof of this last lemma remains true when λ is a negative real number (with $M = 2$). The following result indicates how to obtain a multiple in $Q^+(r)$ of a given element of $\mathbb{R}[x]$ with a factor in $\mathbb{Q}[x]$.

Lemma 6. *Let $A(x), \mathcal{B}(x) \in \mathbb{R}[x]$. If $A(x)\mathcal{B}(x) \in Q^+(r)$, for some $r > 0$, then there exists a polynomial $B(x) \in \mathbb{Q}[x]$ such that $A(x)B(x) \in Q^+(r)$.*

Proof. Writing

$$A(x)\mathcal{B}(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0 \quad (14)$$

we have $c_j > 0$ ($0 \leq j \leq k$), and, by Lemma 2 part (1),

$$0 < c_0 \leq r c_1 \leq r^2 c_2 \leq r^3 c_3 \leq \cdots \leq r^k c_k. \quad (15)$$

For each $j \in \{0, 1, \dots, k\}$, since the j th coefficient c_j depends on the first j coefficients of $A(x)$ and $\mathcal{B}(x)$ and since any real number is a limit of a sequence of rational numbers from both sides, we can successively choose rational numbers $b_0, b_1, \dots, b_m$ so close to the corresponding coefficients of $\mathcal{B}(x)$ in such a way that when the polynomial

$$B(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \in \mathbb{Q}[x] \quad (16)$$

is multiplied to $A(x)$ yielding

$$A(x)B(x) = \tilde{c}_k x^k + \tilde{c}_{k-1} x^{k-1} + \cdots + \tilde{c}_1 x + \tilde{c}_0, \quad (17)$$

these new coefficients still satisfy

$$0 < \tilde{c}_0 \leq r \tilde{c}_1 \leq r^2 \tilde{c}_2 \leq r^3 \tilde{c}_3 \leq \cdots \leq r^k \tilde{c}_k. \quad (18)$$

Lemma 2 part (1) shows then that $A(x)B(x) \in Q^+(r)$. $\square$

The next lemma proves Theorem 3 for the case when the algebraic number γ and all its conjugates lie inside the unit circle.

Lemma 7. *Let γ be a nonzero algebraic number and let $\gamma_1 (= \gamma), \dots, \gamma_n$ be all its conjugates. Assume that*

$$|\gamma| := \max_{1 \leq t \leq n} \{|\gamma_t|\} < r, \quad (19)$$

where $r \in (0, 1]$. Then all conjugates of γ are in $\mathbb{C} \setminus \mathbb{R}^+$ if and only if there exists $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$ such that $p(\gamma) = 0$.

Proof. Assume that the nonzero algebraic number γ and all its conjugates $\gamma_1 (= \gamma), \dots, \gamma_n$ are in $\mathbb{C} \setminus \mathbb{R}^+$. Arrange these numbers in such a way that

$$\begin{aligned} &\{\gamma_1, \gamma_2, \dots, \gamma_n\} \\ &= \{\gamma_j \in \mathbb{R}^-; 1 \leq j \leq J\} \\ &\quad \cup \{\gamma_{J+k} = |\gamma_{J+k}| e^{2\pi i \theta_{J+k}}; \theta_{J+k} \in \mathbb{Q}, 1 \leq k \leq K\} \\ &\quad \cup \{\gamma_{J+K+\ell} = |\gamma_{J+K+\ell}| e^{2\pi i \theta_{J+K+\ell}}; \theta_{J+K+\ell} \in \mathbb{R} \setminus \mathbb{Q}, 1 \leq \ell \leq L\} \end{aligned} \quad (20)$$

with $J + K + L = n$. (In the above arrangement, it is tacitly assumed that J, K , and L are positive integers; indeed, should any one of the three sets be empty, the corresponding value(s) of J, K , or L could be zero. Yet the remaining arguments below still work with only slight adjustments.) Since $|\gamma_t| < r$, for each $1 \leq t \leq n$, there exists an odd positive integer N such that $|\gamma_t|^N < r/n$, for each $1 \leq t \leq n$, and $\gamma_j^N < 0$, for each $1 \leq j \leq J$. Thus, for $1 \leq j \leq J$, we have

$$\begin{aligned} H_j(x) &:= (x - \gamma_j^N) \in Q^+ \left(|\gamma_j|^N \right) \subseteq Q^+ \left(\frac{r}{n} \right) \\ &\quad \text{with } H_j(\gamma_j^N) = 0. \end{aligned} \quad (21)$$

For $1 \leq k \leq K$, by Lemma 5, there exists a polynomial

$$G_k(x) \in Q^+ \left(|\gamma_{J+k}|^N \right) \subseteq Q^+ \left(\frac{r}{n} \right) \quad \text{with } G_k(\gamma_{J+k}^N) = 0. \quad (22)$$

For $1 \leq \ell \leq L$, by Lemma 4 and $|\gamma_{J+K+\ell}|^N < r/n$, there exists a polynomial

$$F_\ell(x) \in Q^+ \left(\frac{r}{n} \right) \quad \text{with } F_\ell(\gamma_{J+K+\ell}^N) = 0. \quad (23)$$

Let

$$E(x) := \prod_{1 \leq j \leq J} H_j(x) \prod_{1 \leq k \leq K} G_k(x) \prod_{1 \leq \ell \leq L} F_\ell(x), \quad (24)$$

and note that $\gamma_1^N, \dots, \gamma_n^N$ are its roots. From Lemma 2 part (6) with $J + K + L = n$, we have $E(x) \in Q^+(r)$. Thus,

$$(x - 1)E(x) \in Q \quad \text{and so} \quad (x^N - 1)E(x^N) \in Q. \quad (25)$$

Consequently,

$$(x - 1)(x^{N-1} + x^{N-2} + \cdots + x + 1)E(x^N) \in Q \quad (26)$$

and so

$$(x^{N-1} + x^{N-2} + \cdots + x + 1)E(x^N) \in Q^+(1). \quad (27)$$

Since $M(x) \mid E(x^N)$ over $\mathbb{R}[x]$, where $M(x) \in \mathbb{Q}[x]$ is the minimal polynomial of γ , there exists $\mathcal{B}(x) \in \mathbb{R}[x]$ such that $E(x^N) = M(x)\mathcal{B}(x)$. Thus,

$$(x^{N-1} + x^{N-2} + \cdots + x + 1)M(x)\mathcal{B}(x) \in Q^+(r). \quad (28)$$

From Lemma 6, there exists $B(x) \in \mathbb{Q}[x]$ such that

$$(x^{N-1} + x^{N-2} + \cdots + x + 1) M(x) B(x) \in Q^+(r) \cap \mathbb{Q}[x]. \quad (29)$$

Putting

$$p(x) := (x^{N-1} + x^{N-2} + \cdots + x + 1) M(x) B(x), \quad (30)$$

we see that $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$ and $p(\gamma) = 0$.

Conversely, assume that there exists $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$ such that $p(\gamma) = 0$. Since $p(x) \in \mathbb{Q}[x]$, all conjugates of γ are zeros of $p(x)$ and, since all coefficients of $p(x)$ are positive, no real conjugate of γ can be positive. $\square$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Assume that all conjugates of $\gamma \in \mathbb{C} \setminus \mathbb{R}^+$. Since $r > |\gamma|$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r_1 \in \mathbb{Q}$ such that $|\gamma| < r_1 < r$. Let

$$\lambda_t := \frac{\gamma_t}{r_1} \quad (1 \leq t \leq n), \quad (31)$$

$$\lambda := \lambda_1. \quad (32)$$

Then $|\lambda_t| < 1$ and $\lambda_t \in \mathbb{C} \setminus \mathbb{R}^+$, for all $1 \leq t \leq n$. Let

$$F(x) := M(r_1 x), \quad (33)$$

where $M(x) \in \mathbb{Q}[x]$ is the minimal polynomial of γ . Then $F(x)$ is the minimal polynomial of λ , and $\lambda_1 (= \lambda), \dots, \lambda_n$ are conjugates of λ . By Lemma 7, there exists $p_1(x) \in Q^+(1) \cap \mathbb{Q}[x]$ such that $p_1(\lambda) = 0$. Then $F(x) \mid p_1(x)$; that is, $M(x) \mid p_1(x/r_1)$. Since $p_1(x) \in Q^+(1) \cap \mathbb{Q}[x]$ and $r_1 \in \mathbb{Q}$, we have $p_1(x/r_1) \in Q^+(r_1) \cap \mathbb{Q}[x]$. Let

$$p(x) := p_1\left(\frac{x}{r_1}\right). \quad (34)$$

Then $p(x) \in Q^+(r_1) \cap \mathbb{Q}[x]$ and $p(\gamma) = 0$. Since $0 < r_1 < r$, by Lemma 2 part (2), we have $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$.

On the other hand, assume that there exists $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$ such that $p(\gamma) = 0$. Since $p(x) \in \mathbb{Q}[x]$, all conjugates of γ are zeros of $p(x)$ and, since all coefficients of $p(x)$ are positive, no real conjugate of γ can be positive. $\square$

It is worth noting that in [7, Lemma 4], the bound for r reduces to $|\gamma|$ if the polynomial $p(x)$ mentioned in Theorem 3 is the minimal polynomial of γ . The next corollary contains the Dubickas-Kuba theorem and some equivalent assertions.

Corollary 8. Let γ be a nonzero algebraic number, let $\gamma_1 (= \gamma), \gamma_2, \dots, \gamma_n$ be all its conjugates, and let $r > |\gamma|$. Then the following statements are equivalent:

- (i) all conjugates of γ are in $\mathbb{C} \setminus \mathbb{R}^+$;
- (ii) there exists $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$ such that $p(\gamma) = 0$;
- (iii) there exists $g(x) \in \mathbb{R}^+[x] \cap \mathbb{Q}[x]$ such that $g(\gamma) = 0$;
- (iv) there exist $q(x) \in \mathbb{Q} \cap \mathbb{Q}[x]$ and $w \in \mathbb{R}^+ \setminus \{\gamma_1, \dots, \gamma_n\}$ such that $q(\gamma) = q(w) = 0$.

Proof. Assertions (i) and (ii) are equivalent by Theorem 3. Clearly, (ii) implies (iii). To show (iii) implies (iv), assume that there exists

$$g(x) = x^m + g_1 x^{m-1} + \cdots + g_{m-1} x + g_m \in \mathbb{R}^+[x] \cap \mathbb{Q}[x] \quad (35)$$

such that $g(\gamma) = 0$, and so $g(\gamma_1) = \cdots = g(\gamma_n) = 0$. Since $g_h > 0$, for all $h \in \{1, \dots, m\}$, none of $\gamma_1, \dots, \gamma_n$ can be real and positive. If $\gamma_j \neq 0$ for some $j \in \{1, 2, \dots, n\}$, choose

$$q(x) = (x^m)^2 - (g_1 x^{m-1} + g_2 x^{m-2} + \cdots + g_{m-1} x + g_m)^2 \in \mathbb{Q}. \quad (36)$$

Since $g(x)$ is a factor of $q(x)$, we have $q(\gamma_1) = \cdots = q(\gamma_n) = 0$. From $\gamma_j \neq 0$ and Lemma 2 part (4), we know that $q(x)$ has a unique positive zero, say w , which must then be distinct from all $\gamma_1, \dots, \gamma_n$, as desired.

That assertion (iv) implies that assertion (i) is again an immediate consequence of Lemma 2 part (4). $\square$

3. Eneström-Kakeya Theorem

To proceed further, we need a new notion.

Definition 9. For

$$f(x) = \sum_{j=0}^J c_j x^j \in \mathbb{R}^+[x] \setminus \mathbb{R}, \quad (37)$$

its lower and upper Eneström-Kakeya quotients are defined, respectively, by

$$\alpha[f] := \min \left\{ \frac{c_0}{c_1}, \frac{c_1}{c_2}, \dots, \frac{c_{J-2}}{c_{J-1}}, \frac{c_{J-1}}{c_J} \right\}, \quad (38)$$

$$\beta[f] := \max \left\{ \frac{c_0}{c_1}, \frac{c_1}{c_2}, \dots, \frac{c_{J-2}}{c_{J-1}}, \frac{c_{J-1}}{c_J} \right\}.$$

Part (1) of Lemma 2 is closely related to the classical Eneström-Kakeya theorem, [8, Theorem 1], which states the following.

Theorem 10. Let $p(x) \in \mathbb{R}[x] \setminus \mathbb{R}$ all of whose coefficients are positive. Then all the zeros of $p(x)$ are contained in the annulus $\alpha[p] \leq |x| \leq \beta[p]$, where $\alpha[p]$ and $\beta[p]$ are, respectively, the lower and upper Eneström-Kakeya quotients of $p(x)$.

In this section, we derive a proposition yielding conditions which are necessary for a product of two polynomials to be in $Q(r)$.

Proposition 11. Let

$$f(x) = c_J x^J + c_{J-1} x^{J-1} + \cdots + c_1 x + c_0 \in \mathbb{R}^+[x], \quad (39)$$

where $J \geq 1$, and let $r \in \mathbb{R}^+$. If

$$c_j > r c_{j+1} \quad (0 \leq j < J), \quad (40)$$

then $f(x)d(x) \notin Q(r)$ for all $d(x) \in \mathbb{R}[x] \setminus \{0\}$.

Proof. Let

$$d(x) = \sum_{k=0}^K d_k x^k \in \mathbb{R}[x] \setminus \{0\}. \quad (41)$$

If $f(x)d(x) \in Q(r)$, then Lemma 2 part (1) gives

$$0 \leq c_0 d_0 \quad (42)$$

$$c_0 d_0 \leq r(c_1 d_0 + c_0 d_1) \quad (43)$$

$$c_1 d_0 + c_0 d_1 \leq r(c_2 d_0 + c_1 d_1 + c_0 d_2) \quad (44)$$

$$\vdots \quad (45)$$

$$\begin{aligned} c_{K-1} d_0 + c_{K-2} d_1 + \cdots + c_0 d_{K-1} \\ \leq r(c_K d_0 + c_{K-1} d_1 + \cdots + c_0 d_K) \end{aligned} \quad (46)$$

$$\begin{aligned} c_K d_0 + c_{K-1} d_1 + \cdots + c_0 d_K \\ \leq r(c_{K+1} d_0 + c_K d_1 + \cdots + c_1 d_K), \end{aligned} \quad (47)$$

where we adopt the convention that $c_j = 0$, for all $j > J$, and $d_k = 0$, for all $k > K$. From $c_0 > 0$, and (42), we get $d_0 \geq 0$. From (43) and (40), we have

$$(c_0 - rc_1) d_0 \leq rc_0 d_1, \quad d_1 \geq 0. \quad (48)$$

From (44) and (40), we get

$$(c_1 - rc_2) d_0 + (c_0 - rc_1) d_1 \leq rc_0 d_2, \quad (49)$$

which together with previous results yield $d_2 \geq 0$. Continuing in the same manner up to (46), we get $d_3 \geq 0$, $d_4 \geq 0$, $\dots$, $d_K \geq 0$. Thus,

$$\begin{aligned} (c_K - rc_{K+1}) d_0 + (c_{K-1} - rc_K) d_1 + \cdots + (c_1 - rc_2) d_{K-1} \\ + (c_0 - rc_1) d_K \leq 0. \end{aligned} \quad (50)$$

Since $c_j > rc_{j+1}$ ($j = 0, 1, 2, \dots, J-1$), the left-hand expression in (50) can be 0 only when $d_0 = d_1 = \cdots = d_K = 0$; that is, $d(x) \equiv 0$, which is not possible. $\square$

The next proposition indicates the significance of the upper and lower Eneström-Kakeya quotients.

Proposition 12. Let $f(x) \in \mathbb{R}^+[x]$.

- (i) The upper Eneström-Kakeya quotient $\beta[f]$ is the smallest $r > 0$ such that $f(x) \in Q^+(r)$.
- (ii) The lower Eneström-Kakeya quotient $\alpha[f]$ has the property that if $p(x) \in Q^+(r)$ with $0 < r < \alpha[f]$, then $f(x) \nmid p(x)$ over $\mathbb{R}[x]$.

Proof. Part (i) follows from Lemma 2 part (1). Part (ii) is immediate from Proposition 11. $\square$

In passing, from the definition of Eneström-Kakeya quotients, it seems natural to ask whether one quotient can be a reciprocal of the other. This is indeed the case when the polynomial is self-reciprocal. Let $g(x)$ be a polynomial. The reciprocal polynomial of $g(x)$ is defined as

$$g^*(x) = x^{\deg g} g\left(\frac{1}{x}\right), \quad (51)$$

and we say that $g(x)$ is *self-reciprocal* if $g(x) = g^*(x)$.

Proposition 13. Let $f(x) \in \mathbb{R}^+[x] \setminus \mathbb{R}$. Then $\beta[f^*] = 1/\alpha[f]$, and $f^*(x) \in Q^+(1/\alpha[f])$. Moreover, if $f(x)$ is self-reciprocal, then $\beta[f] = 1/\alpha[f]$ and $f(x) \in Q^+(1/\alpha[f])$.

Proof. Writing $f(x) = c_J x^J + c_{J-1} x^{J-1} + \cdots + c_1 x + c_0 \in \mathbb{R}^+[x] \setminus \mathbb{R}$, we have

$$f^*(x) = x^J f\left(\frac{1}{x}\right) = c_J + c_{J-1} x + \cdots + c_1 x^{J-1} + c_0 x^J \quad (52)$$

and so

$$\beta[f^*] = \max \left\{ \frac{c_1}{c_0}, \frac{c_2}{c_1}, \dots, \frac{c_{J-1}}{c_{J-2}}, \frac{c_J}{c_{J-1}} \right\} = \frac{1}{\alpha[f]}. \quad (53)$$

From Proposition 12 part (i), we have

$$f^*(x) \in Q^+(\beta[f^*]) = Q^+\left(\frac{1}{\alpha[f]}\right). \quad (54)$$

If $f(x)$ is self-reciprocal, then clearly $\beta[f] = 1/\alpha[f]$ and $f(x) \in Q^+(1/\alpha[f])$. $\square$

Corollary 14. If $f(x) \in \mathbb{R}^+[x] \setminus \mathbb{R}$ is self-reciprocal, then

- (i) $\alpha[f] \leq 1$, and
- (ii) $\alpha[f] = 1$, if and only if $\beta[f] = 1$.

Proof. (i) If $\alpha[f] > 1$, then $1/\alpha[f] < 1 < \alpha[f]$. Since f is self-reciprocal, by Proposition 13, $f(x) \in Q^+(1/\alpha[f])$, which contradicts Proposition 12 part (ii). Part (ii) follows readily from Proposition 13. $\square$

4. Minimal Polynomials with Positive Coefficients

In this section, we aim to show that the bound $|\overline{\gamma}|$, in Theorem 3, is best possible by showing that it cannot be improved for a subclass of the class of algebraic numbers whose minimal polynomials have positive coefficients. Let γ be a nonzero algebraic number, and $M(x) \in \mathbb{Q}[x]$ its minimal polynomial. We say that $M(x)$ is *positively minimal* if all its coefficients are positive. From Proposition 12, we have the following.

Proposition 15. Let γ be an algebraic number and let

$$M(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (55)$$

be its minimal polynomial. If $M(x)$ is positively minimal, then, for any $r \in (0, \alpha[M])$ and any $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$, we have $p(\gamma) \neq 0$.

Proposition 15 leads naturally to the following definition.

Definition 16. Let γ be a nonzero algebraic number, whose minimal polynomial $M(x)$ is positively minimal. Define the *growth factor* of an algebraic number γ with respect to $M(x)$, denoted by r_γ , to be the infimum of the set of nonnegative real numbers with the following property: for any $r > r_\gamma$, there exists $p(x) \in Q^+(r) \cap \mathbb{Q}[x]$ such that $p(\gamma) = 0$.

Remarks. The growth factor enjoys the following basic properties:

- (i) r_γ is unique;
- (ii) $r_\gamma \leq \overline{|\gamma|}$ (Theorem 3);
- (iii) $\alpha[M] \leq r_\gamma \leq \beta[M]$ (Propositions 12 and 15).

For a nonzero algebraic number γ whose minimal polynomial is positively minimal and whose growth factor is r_γ , let

$$\mathcal{P}_\gamma := \{p(x) \in Q^+(r_\gamma) \cap \mathbb{Q}[x]; p(\gamma) = 0\}. \quad (56)$$

A class of algebraic numbers for which the bound $\overline{|\gamma|}$ in Theorem 3 cannot be improved is given in the next theorem.

Theorem 17. Let γ be an algebraic number and assume that its minimal polynomial

$$M(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad (57)$$

is positively minimal.

- (i) If $\alpha[M] = \beta[M]$, then $\alpha[M] = r_\gamma = \beta[M]$ and $M(x) \in \mathcal{P}_\gamma$.
- (ii) We have $\alpha[M] = \beta[M]$, if and only if

$$M(x) = x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^{n-1}x + a^n \quad (58)$$

for some $a \in \mathbb{Q}^+$.

Proof. Assertion (i) follows at once from the preceding remarks.

(ii) If $\alpha[M] = \beta[M]$, then

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \cdots = \frac{a_{n-2}}{a_{n-1}} = a_{n-1}, \quad (59)$$

which gives

$$M(x) = x^n + a_{n-1}x^{n-1} + a_{n-1}^2x^{n-2} + \cdots + a_{n-1}^{n-1}x + a_{n-1}^n \in \mathbb{Q}^+[x]. \quad (60)$$

The converse is trivial. $\square$

Theorem 17 poses a natural question whether a positively minimal polynomial must necessarily belong to $\mathcal{P}_\gamma$. A negative answer is provided by the next proposition.

Proposition 18. Let γ be an algebraic number of degree n over $\mathbb{Q}$. Assume that its minimal polynomial $M(x)$ is positively minimal. If $n \geq 2$ and all the conjugates of $\gamma_1 = \gamma, \dots, \gamma_n$ are negative real numbers, then

$$\alpha[M] \leq r_\gamma \leq \overline{|\gamma|} < \beta[M] = \sum_{t=1}^n |\gamma_t| \quad (61)$$

and $M(x) \notin \mathcal{P}_\gamma$.

Proof. Since all conjugates of γ are negative real numbers, we have

$$x - \gamma_t \in Q^+(|\gamma_t|) \quad (t = 1, \dots, n). \quad (62)$$

Lemma 2 part (6) shows that

$$\begin{aligned} M(x) &= (x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_n) \\ &:= x^n + a_{n-1}x^{n-1} + \cdots + a_1x \\ &\quad + a_0 \in Q^+\left(\sum_{t=1}^n |\gamma_t|\right) \cap \mathbb{Q}[x], \end{aligned} \quad (63)$$

and so Proposition 12 (i) implies that $\beta[M] \leq \sum_{t=1}^n |\gamma_t|$. Since $n \geq 2$ and $a_{n-1} = \sum_{t=1}^n |\gamma_t|$, we have $\overline{|\gamma|} < \sum_{t=1}^n |\gamma_t| = a_{n-1} \leq \beta[M]$. Consequently,

$$\alpha[M] \leq r_\gamma \leq \overline{|\gamma|} < \beta[M] = \sum_{t=1}^n |\gamma_t|. \quad (64)$$

By Proposition 12 (i) and the definition of the upper Eneström-Kakeya quotient, we conclude that $M(x) \notin \mathcal{P}_\gamma$. $\square$

It may be of interest to look at the growth factors and the Eneström-Kakeya quotients for positively minimal polynomials of small degrees.

Proposition 19. Let γ be an algebraic number. Assume that its minimal polynomial $M(x)$ is positively minimal and that all its conjugates are in $\mathbb{C} \setminus \mathbb{R}^+$.

- (1) If $\deg \gamma = 1$, then $\alpha[M] = r_\gamma = \beta[M] = \overline{|\gamma|} = |\gamma|$.
- (2) If $\deg \gamma = 2$, let $\gamma = a + bi$ ($a, b \in \mathbb{R}$) and $M(x) = x^2 + a_1x + a_0$; then,

$$(2.1) \text{ for } b = 0, \text{ we have } a_0/a_1 = \alpha[M] < \overline{|\gamma|} < \beta[M] = a_1, r_\gamma \in [\alpha[M], \overline{|\gamma|}],$$

$$(2.2) \text{ for } b \neq 0 \text{ and } 3a^2 \neq b^2, \text{ we have } \alpha[M] < \overline{|\gamma|} = |\gamma| < \beta[M], r_\gamma \in [\alpha[M], \overline{|\gamma|}],$$

$$(2.3) \text{ for } b \neq 0 \text{ and } 3a^2 = b^2, \text{ we have } \alpha[M] = r_\gamma = \beta[M] = \overline{|\gamma|} = |\gamma|.$$

- (3) If $\deg \gamma = 3$, then $\alpha[M] \leq r_\gamma \leq \overline{|\gamma|} < \beta[M]$.

Proof. Part (1) is trivial. We now consider part (2), that is, $\deg \gamma = 2$. If $b = 0$, then $a_1^2 \geq 4a_0 > 0$, and so

$$\alpha[M] = \frac{a_0}{a_1} < |\gamma| = \frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0} < a_1 = \beta[M]. \quad (65)$$

If $b \neq 0$, without loss of generality, assume that $\gamma = a + bi$ with $b \in \mathbb{R}^+$. Then $|\gamma| = \sqrt{a^2 + b^2}$ and

$$\begin{aligned} M(x) &= (x - a - bi)(x - a + bi) \\ &= x^2 - 2ax + a^2 + b^2 \in \mathbb{Q}[x]. \end{aligned} \quad (66)$$

Since $M(x)$ is positively minimal, we have $a < 0$ and $a^2 + b^2 > 0$. From

$$\sqrt{a^2 + b^2} \geq -2a \iff a^2 + b^2 \geq 4a^2 \iff b^2 \geq 3a^2, \quad (67)$$

we deduce that, if $3a^2 \neq b^2$, then $\alpha[M] < |\gamma| < \beta[M]$ and that, if $3a^2 = b^2$, then $\alpha[M] = r_\gamma = \beta[M] = |\gamma| = |\gamma|$.

Finally consider part (3), that is, $\deg \gamma = 3$. If γ and all its conjugates γ_1, γ_2 , and γ_3 are negative real numbers, the conclusion follows at once from Proposition 18. Assume then that

$$\begin{aligned} \gamma_1 &= -A, & \gamma_2 &= a + bi, & \gamma_3 &= a - bi \\ &\text{with } A \in \mathbb{R}^+ \setminus \mathbb{Q}, & a \in \mathbb{R}, & b \in \mathbb{R}^+. \end{aligned} \quad (68)$$

Since the minimal polynomial,

$$\begin{aligned} M(x) &= (x + A)(x - a - bi)(x - a + bi) \\ &= x^3 + (A - 2a)x^2 + (a^2 + b^2 - 2Aa)x \\ &\quad + A(a^2 + b^2) \in \mathbb{Q}[x], \end{aligned} \quad (69)$$

is positively minimal, we have $A - 2a > 0$, $a^2 + b^2 - 2Aa > 0$ and $A(a^2 + b^2) > 0$, which gives

$$\beta[M] = \max \left\{ \frac{A(a^2 + b^2)}{a^2 + b^2 - 2Aa}, \frac{a^2 + b^2 - 2Aa}{A - 2a}, A - 2a \right\}. \quad (70)$$

Since $\deg \gamma = 3$, $A - 2a$ is positive rational, and A is positive irrational, we must have $a \neq 0$. By the remarks after Proposition 15, there remains only the verification that $|\gamma| < \beta[M]$.

We split our consideration into three cases depending on the maximum absolute value of the conjugates.

(i) Case $|\gamma| = A$.

If $A \leq A - 2a$, or if $A < A(a^2 + b^2)/(a^2 + b^2 - 2Aa)$, then $|\gamma| < \beta[M]$. Otherwise, we have $A \geq A - 2a$ and $A \geq A(a^2 + b^2)/(a^2 + b^2 - 2Aa)$, which gives $a = 0$, a contradiction.

(ii) Case $|\gamma| = \sqrt{a^2 + b^2}$.

If $A(a^2 + b^2)/(a^2 + b^2 - 2Aa) = \sqrt{a^2 + b^2}$, since $A(a^2 + b^2)/(a^2 + b^2 - 2Aa)$ is positive rational, we deduce that $a^2 + b^2$ must be positive rational. Since $A(a^2 + b^2)$ is positive rational, we conclude that A is positive rational, which is a contradiction.

If $A(a^2 + b^2)/(a^2 + b^2 - 2Aa) > \sqrt{a^2 + b^2}$, then $\beta[M] > |\gamma|$.

If $A(a^2 + b^2)/(a^2 + b^2 - 2Aa) < \sqrt{a^2 + b^2}$, then

$$2Aa < \sqrt{a^2 + b^2}(\sqrt{a^2 + b^2} - A). \quad (71)$$

If $a > 0$, by (71), we get

$$\sqrt{a^2 + b^2}(\sqrt{a^2 + b^2} - A) + 2a\sqrt{a^2 + b^2} > 2Aa \quad (72)$$

yielding $(a^2 + b^2 - 2Aa)/(A - 2a) > \sqrt{a^2 + b^2}$; that is, $\beta[M] > |\gamma|$.

Consider now $a < 0$.

If $A - 2a = \sqrt{a^2 + b^2}$, since $A - 2a$ is positive rational, we deduce that $a^2 + b^2$ is positive rational. Since $A(a^2 + b^2)$ is positive rational, we conclude that A is positive rational, which is a contradiction.

If $A - 2a > \sqrt{a^2 + b^2}$, then $\beta[M] > |\gamma|$.

If $A - 2a < \sqrt{a^2 + b^2}$, then $\sqrt{a^2 + b^2}(\sqrt{a^2 + b^2} - A + 2a) > 0 > 2Aa$, yielding $\beta[M] > |\gamma|$.

□

We end this paper with another class of algebraic numbers for which the bound in Theorem 3 is optimal.

Proposition 20. *Let γ be an algebraic number, and let*

$$M(x) = x^n + a \quad (n \in \mathbb{N}, a \in \mathbb{Q}, a > 0) \quad (73)$$

be its minimal polynomial. For $0 < r < |\gamma| = a^{1/n}$, if $q(x) \in \mathbb{Q}^+(r) \cap \mathbb{Q}[x]$, then $q(\gamma) \neq 0$.

Proof. Assume on the contrary that there is $q(x) \in \mathbb{Q}^+(r) \cap \mathbb{Q}[x]$ such that $q(\gamma) = 0$. Thus, $M(x) \mid q(x)$, so that

$$\begin{aligned} q(x) &= (x^n + a) \\ &\quad \times (x^m + c_{m-1}x^{m-1} + \cdots + c_3x^3 + c_2x^2 + c_1x + c_0) \\ &= x^{n+m} + c_{m-1}x^{n+m-1} + \cdots + c_3x^{n+3} \\ &\quad + c_2x^{n+2} + c_1x^{n+1} + c_0x^n \\ &\quad + ax^m + ac_{m-1}x^{m-1} + \cdots + ac_3x^3 \\ &\quad + ac_2x^2 + ac_1x + ac_0. \end{aligned} \quad (74)$$

If $m < n - 1$, by Lemma 2 part (1) we have $0 < a \leq 0$, a contradiction.

If $m > n$, let $m = n + t$ for some $t \in \mathbb{N}$ and the relation (74) becomes

$$\begin{aligned}
 q(x) &= x^{2n+t} + c_{n+t-1}x^{2n+t-1} + c_{n+t-2}x^{2n+t-2} \\
 &\quad + \cdots + c_{t+1}x^{n+t+1} + (c_t + a)x^{n+t} \\
 &\quad + (c_{t-1} + ac_{n+t-1})x^{n+t-1} + (c_{t-2} + ac_{n+t-2})x^{n+t-2} \\
 &\quad + \cdots + (c_4 + ac_{n+4})x^{n+4} \\
 &\quad + (c_3 + ac_{n+3})x^{n+3} + (c_2 + ac_{n+2})x^{n+2} \\
 &\quad + (c_1 + ac_{n+1})x^{n+1} + (c_0 + ac_n)x^n \\
 &\quad + ac_{n-1}x^{n-1} + ac_{n-2}x^{n-2} + \cdots + ac_3x^3 \\
 &\quad + ac_2x^2 + ac_1x + ac_0.
 \end{aligned} \tag{75}$$

Invoking upon Lemma 2 part (1), we get the following chain of inequalities:

$$\begin{aligned}
 0 < ac_0 &\implies c_0 > 0 \\
 ac_0 &\leq rac_1 \implies c_1 > 0, \quad c_0 \leq rc_1 \\
 ac_1 &\leq rac_2 \implies c_2 > 0, \quad c_1 \leq rc_2, \quad c_0 \leq r^2c_2 \\
 &\vdots \\
 ac_{n-2} &\leq rac_{n-1} \\
 \implies c_{n-1} &> 0, \quad c_{n-2} \leq rc_{n-1}, \\
 c_{n-3} &\leq r^2c_{n-1}, \dots, c_0 \leq r^{n-1}c_{n-1} \\
 ac_{n-1} &\leq r(c_0 + ac_n) \leq r^n c_{n-1} + rac_n \\
 &< ac_{n-1} + rac_n \implies 0 < rac_n, \quad c_n > 0 \\
 c_0 + ac_n &\leq r(c_1 + ac_{n+1}) \\
 r(c_0 + ac_n) &\leq r^2c_1 + r^2ac_{n+1} \\
 ac_{n-1} &\leq r^2c_1 + r^2ac_{n+1} \leq r^n c_{n-1} + r^2ac_{n+1} \\
 &< ac_{n-1} + r^2ac_{n+1} \implies 0 < r^2ac_{n+1}, \quad c_{n+1} > 0 \\
 c_1 + ac_{n+1} &\leq r(c_2 + ac_{n+2}) \\
 r(c_1 + ac_{n+1}) &\leq r^2c_2 + r^2ac_{n+2} \\
 c_0 + ac_n &\leq r^2c_2 + r^2ac_{n+2} \\
 r(c_0 + ac_n) &\leq r^3c_2 + r^3ac_{n+2} \\
 ac_{n-1} &\leq r^3c_2 + r^3ac_{n+2} \leq r^n c_{n-1} + r^3ac_{n+2} \\
 &< ac_{n-1} + r^3ac_{n+2} \implies 0 < r^3ac_{n+2}, \quad c_{n+2} > 0.
 \end{aligned} \tag{76}$$

Proceeding in the same manner, we have

$$\begin{aligned}
 c_2 + ac_{n+2} &\leq r(c_3 + ac_{n+3}) \implies c_{n+3} > 0 \\
 c_3 + ac_{n+3} &\leq r(c_4 + ac_{n+4}) \implies c_{n+4} > 0 \\
 &\vdots \\
 c_{t-2} + ac_{n+t-2} &\leq r(c_{t-1} + ac_{n+t-1}) \implies c_{n+t-1} > 0.
 \end{aligned} \tag{77}$$

Since $n + t - 1 \geq t$, we have $c_t > 0$ and

$$\begin{aligned}
 c_{n+t-1} &\leq r, \quad c_{n+t-2} \leq rc_{n+t-1} \leq r^2, \\
 c_{n+t-3} &\leq rc_{n+t-2} \leq r^3, \dots, c_{t+1} \leq r^{n-1}.
 \end{aligned} \tag{78}$$

Thus, $a < c_t + a \leq rc_{t+1} \leq r^n < a$, which is a contradiction. The proofs for the cases $m = n - 1$ and n are similar but simpler. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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