Research Article

A Mean Value Formula for Elliptic Curves

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It is proved in this paper that, for any point on an elliptic curve, the mean value of x -coordinates of its n -division points is the same as its x -coordinate and that of y -coordinates of its n -division points is n times that of its y -coordinate.

1. Introduction

Let K be a field with $\text{char}(K) \neq 2, 3$ and let $\bar{K}$ be the algebraic closure of K . Every elliptic curve E over K can be written as a classical Weierstrass equation as follows:

$$E: y^2 = x^3 + ax + b \quad (1)$$

with coefficients $a, b \in K$. A point Q on E is said to be smooth (or nonsingular) if $(\partial f / \partial x|_Q, \partial f / \partial y|_Q) \neq (0, 0)$, where $f(x, y) = y^2 - x^3 - ax - b$. The point multiplication is the operation of computing

$$nP = \underbrace{P + P + \cdots + P}_n \quad (2)$$

for any point $P \in E$ and a positive integer n . The multiplication-by- n map

$$\begin{aligned} [n] : E &\longrightarrow E \\ P &\longmapsto nP \end{aligned} \quad (3)$$

is an isogeny of degree n^2 . For a point $Q \in E$, any element of $[n]^{-1}(Q)$ is called an n -division point of Q . Assume that $(\text{char}(K), n) = 1$. In this paper, the following result on the mean value of the x, y -coordinates of all the n -division points of any smooth point on an elliptic curve is proved.

Theorem 1. Let E be an elliptic curve defined over K and let $Q = (x_Q, y_Q) \in E$ be a point with $Q \neq \mathcal{O}$. Set

$$\Lambda = \{P = (x_P, y_P) \in E(\bar{K}) \mid nP = Q\}. \quad (4)$$

Then

$$\begin{aligned} \frac{1}{n^2} \sum_{P \in \Lambda} x_P &= x_Q, \\ \frac{1}{n^2} \sum_{P \in \Lambda} y_P &= ny_Q. \end{aligned} \quad (5)$$

According to Theorem 1, let $P_i = (x_i, y_i)$, $i = 1, 2, \dots, n^2$, be all the points such that $nP = Q$ and let λ_i be the slope of the line through P_i and Q ; then $y_Q = \lambda_i(x_Q - x_i) + y_i$. Therefore,

$$n^2 y_Q = \sum_{i=1}^{n^2} \lambda_i \cdot \frac{\sum_{i=1}^{n^2} x_i}{n^2} - \sum_{i=1}^{n^2} \lambda_i x_i + \sum_{i=1}^{n^2} y_i. \quad (6)$$

Thus we have

$$\begin{aligned} y_Q &= \frac{\sum_{i=1}^{n^2} \lambda_i}{n^2} \cdot \frac{\sum_{i=1}^{n^2} x_i}{n^2} - \frac{\sum_{i=1}^{n^2} \lambda_i x_i}{n^2} + \frac{\sum_{i=1}^{n^2} y_i}{n^2} \\ &= \bar{\lambda}_i \cdot \bar{x}_i - \overline{\lambda_i x_i} + \bar{y}_i, \end{aligned} \quad (7)$$

where $\bar{\lambda}_i, \bar{x}_i, \overline{\lambda_i x_i}$, and $\bar{y}_i$ are the average values of the variables $\lambda_i, x_i, \lambda_i x_i$, and y_i , respectively. Therefore,

$$Q = (x_Q, y_Q) = (\bar{x}_i, \bar{\lambda}_i \cdot \bar{x}_i - \overline{\lambda_i x_i} + \bar{y}_i) = \left(\bar{x}_i, \frac{1}{n} \bar{y}_i \right). \quad (8)$$

Remark 2. The discrete logarithm problem in elliptic curve E is to find n by given $P, Q \in E$ with $Q = nP$. The above theorem gives some information on the integer n .

2. Proof of Theorem 1

To prove Theorem 1, define division polynomials [1] $\psi_n \in \mathbb{Z}[x, y, a, b]$ on an elliptic curve $E : y^2 = x^3 + ax + b$ inductively as follows:

$$\begin{aligned}\psi_0 &= 0, \\ \psi_1 &= 1, \\ \psi_2 &= 2y, \\ \psi_3 &= 3x^4 + 6ax^2 + 12bx - a^2, \\ \psi_4 &= 4y(x^6 + 5ax^4 + 20bx^3 - 5a^2x^2 - 4abx - 8b^2 - a^3), \\ \psi_{2n+1} &= \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3, \quad \text{for } n \geq 2, \\ 2y\psi_{2n} &= \psi_n(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2), \quad \text{for } n \geq 3.\end{aligned}\tag{9}$$

It can be checked easily by induction that the ψ_{2n} 's are polynomials. Moreover, $\psi_n \in \mathbb{Z}[x, y^2, a, b]$ when n is odd, and $(2y)^{-1}\psi_n \in \mathbb{Z}[x, y^2, a, b]$ when n is even. Define the polynomial

$$\phi_n = x\psi_n^2 - \psi_{n-1}\psi_{n+1} \tag{10}$$

for $n \geq 1$. Then $\phi_n \in \mathbb{Z}[x, y^2, a, b]$. Since $y^2 = x^3 + ax + b$, replacing y^2 by $x^3 + ax + b$, one has $\phi_n \in \mathbb{Z}[x, a, b]$. So we can denote it by $\phi_n(x)$. Note that $\psi_n\psi_m \in \mathbb{Z}[x, a, b]$ if n and m have the same parity. Furthermore, the division polynomials ψ_n have the following properties.

Lemma 3. Consider

$$\begin{aligned}\psi_n &= nx^{(n^2-1)/2} + \frac{n(n^2-1)(n^2+6)}{60}ax^{(n^2-5)/2} \\ &\quad + \text{lower degree terms},\end{aligned}\tag{11}$$

when n is odd and

$$\begin{aligned}\psi_n &= ny \left(x^{(n^2-4)/2} + \frac{(n^2-1)(n^2+6)-30}{60}ax^{(n^2-8)/2} \right. \\ &\quad \left. + \text{lower degree terms} \right),\end{aligned}\tag{12}$$

when n is even.

Proof. We prove the result by induction on n . It is true for $n < 5$. Assume that it holds for all ψ_m with $m < n$. We give the proof only for the case for odd $n \geq 5$. The case for even n can

be proved similarly. Now let $n = 2k + 1$ be odd, where $k \geq 2$. If k is even, then by induction

$$\begin{aligned}\psi_k &= ky \left(x^{(k^2-4)/2} + \frac{(k^2-1)(k^2+6)-30}{60} \right. \\ &\quad \left. \times ax^{(k^2-8)/2} + \dots \right), \\ \psi_{k+2} &= (k+2)y \\ &\quad \times \left(x^{(k^2+4k)/2} + \frac{(k^2+4k+3)(k^2+4k+10)-30}{60} \right. \\ &\quad \left. \times ax^{(k^2+4k-4)/2} + \dots \right), \\ \psi_{k-1} &= (k-1)x^{(k^2-2k)/2} \\ &\quad + \frac{(k-1)(k^2-2k)(k^2-2k+7)}{60} \\ &\quad \times ax^{(k^2-2k-4)/2} + \dots, \\ \psi_{k+1} &= (k+1)x^{(k^2+2k)/2} \\ &\quad + \frac{(k+1)(k^2+2k)(k^2+2k+7)}{60} \\ &\quad \times ax^{(k^2+2k-4)/2} + \dots.\end{aligned}\tag{13}$$

Substituting y^4 by $(x^3 + ax + b)^2$, we have

$$\begin{aligned}\psi_{k+2}\psi_k^3 &= k^3(k+2) \\ &\quad \times \left(x^{2k^2+2k} + \frac{4(k+1)(k^3+k^2+10k+3)}{60} \right. \\ &\quad \left. \times ax^{2k^2+2k-2} + \dots \right), \\ \psi_{k-1}\psi_{k+1}^3 &= (k-1)(k+1)^3x^{2k^2+2k} \\ &\quad + \frac{4k(k-1)(k^3+2k^2+11k+7)(k+1)^3}{60} \\ &\quad \times ax^{2k^2+2k-2} + \dots.\end{aligned}\tag{14}$$

Therefore,

$$\begin{aligned}
 \psi_{2k+1} &= \psi_{k+2}\psi_k^3 - \psi_{k-1}\psi_{k+1}^3 \\
 &= (2k+1)x^{2k^2+2k} \\
 &\quad + \frac{(2k+1)(4k^2+4k)(4k^2+4k+7)}{60} \\
 &\quad \times ax^{2k^2+2k-2} + \dots \\
 &= (2k+1)x^{((2k+1)^2-1)/2} \\
 &\quad + \frac{(2k+1)((2k+1)^2-1)((2k+1)^2+6)}{60} \\
 &\quad \times ax^{((2k+1)^2-5)/2} + \dots
 \end{aligned} \tag{15}$$

The case when k is odd can be proved similarly. $\square$

The following corollary follows immediately from Lemma 3.

Corollary 4. Consider

$$\begin{aligned}
 \psi_n^2 &= n^2 x^{n^2-1} - \frac{n^2(n^2-1)(n^2+6)}{30} ax^{n^2-3} + \dots, \\
 \phi_n &= x^{n^2} - \frac{n^2(n^2-1)}{6} ax^{n^2-2} + \dots
 \end{aligned} \tag{16}$$

Proof of Theorem 1. Define ω_n as

$$4y\omega_n = \psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2. \tag{17}$$

Then for any $P = (x_P, y_P) \in E$, we have ([1])

$$nP = \left(\frac{\phi_n(x_P)}{\psi_n^2(x_P)}, \frac{\omega_n(x_P, y_P)}{\psi_n(x_P, y_P)^3} \right). \tag{18}$$

If $nP = Q$, then $\phi_n(x_P) - x_Q\psi_n^2(x_P) = 0$. Therefore, for any $P \in \Lambda$, the x -coordinate of P satisfies the equation $\phi_n(x) - x_Q\psi_n^2(x) = 0$. From Corollary 4, we have that

$$\begin{aligned}
 \phi_n(x) - x_Q\psi_n^2(x) \\
 = x^{n^2} - n^2 x_Q x^{n^2-1} + \text{lower degree terms.}
 \end{aligned} \tag{19}$$

Since $\#\Lambda = n^2$, every root of $\phi_n(x) - x_Q\psi_n^2(x)$ is the x -coordinate of some $P \in \Lambda$. Therefore,

$$\sum_{P \in \Lambda} x_P = n^2 x_Q \tag{20}$$

by Vitae's theorem.

Now we prove the mean value formula for y -coordinates. Let K be the complex number field $\mathbb{C}$ first and let ω_1 and ω_2 be complex numbers which are linearly independent over $\mathbb{R}$. Define the lattice

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\} \tag{21}$$

and the Weierstrass $\wp$ -function by

$$\wp(z) = \wp(z, L) = \frac{1}{z^2} + \sum_{\omega \in L, \omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{22}$$

For integers $k \geq 3$, define the Eisenstein series G_k by

$$G_k = G_k(L) = \sum_{\omega \in L, \omega \neq 0} \omega^{-k}. \tag{23}$$

Set $g_2 = 60G_4$ and $g_3 = 140G_6$; then

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{24}$$

Let E be the elliptic curve given by $y^2 = 4x^3 - g_2x - g_3$. Then the map

$$\begin{aligned}
 \frac{\mathbb{C}}{L} &\longrightarrow E(\mathbb{C}) \\
 z &\longmapsto (\wp(z), \wp'(z)), \\
 0 &\longmapsto \infty
 \end{aligned} \tag{25}$$

is an isomorphism of groups $\mathbb{C}/L$ and $E(\mathbb{C})$. Conversely, it is well known [1] that, for any elliptic curve E over $\mathbb{C}$ defined by $y^2 = x^3 + ax + b$, there is a lattice L such that $g_2(L) = -4a, g_3(L) = -4b$ and there is an isomorphism between groups $\mathbb{C}/L$ and $E(\mathbb{C})$ given by $z \mapsto (\wp(z), (1/2)\wp'(z))$ and $0 \mapsto \infty$. Therefore, for any point $(x, y) \in E(\mathbb{C})$, we have $(x, y) = (\wp(z), (1/2)\wp'(z))$ and $n(x, y) = (\wp(nz), (1/2)\wp'(nz))$ for some $z \in \mathbb{C}$.

Let $Q = (\wp(z_Q), (1/2)\wp'(z_Q))$ for a $z_Q \in \mathbb{C}$. Then for any $P_i \in \Lambda, 1 \leq i \leq n^2$, there exist integers j, k with $0 \leq j, k \leq n-1$, such that

$$P_i = \left(\wp\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right), \frac{1}{2}\wp'\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right) \right). \tag{26}$$

Thus,

$$\sum_{j,k=0}^{n-1} \wp\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right) = n^2 \wp(z_Q) \tag{27}$$

which comes from $\sum_{i=1}^{n^2} x_i = n^2 x_Q$. Differentiate with respect to z_Q , we have

$$\sum_{j,k=0}^{n-1} \wp'\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right) = n^3 \wp'(z_Q). \tag{28}$$

That is,

$$\sum_{i=1}^{n^2} y_i = n^3 y_Q. \tag{29}$$

Secondly, let K be a field of characteristic 0 and let E be the elliptic curve over K given by the equation $y^2 = x^3 + ax + b$. Then all of the equations describing the group law are defined

over $\mathbb{Q}(a, b)$. Since $\mathbb{C}$ is algebraically closed and has infinite transcendence degree over $\mathbb{Q}$, $\mathbb{Q}(a, b)$ can be considered as a subfield of $\mathbb{C}$. Therefore we can regard E as an elliptic curve defined over $\mathbb{C}$. Thus the result follows.

At last assume that K is a field of characteristic p . Then the elliptic curve can be viewed as one defined over some finite field $\mathbb{F}_q$, where $q = p^m$ for some integer m . Without loss of generality, let $K = \mathbb{F}_q$ for convenience. Let $K' = \mathbb{Q}_q$ be an unramified extension of the p -adic numbers $\mathbb{Q}_p$ of degree m , and let $\bar{E}$ be an elliptic curve over K' which is a lift of E . Since $(n, p) = 1$, the natural reduction map $\bar{E}[n] \rightarrow E[n]$ is an isomorphism. Now for any point $Q \in E$ with $Q \neq \mathcal{O}$, we have a point $\bar{Q} \in \bar{E}$ such that the reduction point is Q . For any point $P_i \in E(\bar{K})$ with $nP_i = Q$, its lifted point $\bar{P}_i$ satisfies $n\bar{P}_i = \bar{Q}$ and $\bar{P}_i \neq \bar{P}_j$ whenever $P_i \neq P_j$. Thus,

$$\sum_{i=1}^{n^2} y(\bar{P}_i) = n^3 y(\bar{Q}), \quad (30)$$

since K' is a field of characteristic 0. Therefore the formula $\sum_{i=1}^{n^2} y_i = n^3 y_Q$ holds by the reduction from $\bar{E}$ to E . $\square$

Remark 5.

- (1) The result for x -coordinate of Theorem 1 holds also for the elliptic curve defined by the general Weierstrass equation $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$.
- (2) The mean value formula for x -coordinates was given in the first version of this paper [2] with a slightly complicated proof. The formula for y -coordinates was conjectured by Feng and Wu based on [2] and numerical examples in a personal email communication with Moody (June 1, 2010).
- (3) Recently, some mean value formulae for twisted Edwards curves [3, 4] and other alternate models of elliptic curves were given by [5, 6].

3. An Application

Let E be an elliptic curve over K given by the Weierstrass equation $y^2 = x^3 + ax + b$. Then we have a nonzero invariant differential $\omega = dx/y$. Let $\phi \in \text{End}(E)$ be a nonzero endomorphism. Then $\phi^* \omega = \omega \circ \phi = c_\phi \omega$ for some $c_\phi \in \bar{K}(E)$, since the space Ω_E of differential forms on E is a 1-dimensional $\bar{K}(E)$ -vector space. Since $c_\phi \neq 0$ and $\text{div}(\omega) = 0$, we have

$$\text{div}(c_\phi) = \text{div}(\phi^* \omega) - \text{div}(\omega) = \phi^* \text{div}(\omega) - \text{div}(\omega) = 0. \quad (31)$$

Hence c_ϕ has neither zeros nor poles and $c_\phi \in \bar{K}$. Let φ and ψ be two nonzero endomorphisms; then

$$\begin{aligned} c_{\varphi+\psi} \omega &= (\varphi + \psi)^* \omega = \varphi^* \omega + \psi^* \omega = c_\varphi \omega + c_\psi \omega \\ &= (c_\varphi + c_\psi) \omega. \end{aligned} \quad (32)$$

Therefore, $c_{\varphi+\psi} = c_\varphi + c_\psi$. For any nonzero endomorphism ϕ , set $\phi(x, y) = (R_\phi(x), yS_\phi(x))$, where R_ϕ and S_ϕ are rational functions. Then

$$c_\phi = \frac{R'_\phi(x)}{S_\phi(x)}, \quad (33)$$

where $R'_\phi(x)$ is the differential of $R_\phi(x)$. In particular, for any positive integer n , the map $[n]$ on E is an endomorphism. Set $[n](x, y) = (R_n(x), yS_n(x))$. From $c_{[1]} = 1$ and $[n] = [1] + [(n-1)]$, we have

$$c_{[n]} = \frac{R'_n(x)}{S_n(x)} = n. \quad (34)$$

For any $Q = (x_Q, y_Q) \in E$ and any

$$P = (x_P, y_P) \in \Lambda = \{P = (x_P, y_P) \in E(\bar{K}) \mid nP = Q\}, \quad (35)$$

we have $y_P = y_Q/S_n(x_P)$. Therefore, Theorem 1 gives

$$\sum_{P \in \Lambda} \frac{1}{S_n(x_P)} = \sum_{P \in \Lambda} \frac{y_P}{y_Q} = \frac{1}{y_Q} \sum_{P \in \Lambda} y_P = n^3. \quad (36)$$

Thus,

$$\begin{aligned} \sum_{P \in \Lambda} \frac{1}{R'_n(x_P)} &= \sum_{P \in \Lambda} \frac{1}{n \cdot S_n(x_P)} = \frac{1}{n} \sum_{P \in \Lambda} \frac{1}{S_n(x_P)} = n^2, \\ \sum_{P \in \Lambda} \frac{x_Q}{R'_n(x_P)} &= x_Q \sum_{P \in \Lambda} \frac{1}{R'_n(x_P)} = n^2 x_Q = \sum_{P \in \Lambda} x_P. \end{aligned} \quad (37)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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