Hindawi Publishing Corporation
International Journal of Analysis
Volume 2014, Article ID 301709, 9 pages
http://dx.doi.org/10.1155/2014/301709

Research Article

Weighted Fractional Differentiation Composition Operators from Mixed-Norm Spaces to Weighted-Type Spaces

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Received 3 December 2013; Accepted 10 January 2014; Published 27 February 2014

Academic Editor: Frédéric Robert

Let $\mathbb{D}$ be an open unit disc in the complex plane $\mathbb{C}$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ as well as $u : \mathbb{D} \to \mathbb{C}$ be analytic maps. For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ the weighted fractional differentiation composition operator is defined as $(D_{\varphi,u}^{\beta} f)(z) = u(z) f^{[\beta]}(\varphi(z))$, where $\beta \geq 0$, $f^{[\beta]}(z) = \sum_{n=0}^{\infty} (\Gamma(n+1+\beta)/\Gamma(n+1)) a_n z^n$, and $f^{[0]}(z) = f(z)$. In this paper, we obtain a characterization of boundedness and compactness of weighted fractional differentiation composition operator from mixed-norm space $H(p,q,\phi)$ to weighted-type space H_{μ}^{∞}.

1. Introduction

The classical/Gaussian hypergeometric series is defined by the power series expansion

$$\begin{aligned} {}_2F_1(a,b;c;z) &\equiv F(a,b;c;z) \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (|z| < 1). \end{aligned} \tag{1}$$

Here a, b, c are complex numbers such that $c \neq -m$, $m = 0, 1, 2, 3, \ldots$, and $(a)_n$ is Pochhammer's symbol/shifted factorial defined by Appel's symbol

$$(a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{N}, \tag{2}$$

and $(a)_0 = 1$ for $a \neq 0$.

Obviously, $F(a,b;c;z) \in H(\mathbb{D})$. Many properties of the hypergeometric series including the Gauss and Euler transformations are found in standard textbooks such as [1, 2].

For any two analytic functions f and g represented by their power series expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n \tag{3}$$

in $|z| < R$, the Hadamard product (or convolution) of f and g denoted by $f * g$ and is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \tag{4}$$

in $|z| < R^2$. Moreover,

$$(f * g)(z) = \frac{1}{2\pi} \int_{|w|=\rho} f(w)\, g\left(\frac{z}{w}\right) \frac{dw}{w}, \qquad |z| < \rho R < R^2. \tag{5}$$

In particular, if $f, g \in H(\mathbb{D})$, we have

$$(f * g)(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(\rho e^{it}\right) g\left(z e^{-it}\right) dt, \qquad 0 < \rho < 1. \tag{6}$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ and $\beta > 0$, then the fractional derivative $f^{[\beta]}$ (see [3]) of order β is defined by

$$f^{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\beta)}{\Gamma(n+1)} a_n z^n. \tag{7}$$

In terms of convolution, we also have

$$f^{[\beta]}(z) = \Gamma(1+\beta)\left[f(z) * F(1, 1+\beta; 1; z)\right]. \quad (8)$$

For $\beta = 0$, we define

$$f^{[0]}(z) = f(z). \quad (9)$$

It is obvious to find that the fractional derivative and the ordinary derivative satisfy

$$f^{[k]}(z) = \frac{d^k}{dz^k}\left(z^k f(z)\right), \quad k = 0, 1, 2, \ldots. \quad (10)$$

Let $u(z) \in H(\mathbb{D})$ be fixed and let $\varphi(z)$ be nonconstant analytic self-map of $\mathbb{D}$. For $\beta \geq 0$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we can define an operator $D^{\beta}_{\varphi,u}$ on $H(\mathbb{D})$, called a weighted fractional differentiation composition operator, by

$$D^{\beta}_{\varphi,u}(f(z)) = u(z) f^{[\beta]}(\varphi(z)). \quad (11)$$

We can regard this operator as a generalization of a multiplication operator and a weighted composition operator. In this paper we study the boundedness and the compactness of weighted fractional differentiation composition operator from mixed-norm spaces to weighted-type spaces. Recall that a positive continuous function ϕ on $[0, 1)$ is called normal if there is $\delta \in [0, 1)$ and s and t with $0 < s < t$ such that

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [\delta, 1), \qquad \lim_{r\to 1}\frac{\phi(r)}{(1-r)^s} = 0,$$
$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [\delta, 1), \qquad \lim_{r\to 1}\frac{\phi(r)}{(1-r)^t} = \infty. \quad (12)$$

Let $dm(z) = (1/\pi) r\, dr\, d\theta$ be the normalized Lebesgue area measure on $\mathbb{D}$ and let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$.

For $0 < p, q \leq \infty$, and ϕ is normal we denote by $H(p, q, \phi)$ the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f, r)\frac{\phi^p(r)}{1-r}\, dr\right)^{1/p} < \infty, \quad (13)$$

where $p \in (0, \infty)$ and

$$\|f\|_{H(\infty,q,\phi)} = \sup_{0<r<1} \phi(r) M_q(f, r) < \infty, \quad (14)$$

where

$$M_q(f, r) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^q d\theta\right)^{1/q} \quad \text{if } q \in (0, \infty),$$
$$M_\infty(f, r) = \sup_{\theta\in[0,2\pi]}\left|f\left(re^{i\theta}\right)\right|. \quad (15)$$

When $p = q$ and $\phi(r) = (1-r)^{(\alpha+1)/p}$, $\alpha > -1$, then $H(p, q, \phi) \equiv A^p_\alpha(\mathbb{D})$ (classical weighted Bergman space), defined by

$$A^p_\alpha = \left\{f \in H(\mathbb{D}) : \|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha dm(z) < \infty\right\}. \quad (16)$$

Further, when $\alpha \to -1+0$ then the natural limit to the weighted Bergman space is the Hardy space H^p; that is, $A^p_{-1} \equiv H^p$. For more details on Hardy space, see [4].

Suppose $\mu : \mathbb{D} \to \mathbb{C}$ is normal and radial; that is, $\mu(z) = \mu(|z|)$. The weighted-type space H^∞_μ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{D}} \mu(z)|f(z)| < \infty. \quad (17)$$

A little version of H^∞_μ is denoted by $H^\infty_{\mu,0}$ as the subset of H^∞_μ consisting of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0. \quad (18)$$

Note that $H^\infty_{\mu,0}$ is a closed subspace of H^∞_μ. For the space H^∞_μ, see [5].

For $\mu \equiv 1$, we have the space of bounded analytic functions H^∞, where

$$\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty. \quad (19)$$

For $\beta = 0$, $D^{\beta}_{\varphi,u}$ equals the weighted composition operator defined by $(uC_\varphi)(f)(z) = u(z)f(\varphi(z))$, $z \in \mathbb{D}$, which reduces to the composition operator C_φ for $u(z) \equiv 1$. During the last century, composition operators were studied between different spaces of analytic functions. There are many papers about these operators. Among other things, they deal with boundedness and compactness of these operators. But we will discuss mostly the mixed-norm spaces. For some recent papers on weighted composition operators on some H^∞-type spaces, see [5–14].

If $\beta = 1$, we get the operator $D^1_{\varphi,u} = M_u C_\varphi + M_{u\varphi} C_\varphi D$ which for $u(z) \equiv 1$ gives $D^1_{\varphi,1} = C_\varphi + \varphi C_\varphi D$ which for $\varphi(z) = z$ gives $D^1_{z,u} = M_u + zM_u D$ and $u(z) = \varphi'(z)$ gives $D^1_{\varphi,\varphi'} = M_{\varphi'} C_\varphi + \varphi D C_\varphi$. For particular choices of β, u, and φ, we obtain many operators which are product, addition, and composition of multiplication and differentiation operators. For more details of these types of operators see [15–21].

Recall that, in [22], Stević characterized the boundedness and compactness of weighted differentiation composition operators from mixed-norm spaces to the weighted-type spaces. Our results can be viewed as generalizations of their results.

Throughout this paper, C denotes a positive constant which may vary for different lines. The notation $A \asymp B$ means there is a positive constant C such that $B/C \leq A \leq CB$.

2. Some Lemmas

We collect some basic lemmas which are useful in the proof of the main results.

Lemma 1 (see [23, Lemma 6]). *For $\gamma > -1$ and $m > 1+\gamma$ one has*

$$\int_0^1 (1-\rho r)^{-m}(1-r)^{\gamma} \le c(1-\rho)^{1+\gamma-m}, \quad 0<\rho<1. \tag{20}$$

Lemma 2 (see [24, Proposition 1.4.10]). *For $\gamma > 1$ one has*

$$\int_0^{2\pi} \frac{1}{|1-z|^{\gamma}} \le C\frac{1}{(1-|z|)^{\gamma-1}}. \tag{21}$$

Lemma 3 (see [22, Lemma 1]). *Assume $0 < p, q \le \infty$, ϕ is normal, and $f \in H(p,q,\phi)$. Then, for every $n \in N_0$, there is a positive constant C independent of f such that*

$$\left|f^{(n)}(z)\right| \le C\frac{\|f\|_{H(p,q,\phi)}}{\phi(|z|)\left(1-|z|^2\right)^{(1/q)+n}}, \quad z \in \mathbb{D}. \tag{22}$$

3. Main Results

In the following results we show boundedness of the operator $D^{\beta}_{\varphi,u}$ from mixed-norm space $H(p,q,\phi)$ to weighted-type space H^{∞}_{μ}. As a consequence we obtain many new results in the form of corollaries.

Lemma 3 is true for the nth order derivative of $f \in H(p,q,\phi)$. Here we will give a more general result which involves fractional derivative $f^{[\beta]}$ of $f \in H(p,q,\phi)$.

Proposition 4. *Assume $0 < p \le \infty$, $1 \le q \le \infty$, ϕ is normal, and $f \in H(p,q,\phi)$. Then, for every $\beta \ge 0$, there is a positive constant C independent of f such that*

$$\left|f^{[\beta]}(z)\right| \le C\frac{\|f\|_{H(p,q,\phi)}}{\left(1-|z|^2\right)^{(1/q)+\beta}\phi(|z|)} \quad \forall z \in \mathbb{D}. \tag{23}$$

Theorem 5. *For $\beta \ge 0$, $0 < p \le \infty$, $1 \le q \le \infty$, $u \in H(\mathbb{D})$, ϕ and μ are normal, and φ is an analytic self-map of $\mathbb{D}$. Then $D^{\beta}_{\varphi,u} : H(p,q,\phi) \to H^{\infty}_{\mu}$ is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+\beta}} < \infty. \tag{24}$$

Moreover, if $D^{\beta}_{\varphi,u} : H(p,q,\phi) \to H^{\infty}_{\mu}$ is bounded, then the following asymptotic relation holds:

$$\left\|D^{\beta}_{\varphi,u}\right\|_{H(p,q,\phi)\to H^{\infty}_{\mu}} \asymp M. \tag{25}$$

Proof of Proposition 4. Let $\beta > 0$.

Using integral representation of convolution we have, for $1 \le q < \infty$,

$$\begin{aligned} M_q^q\left(f^{[\beta]},\rho r\right) &= \frac{1}{2\pi}\int_0^{2\pi}\left|f^{[\beta]}\left(\rho re^{i\theta}\right)\right|^q d\theta \\ &= \frac{\Gamma(1+\beta)^q}{2\pi} \\ &\quad\times \int_0^{2\pi}\left|\left(f * F(1,1+\beta;1)\right)\left(\rho re^{i\theta}\right)\right|^q d\theta, \\ M_q\left(f^{[\beta]},\rho r\right) &= \frac{\Gamma(1+\beta)}{2\pi^{1/q}} \\ &\quad\times\left[\int_0^{2\pi}\left|\frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}e^{-it}\right)\right.\right. \\ &\qquad\left.\left.\times F\left(1,1+\beta;1;\rho e^{it}\right)dt\right|^q d\theta\right]^{1/q}. \end{aligned} \tag{26}$$

Since $1 \le q < \infty$, Minkowski's inequality gives

$$\begin{aligned} M_q\left(f^{[\beta]},\rho r\right) &\le \frac{\Gamma(1+\beta)}{2\pi^{1/q}} \\ &\quad\times\left[\int_0^{2\pi}\frac{1}{2\pi}\left(\int_0^{2\pi}\left|f\left(re^{i(\theta-t)}\right)\right.\right.\right. \\ &\qquad\left.\left.\left.\times F\left(1,1+\beta;1;\rho e^{it}\right)\right|^q d\theta\right)^{1/q} dt\right] \\ &= C\left[\frac{1}{2\pi}\left(\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^q d\theta\right)^{1/q}\right. \\ &\qquad\left.\times\int_0^{2\pi}\left|F\left(1,1+\beta;1;\rho e^{it}\right)\right| dt\right] \\ &= CM_q(f,r)\,M_1(F,\rho), \end{aligned} \tag{27}$$

where $F(z) \equiv F(1,1+\beta;1;z)$.

For $q = \infty$, we have

$$\begin{aligned} M_{\infty}\left(f^{[\beta]},\rho r\right) &= \sup_{\theta\in[0,2\pi]}\left|f^{[\beta]}\left(\rho re^{i\theta}\right)\right| \\ &= \sup_{\theta\in[0,2\pi]}\left|\left(f * F(1,1+\beta;1)\right)\left(\rho re^{i\theta}\right)\right| \end{aligned}$$

$$= \sup_{\theta\in[0,2\pi]} \left| \frac{1}{2\pi} \int_0^{2\pi} f\left(re^{i\theta}e^{-it}\right) \times F\left(1, 1+\beta; 1; \rho e^{it}\right) dt \right| \le M_\infty(f,r) M_1(F,\rho). \tag{28}$$

Putting $\rho = r$, we have, for any $q \in [1,\infty]$,

$$M_q\left(f^{[\beta]}, r^2\right) \le C M_q(f,r) M_1(F,r). \tag{29}$$

Since

$$\begin{aligned} M_1(F,r) &= \frac{1}{2\pi}\int_0^{2\pi} \left|F\left(1,1+\beta;1;re^{it}\right)\right| dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\left|1-re^{it}\right|^{\beta+1}} dt \\ &\le \frac{C}{2\pi(1-r)^\beta} \quad (\text{by Lemma } 2), \end{aligned} \tag{30}$$

therefore

$$\begin{aligned} M_q\left(f^{[\beta]}, r^2\right) &\le C M_q(f,r) \frac{1}{(1-r)^\beta} \\ (1-r)^\beta M_q\left(f^{[\beta]}, r^2\right) &\le C M_q(f,r). \end{aligned} \tag{31}$$

For $0 < p < \infty$

$$\begin{aligned} \|f\|^p_{H(p,q,\phi)} &= \int_0^1 M_q^p(f,r) \frac{\phi^p(r)}{1-r} dr \\ &\ge \int_0^1 M_q^p\left(f^{[\beta]}, r^2\right)(1-r)^{p\beta} \times \frac{\phi^p(r)}{1-r} dr \quad (\text{by } (31)) \\ &= \int_0^1 M_q^p\left(f^{[\beta]}, r^2\right) \frac{\left(1-r^2\right)^{p\beta}}{(1+r)^{p\beta}} \frac{\phi^p(r)}{1-r} dr \\ &\ge C\int_0^1 M_q^p\left(f^{[\beta]}, r^2\right)\left(1-r^2\right)^{p\beta} \frac{\phi^p(r)}{1-r} dr. \end{aligned} \tag{32}$$

Putting $r^2 = u$, we have

$$\begin{aligned} \|f\|^p_{H(p,q,\phi)} &\ge C\int_0^1 M_q^p\left(f^{[\beta]}, u\right)(1-u)^{p\beta} \times \frac{\phi^p(\sqrt{u})}{1-\sqrt{u}} \cdot \frac{1}{\sqrt{u}} du \\ &\ge C\int_{(1+|z|)/2}^{(3+|z|)/4} M_q^p\left(f^{[\beta]}, u\right)(1-u)^{p\beta} \times \frac{\phi^p(\sqrt{u})}{1-\sqrt{u}} \cdot \frac{1}{\sqrt{u}} du \\ &\ge C M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right) \times \int_{(1+|z|)/2}^{(3+|z|)/4} (1-u)^{p\beta} \frac{\phi^p(\sqrt{u})}{1-\sqrt{u}} \cdot \frac{1}{\sqrt{u}} du \\ &\ge C M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right) \times \int_{(1+|z|)/2}^{(3+|z|)/4} (1-u)^{p\beta} \frac{\phi^p(\sqrt{u})}{1-\sqrt{u}} du \\ &= C M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right) \times \int_{(1+|z|)/2}^{(3+|z|)/4} (1-u)^{p\beta} \frac{\phi^p(\sqrt{u})}{1-u} \left(1+\sqrt{u}\right) du \\ &\ge C M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right) \times \int_{(1+|z|)/2}^{(3+|z|)/4} (1-u)^{p\beta} \frac{\phi^p(\sqrt{u})}{1-u} du. \end{aligned} \tag{33}$$

By using the following asymptotic relations:

$$\begin{aligned} \phi(|z|) &\asymp \phi(|w|), \quad w \in B\left(z, \sqrt{\frac{3+|z|}{4}} - |z|\right), \\ 1-r &\asymp 1-|z|, \quad r \in \left[\frac{1+|z|}{2}, \frac{3+|z|}{4}\right], \end{aligned} \tag{34}$$

we have

$$\begin{aligned} \|f\|^p_{H(p,q,\phi)} &\ge C M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right) \times \int_{(1+|z|)/2}^{(3+|z|)/4} (1-|z|)^{p\beta} \frac{\phi^p(|z|)}{1-|z|} du \\ &\ge C M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right)(1-|z|)^{p\beta-1}\phi^p(|z|) \times \int_{(1+|z|)/2}^{(3+|z|)/4} du \\ &= C M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right)(1-|z|)^{p\beta}\phi^p(|z|). \end{aligned} \tag{35}$$

A simple calculation using Cauchy's integral formula gives

$$M_q^p\left(f^{[\beta]}, \frac{1+|z|}{2}\right) \ge C\left|f^{[\beta]}(z)\right|^p (1-|z|)^{p/q}. \tag{36}$$

The above inequality can be written as the following:

$$\begin{aligned} \|f\|^p_{H(p,q,\phi)} &\ge C\left|f^{[\beta]}(z)\right|^p (1-|z|)^{p/q}(1-|z|)^{p\beta}\phi^p(|z|), \\ \|f\|_{H(p,q,\phi)} &\ge C\left|f^{[\beta]}(z)\right| (1-|z|)^{1/q}(1-|z|)^{\beta}\phi(|z|), \end{aligned}$$

$$\left|f^{[\beta]}(z)\right| \le C\frac{\|f\|_{H(p,q,\phi)}}{(1-|z|)^{\beta+(1/q)}\phi(|z|)},$$
$$\left|f^{[\beta]}(z)\right| \le C\frac{\|f\|_{H(p,q,\phi)}}{\left(1-|z|^2\right)^{\beta+(1/q)}\phi(|z|)}. \tag{37}$$

When $p=\infty$, from inequality (31) we have

$$\begin{aligned}\|f\|_{H(\infty,q,\phi)} &= \sup_{0<r<1}\phi(r)M_q(f,r)\\ &\ge C\sup_{0<r<1}\phi(r)(1-r)^{\beta}M_q\left(f^{[\beta]},r^2\right)\\ &\ge CM_q\left(f^{[\beta]},\frac{1+|z|}{2}\right)\\ &\quad\times\phi(|z|)(1-|z|)^{\beta}\quad(\text{where, } 0<|z|<1)\\ &\ge C\left|f^{[\beta]}(z)\right|(1-|z|)^{1/q}\phi(|z|)(1-|z|)^{\beta}\\ \left|f^{[\beta]}(z)\right| &\le C\frac{\|f\|_{H(\infty,q,\phi)}}{(1-|z|)^{(1/q)+\beta}\phi(|z|)}\\ \left|f^{[\beta]}(z)\right| &\le C\frac{\|f\|_{H(\infty,q,\phi)}}{\left(1-|z|^2\right)^{(1/q)+\beta}\phi(|z|)}.\end{aligned} \tag{38}$$

For $\beta=0$ we have $f^{[\beta]}=f$ and so $M_q(f^{[\beta]})=M_q(f)$. Therefore the proof is similar to the case, $\beta>0$. □

Proof of Theorem 5. Suppose (24) holds. We have

$$\begin{aligned}\left\|\left(D^{\beta}_{\varphi,u}f\right)(z)\right\|_{H^{\infty}_{\mu}} &= \sup_{z\in\mathbb{D}}\mu(z)\left|\left(D^{\beta}_{\varphi,u}f\right)(z)\right|\\ &= \sup_{z\in\mathbb{D}}\mu(z)|u(z)|\left|f^{[\beta]}(\varphi(z))\right|\\ &\le C\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)|}{\phi(\varphi(z))\left(1-|\varphi(z)|^2\right)^{(1/q)+\beta}}\\ &\quad\times\|f\|_{H(p,q,\phi)}\quad(\text{By Proposition 4})\\ &\le CM\|f\|_{H(p,q,\phi)}.\end{aligned} \tag{39}$$

Therefore,

$$\left\|D^{\beta}_{\varphi,u}\right\|_{H(p,q,\phi)\to H^{\infty}_{\mu}} \le CM. \tag{40}$$

Conversely, assume that $D^{\beta}_{\varphi,u}: H(p,q,\phi)\to H^{\infty}_{\mu}$ is bounded. For a fixed $w\in\mathbb{D}$, let

$$f_w(z)=\frac{\left(1-|w|^2\right)^{t+1}F((1/q)+\beta+t+1,1;1+\beta;\overline{w}z)}{\phi(|w|)}, \tag{41}$$

where the constant t is from the definition of the normality of the function ϕ:

$$\begin{aligned}M^q_q(f_w,r) &= \frac{1}{2\pi}\int_0^{2\pi}\left|f_w\left(re^{i\theta}\right)\right|^q d\theta\\ &= \frac{\left(1-|w|^2\right)^{(t+1)q}}{\phi^q(|w|)}\cdot\frac{1}{2\pi}\\ &\quad\times\int_0^{2\pi}\left|F\left(\frac{1}{q}+\beta+t+1,1;1+\beta;\overline{w}re^{i\theta}\right)\right|^q d\theta.\end{aligned} \tag{42}$$

In view of the well-known Gauss identity [1], of hypergeometric function

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z), \tag{43}$$

we can rewrite the previous equation to obtain the following:

$$\begin{aligned}M^q_q(f_w,r) &= \frac{\left(1-|w|^2\right)^{(t+1)q}}{\phi^q(|w|)}\cdot\frac{1}{2\pi}\\ &\quad\times\int_0^{2\pi}\left|\left(1-\overline{w}re^{i\theta}\right)^{((-1/q)-t-1)}\right.\\ &\qquad\left.\times F\left(-\frac{1}{q}-t,\beta;1+\beta;\overline{w}re^{i\theta}\right)\right|^q d\theta.\end{aligned} \tag{44}$$

We know the hypergeometric function $F(a,b;c;z)$ is bounded if $c-a-b>0$ on $|z|\le 1$ (see [1]). The hypergeometric function $F((-1/q)-t,\beta;1+\beta;\overline{w}re^{i\theta})$ is bounded on $|z|\le 1$ since $1+\beta-((-1/q)-t)-\beta=1+(1/q)+t>0$; therefore, using Lemma 2 gives

$$\begin{aligned}M^q_q(f_w,r) &\le C\frac{\left(1-|w|^2\right)^{(t+1)q}}{\phi^q(|w|)}\\ &\quad\cdot\frac{1}{2\pi}\int_0^{2\pi}\left|\left(1-\overline{w}re^{i\theta}\right)^{((-1/q)-t-1)}\right|^q d\theta\\ &\le C\frac{\left(1-|w|^2\right)^{(t+1)q}}{\phi^q(|w|)}\cdot(1-|w|r)^{(-tq-q)},\\ M_q(f_w,r) &\le C\frac{\left(1-|w|^2\right)^{t+1}}{\phi(|w|)}\cdot\frac{1}{(1-|w|r)^{t+1}}.\end{aligned} \tag{45}$$

Since ϕ is normal and using Lemma 1, we get $\sup_{w\in\mathbb{D}}\|f_w\|_{H(p,q,\phi)} < \infty$. Now

$$\begin{aligned} f_w^{[\beta]}(z) &= \Gamma(1+\beta)\left[f_w * F(1,1+\beta;1;z)\right] \\ &= \Gamma(1+\beta)\left[\frac{\left(1-|w|^2\right)^{(t+1)}}{\phi(|w|)} \times F\left(\frac{1}{q}+\beta+t+1,1;1+\beta;\overline{w}z\right) * F(1,1+\beta;1;z)\right] \\ &= \Gamma(1+\beta)\frac{\left(1-|w|^2\right)^{(t+1)}}{\phi(|w|)} \times \left[\sum_{n=0}^{\infty}\frac{((1/q)+\beta+t+1)_n(1)_n}{(1+\beta)_n(1)_n}(\overline{w}z)^n * \sum_{n=0}^{\infty}\frac{(1+\beta)_n}{(1)_n}z^n\right] \\ &= \frac{\Gamma(1+\beta)\left(1-|w|^2\right)^{(t+1)}}{\phi(|w|)} \times \left[\sum_{n=0}^{\infty}\frac{((1/q)+\beta+t+1)_n}{(1)_n}(\overline{w}z)^n\right] \\ &= \frac{\Gamma(1+\beta)\left(1-|w|^2\right)^{(t+1)}}{\phi(|w|)(1-\overline{w}z)^{((1/q)+\beta+t+1)}}. \end{aligned} \tag{46}$$

For every $\lambda \in \mathbb{D}$, we have

$$\begin{aligned} C\left\|D_{\varphi,u}^{\beta}\right\|_{H(p,q,\phi)\to H_\mu^\infty} &\geq \left\|D_{\varphi,u}^{[\beta]}f_{\varphi(\lambda)}\right\|_{H_\mu^\infty} \\ &= \sup_{\lambda\in\mathbb{D}}\frac{\mu(\lambda)|u(\lambda)|(1+\beta)}{\phi(|\varphi(\lambda)|)\left(1-|\varphi(\lambda)|^2\right)^{(1/q)+\beta}}, \end{aligned} \tag{47}$$

$$C\left\|D_{\varphi,u}^{\beta}\right\|_{H(p,q,\phi)\to H_\mu^\infty} \geq \sup_{\lambda\in\mathbb{D}}\frac{\mu(\lambda)|u(\lambda)|}{\phi(|\varphi(\lambda)|)\left(1-|\varphi(\lambda)|^2\right)^{(1/q)+\beta}}.$$

Therefore,

$$\left\|D_{\varphi,u}^{\beta}\right\|_{H(p,q,\phi)\to H_\mu^\infty} \geq CM. \tag{48}$$

From (40) and (48) we get the asymptotic relation (25). □

If we take $\mu(z) \equiv 1$ in the proof of Theorem 5, we get the following result.

Theorem 6. *For $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $u \in H(\mathbb{D})$, ϕ and μ are normal, and φ is an analytic self-map of $\mathbb{D}$. Then $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H^\infty$ is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}}\frac{|u(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+\beta}} < \infty. \tag{49}$$

Moreover, if $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H^\infty$ is bounded, then the following asymptotic relation holds:

$$\left\|D_{\varphi,u}^{\beta}\right\|_{H(p,q,\phi)\to H^\infty} \asymp M. \tag{50}$$

Theorem 7. *For $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $u \in H(\mathbb{D})$, ϕ and μ are normal, and φ is an analytic self-map of $\mathbb{D}$. Then $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is bounded if and only if $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H_\mu^\infty$ is bounded and $u \in H_{\mu,0}^\infty$.*

Proof. Suppose $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is bounded. Then it is clear that $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H_\mu^\infty$ is also bounded, since

$$\|u\|_{H_\mu^\infty} = \left\|D_{\varphi,u}^{\beta}h\right\|_{H_\mu^\infty} \leq \left\|D_{\varphi,u}^{\beta}h\right\|_{H(p,q,\phi)\to H_{\mu,0}^\infty}\|h\|_{H(p,q,\phi)} \quad \text{where } h(z) \equiv 1 \ \forall z \in \mathbb{D}. \tag{51}$$

Therefore, $u \in H_{\mu,0}^\infty$.

Conversely, assume that $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H_\mu^\infty$ is bounded and $u \in H_{\mu,0}^\infty$. Let p be any polynomial. Then,

$$\begin{aligned} \mu(z)\left|\left(D_{\varphi,u}^{\beta}p\right)(z)\right| &= \mu(z)|u(z)|\left|p^{[\beta]}(\varphi(z))\right| \\ &\leq \mu(z)|u(z)|\left\|p^{[\beta]}\right\|_\infty. \end{aligned} \tag{52}$$

Let $p(z) = \sum_{n=0}^{k} a_n z^n$, where $k < \infty$. Consider the following:

$$\begin{aligned} p^{[\beta]}(z) &= \Gamma(1+\beta)\left[p(z) * F(1,1+\beta;1;z)\right] \\ &= \Gamma(1+\beta)\left[\sum_{n=0}^{k}a_n z^n * \sum_{n=0}^{\infty}\frac{(1+\beta)_n}{(1)_n};z^n\right] \\ &= \Gamma(1+\beta)\left[\sum_{n=0}^{k}\frac{(1+\beta)_n}{(1)_n}a_n z^n\right]. \end{aligned} \tag{53}$$

Obviously, $p^{[\beta]}(z)$ is bounded in $|z| < 1$. Also, $u \in H_{\mu,0}^\infty$; therefore $p^{[\beta]} \in H_{\mu,0}^\infty$. Since the set of polynomials is dense in $H(p,q,\phi)$, therefore we have from the proof of Theorem 7 of [22] that

$$D_{\varphi,u}^{\beta}(H(p,q,\phi)) \subseteq H_{\mu,0}^\infty, \tag{54}$$

from which the boundedness of $D_{\varphi,u}^{\beta} : H(p,q,\phi) \to H_{\mu,0}^\infty$ follows. □

As consequences of the above theorems we have the following important corollary.

Corollary 8. *For $\beta = 1$, $0 < p \le \infty$, $1 \le q \le \infty$, $u \in H(\mathbb{D})$, ϕ and μ are normal, and φ is an analytic self-map of $\mathbb{D}$.*

(a) *Then $D^1_{\varphi,u} : H(p,q,\phi) \to H^\infty_\mu$ is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}} \frac{\mu(z)\,|u(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+1}} < \infty. \tag{55}$$

(b) *When $u = 1$, one gets that $D^1_{\varphi,1} : H(p,q,\phi) \to H^\infty_\mu$ is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}} \frac{\mu(z)}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+1}} < \infty. \tag{56}$$

(c) *For $\varphi(z) = z$, then $D^1_{z,u} : H(p,q,\phi) \to H^\infty_\mu$ is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}} \frac{\mu(z)\,|u(z)|}{\phi(|z|)\left(1-|z|^2\right)^{(1/q)+1}} < \infty. \tag{57}$$

(d) *For $u = \varphi'$, $D^1_{\varphi,\varphi'} : H(p,q,\phi) \to H^\infty_\mu$ is bounded if and only if*

$$M = \sup_{z\in\mathbb{D}} \frac{\mu(z)\,|\varphi'(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+1}} < \infty. \tag{58}$$

A simple calculation shows that $D^1_{\varphi,1} - D^0_{\varphi,1} = \varphi C_\varphi D$, $D^1_{z,u} - D^0_{z,u} = zM_u D$, and $D^1_{\varphi,\varphi'} - D^0_{\varphi,\varphi'} = \varphi D C_\varphi$. Thus the corollaries of Stević's paper [22] on boundedness of these operators follow easily.

4. Compactness of $D^\beta_{\varphi,u}$

In this section we will characterize when weighted composition operators $D^\beta_{\varphi,u}$ acting between mixed-norm spaces and weighted-type space are compact. Before stating the results, we show the following lemma whose proof can be obtained by adapting the proof of Lemma 4 of [25].

Lemma 9. *Suppose $\beta \ge 0$, $0 < p \le \infty$, $1 \le q \le \infty$, ϕ and μ are normal, and φ is a holomorphic self-map of $\mathbb{D}$. Then the operator $D^\beta_{\varphi,u} : H(p,q,\phi) \to H^\infty_\mu$ is compact if and only if $D^\beta_{\varphi,u} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $H(p,q,\phi)$ which converges to zero uniformly on compact subsets of D, we have*

$$\left\|D^\beta_{\varphi,u} f_k\right\|_{H^\infty_\mu} \longrightarrow 0, \tag{59}$$

as $k \to \infty$.

Theorem 10. *Suppose $\beta \ge 0$, $0 < p \le \infty$, $1 \le q \le \infty$, ϕ and μ are normal, and φ is a holomorphic self-map of $\mathbb{D}$. Then the operator $D^\beta_{\varphi,u} : H(p,q,\phi) \to H^\infty_\mu$ is compact if and only if $D^\beta_{\varphi,u} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\,|u(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+\beta}} = 0. \tag{60}$$

Proof. Suppose, $D^\beta_{\varphi,u} : H(p,q,\phi) \to H^\infty_\mu$ is compact. Then it is bounded. Suppose (67) is not true. Then there is a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $\varphi(z_k) \to 1$ as $z_k \to \infty$ and $\delta > 0$ such that

$$\frac{\mu(z_k)\,|u(z_k)|}{\phi(|\varphi(z_k)|)\left(1-|\varphi(z_k)|^2\right)^{(1/q)+\beta}} \ge \delta, \quad k \in \mathbb{N}. \tag{61}$$

Let $g_k(z) = f_{\varphi(z_k)}$, $k \in \mathbb{N}$.

It is obvious that

$$|g_k(z)| = \left|f_{\varphi(z_k)}\right| \le C \frac{\left(1-|\varphi(z_k)|^2\right)^{t+1}}{\phi(|\varphi(z_k)|)\left(1-\overline{\varphi(z_k)}z\right)^{(1/q)+t+1}}. \tag{62}$$

As $k \to \infty$ we have $\varphi(z_k) \to 1$. From the above inequality we have $g_k \to 0$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{D}$. Therefore,

$$\lim_{k\to\infty} \left\|D^\beta_{\varphi,u} g_k\right\|_{H^\infty_\mu} = 0. \tag{63}$$

But from our assumption we have

$$\begin{aligned}
\left\|D^\beta_{\varphi,u} g_k\right\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{D}} \mu(z)\,|u(z)|\left|g_k^{[\beta]}(\varphi(z))\right| \\
&\ge \mu(z_k)\,|u(z_k)|\left|g_k^{[\beta]}(\varphi(z_k))\right| \\
&\ge C \frac{\mu(z_k)\,|u(z_k)|}{\phi(|\varphi(z_k)|)\left(1-|\varphi(z_k)|^2\right)^{(1/q)+\beta}} \\
&\ge C\delta > 0,
\end{aligned} \tag{64}$$

when $k \to \infty$, which is a contradiction.

Suppose $D^\beta_{\varphi,u} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and (67) holds. Then for every $\varepsilon > 0$ there exist $\rho \in (0,1)$ such that, when $\rho < |\varphi(z)| < 1$,

$$\frac{\mu(z)\,|u(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+\beta}} < \varepsilon. \tag{65}$$

Consider $h(z) = 1/\Gamma(1+\beta)$. Obviously, $h(z) \in H(p,q,\phi)$:

$$\begin{aligned}
\|u\|_{H^\infty_\mu} &= \left\|D^\beta_{\varphi,u} h\right\|_{H^\infty_\mu} \\
&\le \left\|D^\beta_{\varphi,u}\right\|_{H(p,q,\phi)\to H^\infty_\mu} \|h\|_{H(p,q,\phi)} \\
&< \infty.
\end{aligned} \tag{66}$$

Assume that $\{h_k\}_{k\in\mathbb{N}}$ is bounded sequence in $H(p,q,\phi)$, that is, $\sup_k \|h_k\|_{H(p,q,\phi)} < L$, and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. For $\rho < |\varphi(z)| < 1$,

$$\mu(z)|u(z)|\left|h_k^{[\beta]}(\varphi(z))\right| \le C\frac{\mu(z)|u(z)|}{\phi(\varphi(z))\left(1-|\varphi(z)|^2\right)^{(1/q)+\beta}} \times \|h\|_{H(p,q,\phi)} < LC\varepsilon. \tag{67}$$

Now, consider the case $|\varphi(z)| \le \rho$. We have

$$h_k^{[\beta]}(\varphi(z)) = \frac{\Gamma(1+\beta)}{2\pi i}\int_{|w|=\xi} h_k(w)\, F\left(1,1+\beta;1;\frac{\varphi(z)}{w}\right) \cdot \frac{1}{w}\,dw, \quad |\varphi(z)| < |w|. \tag{68}$$

Putting $w = \xi e^{i\psi}$ gives us

$$\begin{aligned} h_k^{[\beta]}(\varphi(z)) &= \frac{\Gamma(1+\beta)}{2\pi} \times \int_0^{2\pi} h_k(w)\, F\left(1,1+\beta;1;\frac{\varphi(z)}{w}\right) d\psi \\ &= \frac{\Gamma(1+\beta)}{2\pi} \times \int_0^{2\pi} h_k(w)\frac{1}{(1-(\varphi(z)/w))^{1+\beta}}\,d\psi, \\ \left|h_k^{[\beta]}(\varphi(z))\right| &\le \frac{\Gamma(1+\beta)}{2\pi} \times \int_0^{2\pi} |h_k(w)|\frac{1}{|1-(\varphi(z)/w)|^{1+\beta}}\,d\psi \\ &\le \frac{\Gamma(1+\beta)}{2\pi} \times \int_0^{2\pi} |h_k(w)|\frac{1}{(1-(|\varphi(z)|/|w|))^{1+\beta}}\,d\psi \\ &\le \frac{\Gamma(1+\beta)}{2\pi}\int_0^{2\pi} |h_k(w)|\frac{1}{(1-(r/\xi))^{1+\beta}}\,d\psi \\ &= \frac{\Gamma(1+\beta)}{2\pi(1-(r/\xi))^{1+\beta}}\int_0^{2\pi} |h_k(w)|\,d\psi \\ &= \frac{\xi\Gamma(1+\beta)}{2\pi(\xi-r)^{1+\beta}}\int_0^{2\pi} |h_k(w)|\,d\psi \\ &\le \frac{\Gamma(1+\beta)}{2\pi(\xi-r)^{1+\beta}}\int_0^{2\pi} |h_k(w)|\,d\psi, \end{aligned} \tag{69}$$

where $|\varphi(z)| = r$. Therefore,

$$\left|h_k^{[\beta]}(\varphi(z))\right| \le C\int_0^{2\pi} |h_k(w)|\,d\psi \tag{70}$$

in compact subsets of $\mathbb{D}$. Therefore we have

$$\begin{aligned} \mu(z)|u(z)|\left|h_k^{[\beta]}(\varphi(z))\right| &\le \|u\|_{H_\mu^\infty}\left|h_k^{[\beta]}(\varphi(z))\right| \\ &\le \left|h_k^{[\beta]}(\varphi(z))\right| \\ &\le C\int_0^{2\pi} |h_k(w)|\,d\psi. \end{aligned} \tag{71}$$

Since, on compact subsets of $\mathbb{D}$, $h_k \to 0$ uniformly as $k \to \infty$, therefore

$$\begin{aligned} &\mu(z)|u(z)|\left|h_k^{[\beta]}(\varphi(z))\right| \\ &\quad \le C\int_0^{2\pi} |h_k(w)|\,d\psi \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{72}$$

From (72) and (73) it follows that $\|D_{\varphi,u}^\beta h_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$. Therefore the operator $D_{\varphi,u}^\beta : H(p,q,\phi) \to H_\mu^\infty$ is compact. □

The following result is obvious as can be seen from the above theorem.

Theorem 11. *Suppose $\beta \ge 0$, $0 < p \le \infty$, $1 \le q \le \infty$, ϕ and μ are normal, and φ is a holomorphic self-map of $\mathbb{D}$. Then the operator $D_{\varphi,u}^\beta : H(p,q,\phi) \to H^\infty$ is compact if and only if $D_{\varphi,u}^\beta : H(p,q,\phi) \to H^\infty$ is bounded and*

$$\lim_{|\varphi(z)|\to 1}\frac{|u(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{(1/q)+\beta}} = 0. \tag{73}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The first author would like to acknowledge the financial support as Junior Research Fellow and the second author as Principal Investigator of the Project NBHM (National Board for Higher Mathematics, India) (2/48(9)/2012/NBHM (R.P.)/RD-II/2936).

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