Research Article

Coefficient Estimates for a New Subclass of Analytic and Bi-Univalent Functions Defined by Hadamard Product

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We introduce and investigate a new general subclass $\mathcal{H}_{\Sigma}^{\lambda, \mu}(\varphi; \Theta)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}. \quad (2)$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [1] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}), \quad (3)$$

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (4)$$

Denote by $f * \Theta$ the Hadamard product (or convolution) of the functions f and Θ ; that is, if f is given by (1) and Θ is given by

$$\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0), \quad (5)$$

then

$$(f * \Theta)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (6)$$

For two functions f and Θ , analytic in $\mathbb{U}$, we say that the function f is subordinate to Θ in $\mathbb{U}$ and write

$$f(z) \prec \Theta(z) \quad (z \in \mathbb{U}), \quad (7)$$

if there exists a Schwarz function ω , which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0, \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}) \quad (8)$$

such that

$$f(z) = \Theta(\omega(z)) \quad (z \in \mathbb{U}). \quad (9)$$

Indeed, it is known that

$$f(z) \prec \Theta(z) \quad (z \in \mathbb{U}) \implies f(0) = \Theta(0), \quad f(\mathbb{U}) \subset \Theta(\mathbb{U}). \quad (10)$$

Furthermore, if the function Θ is univalent in $\mathbb{U}$, then we have the following equivalence:

$$\begin{aligned} f(z) < \Theta(z) \\ (z \in \mathbb{U}) \iff f(0) = \Theta(0), \quad f(\mathbb{U}) \subset \Theta(\mathbb{U}). \end{aligned} \quad (11)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class Σ , see [2] (see also [3]). In fact, the aforementioned work of Srivastava et al. [2] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Tang et al. [4], El-Ashwah [5], Frasin and Aouf [6], Aouf et al. [7], and others (see, e.g., [2, 8–15]).

Throughout this paper, we assume that φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad (B_1 > 0). \quad (12)$$

With this assumption on φ , we now introduce the following subclass of bi-univalent functions.

Definition 1. Let the function f , defined by (1), be in the analytic function class $\mathcal{A}$ and let $\Theta \in \Sigma$. We say that

$$f \in \mathcal{H}_{\Sigma}^{\lambda, \mu}(\varphi; \Theta) \quad (\lambda \geq 1, \mu \geq 0) \quad (13)$$

if the following conditions are satisfied:

$$\begin{aligned} f &\in \Sigma, \\ (1 - \lambda) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu} + \lambda (f * \Theta)'(z) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu-1} &< \varphi(z) \quad (z \in \mathbb{U}), \\ (1 - \lambda) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu} + \lambda ((f * \Theta)^{-1})'(w) &\times \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu-1} < \varphi(w) \quad (w \in \mathbb{U}), \end{aligned} \quad (14)$$

where the function $(f * \Theta)^{-1}$ is given by

$$\begin{aligned} (f * \Theta)^{-1}(w) &= w - a_2 b_2 w^2 + (2a_2^2 b_2^2 - a_3 b_3) w^3 \\ &\quad - (5a_2^3 b_2^3 - 5a_2 b_2 a_3 b_3 + a_4 b_4) w^4 + \cdots. \end{aligned} \quad (15)$$

Remark 2. If we let

$$\Theta(z) = \frac{z}{1-z}, \quad (16)$$

then the class $\mathcal{H}_{\Sigma}^{\lambda, \mu}(\varphi; \Theta)$ reduces to the class denoted by $\mathcal{H}_{\Sigma}^{\mu}(\lambda, \varphi)$ which is the subclass of the functions $f \in \Sigma$ satisfying

$$\begin{aligned} (1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} &< \varphi(z), \\ (1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} &< \varphi(w), \end{aligned} \quad (17)$$

where the function g is defined by

$$\begin{aligned} g(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots, \end{aligned} \quad (18)$$

which was introduced and studied recently by Tang et al. [4].

Remark 3. If we let

$$\varphi(z) = \left(\frac{1+z}{1-z} \right)^{\alpha} \quad (0 < \alpha \leq 1), \quad (19)$$

then the class $\mathcal{H}_{\Sigma}^{\lambda, \mu}(\varphi; \Theta)$ reduces to the new class denoted by $\mathcal{H}_{\Sigma}^{\lambda, \mu}(\alpha; \Theta)$ which is the subclass of the functions $f \in \Sigma$ satisfying

$$\begin{aligned} \left| \arg \left\{ (1 - \lambda) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu} \right. \right. \\ \left. \left. + \lambda (f * \Theta)'(z) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu-1} \right\} \right| &< \frac{\alpha\pi}{2}, \\ \left| \arg \left\{ (1 - \lambda) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu} \right. \right. \\ \left. \left. + \lambda ((f * \Theta)^{-1})'(w) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu-1} \right\} \right| &< \frac{\alpha\pi}{2}, \end{aligned} \quad (20)$$

where the function $(f * \Theta)^{-1}$ is defined by (15). Also we have the following classes:

$$(i) \quad \mathcal{H}_{\Sigma}^{\lambda, 1}(\alpha; \Theta) = \mathcal{B}_{\Sigma}(\Theta, \alpha, \lambda) \quad (21)$$

introduced and studied recently by El-Ashwah [5];

$$(ii) \quad \mathcal{H}_{\Sigma}^{\lambda, \mu} \left(\alpha; \frac{z}{1-z} \right) = \mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda) \quad (22)$$

introduced and studied recently by Çağlar et al. [10];

(iii)

$$\mathcal{H}_{\Sigma}^{\lambda,1}\left(\alpha; \frac{z}{1-z}\right) = \mathcal{B}_{\Sigma}(\alpha, \lambda) \quad (23)$$

introduced and studied by Frasin and Aouf [6];

(iv)

$$\mathcal{H}_{\Sigma}^{1,0}\left(\alpha; \frac{z}{1-z}\right) = \mathcal{S}_{\Sigma}^*[\alpha] \quad (24)$$

of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) introduced and studied by Brannan and Taha [3];

(v)

$$\mathcal{H}_{\Sigma}^{\lambda,1}\left(\alpha; z + \sum_{n=2}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n} \frac{1}{n!} z^n\right) = \mathcal{T}_{q,s}^{\Sigma}[a_1, b_1, \alpha, \lambda] \quad (25)$$

introduced and studied recently by Aouf et al. [7].

Remark 4. If we let

$$\varphi(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \leq \beta < 1), \quad (26)$$

then the class $\mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi; \Theta)$ reduces to the new class denoted by $\mathcal{H}_{\Sigma}^{\lambda,\mu}(\beta; \Theta)$ which is the subclass of the functions $f \in \Sigma$ satisfying

$$\begin{aligned} & \Re \left\{ (1-\lambda) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu} \right. \\ & \quad \left. + \lambda (f * \Theta)'(z) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu-1} \right\} > \beta, \\ & \Re \left\{ (1-\lambda) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu} \right. \\ & \quad \left. + \lambda ((f * \Theta)^{-1})'(w) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu-1} \right\} > \beta, \end{aligned} \quad (27)$$

where the function $(f * \Theta)^{-1}$ is defined by (15). Also we have the following classes:

(i)

$$\mathcal{H}_{\Sigma}^{\lambda,1}(\beta; \Theta) = \mathcal{B}_{\Sigma}(\Theta, \beta, \lambda) \quad (28)$$

introduced and studied recently by El-Ashwah [5];

(ii)

$$\mathcal{H}_{\Sigma}^{\lambda,\mu}\left(\beta; \frac{z}{1-z}\right) = \mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda) \quad (29)$$

introduced and studied recently by Çağlar et al. [10];

(iii)

$$\mathcal{H}_{\Sigma}^{\lambda,1}\left(\beta; \frac{z}{1-z}\right) = \mathcal{B}_{\Sigma}(\beta, \lambda) \quad (30)$$

introduced and studied by Frasin and Aouf [6];

(iv)

$$\mathcal{H}_{\Sigma}^{1,0}\left(\beta; \frac{z}{1-z}\right) = \mathcal{S}_{\Sigma}^*(\beta) \quad (31)$$

of bi-starlike functions of order β ($0 \leq \beta < 1$) introduced and studied by Brannan and Taha [3];

(v)

$$\mathcal{H}_{\Sigma}^{\lambda,1}\left(\alpha; z + \sum_{n=2}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n} \frac{1}{n!} z^n\right) = \mathcal{T}_{q,s}^{\Sigma}[a_1, b_1, \beta, \lambda] \quad (32)$$

introduced and studied recently by Aouf et al. [7].

For more results see also [2, 8, 13, 15–17].

Firstly, in order to derive our main results, we need to go to following lemma.

Lemma 5 (see [18]). *If $p \in \mathcal{P}$, then $|c_n| \leq 2$ for each n , where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which*

$$\Re(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (33)$$

for $z \in \mathbb{U}$.

2. A Set of General Coefficient Estimates

In this section, we state and prove our general results involving the bi-univalent function class $\mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi; \Theta)$ given by Definition 1.

Theorem 6. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class*

$$\mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi; \Theta) \quad (\lambda \geq 1, \mu \geq 0) \quad (34)$$

with

$$\Theta \in \Sigma, \quad \Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0). \quad (35)$$

Then

$$|a_2| \leq \frac{1}{b_2} \min \left\{ \frac{B_1}{\lambda + \mu}, \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(\mu + 1)(2\lambda + \mu)}}, \frac{B_1 \sqrt{2B_1}}{\sqrt{|B_1^2(\mu + 1)(2\lambda + \mu) - 2(B_2 - B_1)(\lambda + \mu)^2|}} \right\}, \quad (36)$$

$$|a_3| \leq \frac{1}{b_3} \min \left\{ \frac{B_1^2}{(\lambda + \mu)^2} + \frac{B_1}{2\lambda + \mu}, \frac{B_1[(\mu + 3) + |1 - \mu|]}{2(\mu + 1)(2\lambda + \mu)} + \frac{2|B_2 - B_1|}{(\mu + 1)(2\lambda + \mu)}, \frac{2B_1^3}{|B_1^2(\mu + 1)(2\lambda + \mu) - 2(B_2 - B_1)(\lambda + \mu)^2|} + \frac{B_1}{2\lambda + \mu} \right\}. \quad (37)$$

Proof. Let $f \in \mathcal{H}_{\Sigma}^{\lambda, \mu}(\varphi; \Theta)$. Then there are analytic functions $u, v: \mathbb{U} \rightarrow \mathbb{U}$, with $u(0) = v(0) = 0$, satisfying

$$\begin{aligned} (1 - \lambda) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu} + \lambda (f * \Theta)'(z) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu-1} \\ = \varphi(u(z)), \\ (1 - \lambda) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu} \\ + \lambda ((f * \Theta)^{-1})'(w) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu-1} = \varphi(v(w)). \end{aligned} \quad (38)$$

Define the functions p and q by

$$\begin{aligned} p(z) &= \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \dots, \\ q(z) &= \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + \dots, \end{aligned} \quad (39)$$

or, equivalently,

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \dots \right], \quad (40)$$

$$v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[q_1 z + \left(q_2 - \frac{q_1^2}{2} \right) z^2 + \dots \right]. \quad (41)$$

Then p and q are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Since $u, v: \mathbb{U} \rightarrow \mathbb{U}$, the functions p and q have a positive real part in $\mathbb{U}$, and by Lemma 5 $|p_n| \leq 2$ and $|q_n| \leq 2$ for each n . Also it is clear that

$$\begin{aligned} (1 - \lambda) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu} + \lambda (f * \Theta)'(z) \left(\frac{(f * \Theta)(z)}{z} \right)^{\mu-1} \\ = \varphi \left(\frac{p(z) - 1}{p(z) + 1} \right), \\ (1 - \lambda) \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu} + \lambda ((f * \Theta)^{-1})'(w) \\ \times \left(\frac{(f * \Theta)^{-1}(w)}{w} \right)^{\mu-1} = \varphi \left(\frac{q(w) - 1}{q(w) + 1} \right). \end{aligned} \quad (42)$$

Using (40) and (41) together with (12), we get

$$\begin{aligned} \varphi \left(\frac{p(z) - 1}{p(z) + 1} \right) \\ = 1 + \frac{1}{2} B_1 p_1 z + \left(\frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \dots, \\ \varphi \left(\frac{q(w) - 1}{q(w) + 1} \right) \\ = 1 + \frac{1}{2} B_1 q_1 w + \left(\frac{1}{2} B_1 \left(q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} B_2 q_1^2 \right) w^2 + \dots. \end{aligned} \quad (43)$$

Now, upon equating the coefficients in (42)-(43), we get

$$(\lambda + \mu) b_2 a_2 = \frac{1}{2} B_1 p_1, \quad (44)$$

$$\begin{aligned} (2\lambda + \mu) b_3 a_3 + (\mu - 1) \left(\lambda + \frac{\mu}{2} \right) b_2^2 a_2^2 \\ = \frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2, \end{aligned} \quad (45)$$

$$-(\lambda + \mu) b_2 a_2 = \frac{1}{2} B_1 q_1, \quad (46)$$

$$\begin{aligned} -(2\lambda + \mu) b_3 a_3 + (\mu + 3) \left(\lambda + \frac{\mu}{2} \right) b_2^2 a_2^2 \\ = \frac{1}{2} B_1 \left(q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} B_2 q_1^2. \end{aligned} \quad (47)$$

From (44) and (46), it follows that

$$p_1 = -q_1, \quad (48)$$

$$2(\lambda + \mu)^2 b_2^2 a_2^2 = \frac{1}{4} B_1^2 (p_1^2 + q_1^2). \quad (49)$$

Now (45), (47), and (48) yield

$$a_2^2 = \frac{B_1 (p_2 + q_2) + (B_2 - B_1) p_1^2}{2(\mu + 1)(2\lambda + \mu) b_2^2}. \quad (50)$$

Also from (45), (47), and (49), we get

$$a_2^2 = \frac{B_1^3 (p_2 + q_2)}{2 [B_1^2 (\mu + 1)(2\lambda + \mu) - 2(B_2 - B_1)(\lambda + \mu)^2] b_2^2}. \quad (51)$$

Applying Lemma 5 for (44), (50), and (51), we get the desired estimate on the coefficient $|a_2|$ as asserted in (36).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (47) from (45), we obtain

$$2(2\lambda + \mu) b_3 a_3 - 2(2\lambda + \mu) b_2^2 a_2^2 = \frac{1}{2} B_1 (p_2 - q_2). \quad (52)$$

Upon substituting the value of a_2^2 from (44), (50), and (51) into (52), it follows that

$$a_3 = \frac{B_1^2 p_1^2}{4(\lambda + \mu)^2 b_3} + \frac{B_1 (p_2 - q_2)}{4(2\lambda + \mu) b_3}, \quad (53)$$

$$a_3 = \frac{B_1 (p_2 + q_2) + (B_2 - B_1) p_1^2}{2(\mu + 1)(2\lambda + \mu) b_3} + \frac{B_1 (p_2 - q_2)}{4(2\lambda + \mu) b_3} \quad (54)$$

$$= \frac{B_1 [(\mu + 3) p_2 + (1 - \mu) q_2] + 2(B_2 - B_1) p_1^2}{4(\mu + 1)(2\lambda + \mu) b_3},$$

$$a_3 = \frac{B_1^3 (p_2 + q_2)}{2 [B_1^2 (\mu + 1)(2\lambda + \mu) - 2(B_2 - B_1)(\lambda + \mu)^2] b_3} + \frac{B_1 (p_2 - q_2)}{4(2\lambda + \mu) b_3}, \quad (55)$$

respectively. Applying Lemma 5 for (53), (54), and (55), we get the desired estimate on the coefficient $|a_3|$ as asserted in (37). $\square$

If we take

$$\Theta(z) = \frac{z}{1 - z} \quad (56)$$

in Theorem 6, then we have the following corollary.

Corollary 7. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class

$$\mathcal{H}_\Sigma^\mu(\lambda, \varphi) \quad (\lambda \geq 1, \mu \geq 0). \quad (57)$$

Then

$$|a_2| \leq \min \left\{ \frac{B_1}{\lambda + \mu}, \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(\mu + 1)(2\lambda + \mu)}}, \frac{B_1 \sqrt{2B_1}}{\sqrt{|B_1^2 (\mu + 1)(2\lambda + \mu) - 2(B_2 - B_1)(\lambda + \mu)^2|}} \right\},$$

$$|a_3| \leq \min \left\{ \frac{B_1^2}{(\lambda + \mu)^2} + \frac{B_1}{2\lambda + \mu}, \frac{B_1 [(\mu + 3) + |1 - \mu|]}{2(\mu + 1)(2\lambda + \mu)} + \frac{2|B_2 - B_1|}{(\mu + 1)(2\lambda + \mu)}, \frac{2B_1^3}{|B_1^2 (\mu + 1)(2\lambda + \mu) - 2(B_2 - B_1)(\lambda + \mu)^2|} + \frac{B_1}{2\lambda + \mu} \right\}. \quad (58)$$

Remark 8. Corollary 7 is an improvement of the estimates which were given by Tang et al. [4, Theorem 2.1].

Remark 9. If we set $\lambda = 1$ and $\mu = 1$ in Corollary 7, then we have an improvement of the estimates which were given by Ali et al. [16, Theorem 2.1].

If we take

$$\varphi(z) = \left(\frac{1+z}{1-z} \right)^\alpha \quad (0 < \alpha \leq 1) \quad (59)$$

in Theorem 6, then we have the following corollary.

Corollary 10. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class

$$\mathcal{H}_\Sigma^{\lambda, \mu}(\alpha; \Theta) \quad (\lambda \geq 1, \mu \geq 0, 0 < \alpha \leq 1) \quad (60)$$

with

$$\Theta \in \Sigma, \quad \Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0). \quad (61)$$

Then

$$|a_2| \leq \frac{1}{b_2} \min \left\{ \frac{2\alpha}{\lambda + \mu}, \sqrt{\frac{4\alpha(2-\alpha)}{(\mu+1)(2\lambda+\mu)}}, \right. \\ \left. \frac{2\alpha}{\sqrt{(\lambda+\mu)^2 + \alpha(\mu+2\lambda-\lambda^2)}} \right\},$$

$$|a_3| \leq \frac{1}{b_3} \min \left\{ \frac{4\alpha^2}{(\lambda+\mu)^2} + \frac{2\alpha}{2\lambda+\mu}, \right. \\ \frac{2\alpha[(\mu+3)+|1-\mu|]}{2(\mu+1)(2\lambda+\mu)} + \frac{4\alpha(1-\alpha)}{(\mu+1)(2\lambda+\mu)}, \\ \left. \frac{4\alpha^2}{(\lambda+\mu)^2 + \alpha(\mu+2\lambda-\lambda^2)} + \frac{2\alpha}{2\lambda+\mu} \right\}. \quad (62)$$

Remark 11. In Corollary 10,

- (i) if we set $\mu = 1$, then we have an improvement of the estimates which were given by El-Ashwah [5, Theorem 1];
- (ii) if we set $\Theta(z) = z/(1-z)$, then we have an improvement of the estimates which were given by Çağlar et al. [10, Theorem 2.1];
- (iii) if we set $\Theta(z) = z/(1-z)$ and $\mu = 1$, then we have an improvement of the estimates which were given by Frasin and Aouf [6, Theorem 2.2];
- (iv) if we set $\Theta(z) = z + \sum_{n=2}^{\infty} (((a_1)_n \cdots (a_q)_n / (b_1)_n \cdots (b_s)_n) (1/n!) z^n)$ and $\mu = 1$, then we have an improvement of the estimates which were given by Aouf et al. [7, Theorem 4].

If we take

$$\varphi(z) = \frac{1 + (1-2\beta)z}{1-z} \quad (0 \leq \beta < 1) \quad (63)$$

in Theorem 6, then we have the following corollary.

Corollary 12. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class

$$\mathcal{B}_{\Sigma}(\Theta, \beta, \lambda) \quad (\lambda \geq 1, 0 \leq \beta < 1) \quad (64)$$

with

$$\Theta \in \Sigma, \quad \Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n > 0). \quad (65)$$

Then

$$|a_2| \leq \frac{1}{b_2} \min \left\{ \frac{2(1-\beta)}{\lambda+\mu}, \sqrt{\frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}} \right\},$$

$$|a_3| \leq \begin{cases} \frac{1}{b_3} \min \left\{ \frac{4(1-\beta)^2}{(\lambda+\mu)^2} + \frac{2(1-\beta)}{2\lambda+\mu}, \frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)} \right\}, & 0 \leq \mu < 1, \\ \frac{2(1-\beta)}{(2\lambda+\mu)b_3}, & \mu \geq 1. \end{cases} \quad (66)$$

Remark 13. In Corollary 12,

- (i) if we set $\mu = 1$, then we have an improvement of the estimates which were given by El-Ashwah [5, Theorem 2];
- (ii) if we set $\Theta(z) = z/(1-z)$, then we have the estimates which were given by Çağlar et al. [10, Theorem 3.1];
- (iii) if we set $\Theta(z) = z/(1-z)$ and $\mu = 1$, then we have an improvement of the estimates which were given by Frasin and Aouf [6, Theorem 3.2];
- (iv) if we set $\Theta(z) = z + \sum_{n=2}^{\infty} (((a_1)_n \cdots (a_q)_n / (b_1)_n \cdots (b_s)_n) (1/n!) z^n)$ and $\mu = 1$, then we have an improvement of the estimates which were given by Aouf et al. [7, Theorem 8].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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