Research Article

Existence and Nonexistence of a Solution for a Nonlinear $p(x)$ -Elliptic Problem with Right-Hand Side Measure

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We discuss the existence and nonexistence of solution of a nonlinear problem $p(x)$ -elliptic- $\operatorname{div}(a(x, \nabla u)) + g(x, u, \nabla u) = \mu$, where μ is a Radon measure with bounded total variation, by considering the Sobolev spaces with variable exponents. This study is done in two cases: (i) μ is absolutely continuous with respect to $p(x)$ -capacity, and (ii) μ is concentrated on a Borel set of null $p(x)$ -capacity.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, let $N \geq 2$, and let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ (see definition below). Consider the following nonlinear $p(x)$ -elliptic problem:

$$\begin{aligned} Au + g(x, u, \nabla u) &= \mu \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (\mathcal{P})$$

where $Au = -\operatorname{div}(a(x, \nabla u))$ is a Leray-Lions operator from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$. The nonlinear term $g(x, u, \nabla u)$ satisfies a growth condition of the following form:

$$|g(x, s, \xi)| \leq b(|s|) \left(d(x) + |\xi|^{p(x)} \right), \quad (1)$$

where $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous increasing function and $d : \Omega \rightarrow \mathbb{R}^+$, $d \in L^1(\Omega)$ and assuming also that $g(x, s, \xi) \operatorname{sgn}(s) \geq \rho |\xi|^{p(x)}$, a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^N$, and $\forall s \in \mathbb{R}$ for $|s| \geq \sigma$, where σ and ρ are two positive real constants. The second term μ is a Radon measure on Ω .

We are interested in the existence and nonexistence of solution of the problem $(\mathcal{P})$. We prove the existence of solution $(\mathcal{P})$ if μ does not charge the sets E of null $p(x)$ -capacity, in other words, if and only if μ is absolutely continuous with respect to $p(x)$ -capacity. We give a necessary and sufficient condition that allows us the existence of solution

$(\mathcal{P})$. Thus, if μ is concentrated on a Borel set E of Ω of null $p(x)$ -capacity, we will show that the problem $(\mathcal{P})$ admits no solution. Boccardo et al. [1] treated $(\mathcal{P})$ for p constant. We can also see other variations of this problem; for example, if the nonlinear term is independent of ∇u and behaves as $|u|^{q-1}u$, with $q > 1$, the results of the existence and nonexistence of solution depend on the measure μ ; see Baras and Pierre [2], Brezis [3], and Gallouët and Morel [4].

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. The function $p(\cdot)$ satisfies the log-Hölder continuity on Ω if

$$\begin{aligned} |p(x) - p(y)| &\leq \frac{C}{|\log|x - y||}, \\ \forall x, y \in \overline{\Omega} \text{ such that } |x - y| &< \frac{1}{2}, \end{aligned} \quad (2)$$

with C being a positive constant.

Let $C_+(\overline{\Omega}) = \{p : \overline{\Omega} \rightarrow \mathbb{R} \text{ be log-Hölder continuous such that } 1 < p_- \leq p_+ < N\}$, where $p_- = \inf_{x \in \overline{\Omega}} p(x)$ and $p_+ = \sup_{x \in \overline{\Omega}} p(x)$.

For $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ as measurable} \right. \\ \left. \text{such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}, \quad (3)$$

normed by $\|u\|_{p(x)} = \inf\{\lambda > 0, \int_{\Omega} |u(x)/\lambda|^{p(x)} dx \leq 1\}$. The space $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space, and its dual space is isomorphic to $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$ (see [5, 6]).

Proposition 1 (see [5, 6]). (i) For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, one has the following:

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|u\|_{p(x)} \|v\|_{p'(x)}. \quad (4)$$

(ii) For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ a.e. in Ω ; then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2 (see [5, 6]). If one denotes $\rho(u) = \int_{\Omega} |u(x)|^{p(x)} dx$, $\forall u \in L^{p(x)}(\Omega)$, then

$$\min \left\{ \|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+} \right\} \leq \rho(u) \leq \max \left\{ \|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+} \right\}. \quad (5)$$

Define now the Sobolev space with variable exponent by

$$W^{1,p(x)}(\Omega) = \left\{ u \in \frac{L^{p(x)}(\Omega)}{|\nabla u|} \in L^{p(x)}(\Omega) \right\}, \quad (6)$$

normed by

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \quad (7)$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and let $p^*(x) = Np(x)/(N - p(x))$ for $p(x) < N$.

Proposition 3 (see [5, 7]). (i) Assuming that $1 < p_- \leq p_+ < \infty$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

(ii) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$, for any $x \in \Omega$, then embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.

(iii) The Poincaré inequality: there exists a constant $C > 0$, such that

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \quad (8)$$

(iv) The Sobolev inequality: there exists another constant $C > 0$, such that

$$\|u\|_{p^*(x)} \leq C \|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \quad (9)$$

Remark 4. By (iii) of the Proposition 3, we deduce that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$.

Proposition 5 (see [8]). One denotes the dual space of the Sobolev space $W_0^{1,p(x)}(\Omega)$ by $W^{-1,p'(x)}(\Omega)$, and, for each $F \in W^{-1,p'(x)}(\Omega)$, there exist $f_0, f_1, \dots, f_N \in L^{p'(x)}(\Omega)$ such that $F = f_0 + \sum_{i=1}^N (\partial f_i / \partial x_i)$, and, for all $u \in W_0^{1,p(x)}(\Omega)$, one has the following:

$$\langle F, u \rangle = \int_{\Omega} f_0 u dx - \sum_{i=1}^N \int_{\Omega} f_i \frac{\partial u}{\partial x_i} dx. \quad (10)$$

Moreover,

$$\|F\|_{-1,p'(x)} = \sum_{i=0}^N \|f_i\|_{p'(x)}. \quad (11)$$

For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(\cdot)$ can be defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k \frac{s}{|s|}, & \text{if } |s| > k. \end{cases} \quad (12)$$

Let K be a compact Ω ; the $p(x)$ -capacity of K over Ω is defined by

$$\text{cap}_{p(x)}(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^{p(x)} dx, u \in C_0^\infty(\Omega), u \geq \chi_K \right\}, \quad (13)$$

where χ_K is the characteristic function of K .

If U is open in Ω , the $p(x)$ -capacity of U from Ω is defined by

$$\text{cap}_{p(x)}(U, \Omega) = \sup \left\{ \text{cap}_{p(x)}(K, \Omega), K \text{ compact} \subseteq U \right\}. \quad (14)$$

If B is any set of Ω , we define the $p(x)$ -capacity variational B over Ω by

$$\text{cap}_{p(x)}(B, \Omega) = \inf \left\{ \text{cap}_{p(x)}(U, \Omega), U \text{ open} \supseteq B \right\}. \quad (15)$$

Denote by $M_b(\Omega)$ the space of signed measures on Ω and

$$M_b^{p(x)}(\Omega) = \left\{ \mu \in M_b(\Omega) : \mu(E) = 0, \right. \\ \left. \forall E \subset \Omega; \text{cap}_{p(x)}(E, \Omega) = 0 \right\}. \quad (16)$$

3. Measures and $p(x)$ -Variational Capacity

In this section we will give some results concerning the measures and $p(x)$ -capacity variational capacity of any set of Ω .

Theorem 6. Let $\mu \in M_b(\Omega)$, $\mu \geq 0$; then there exist $\gamma \in W^{-1,p'(x)}(\Omega)$ and $h \in L^1(\Omega, d\gamma)$ such that

$$\mu = h\gamma. \quad (17)$$

Proof. Let $u \in W_0^{1,p(x)}(\Omega)$ and let $\tilde{u}$ be measurable representation $\text{cap}_{p(x)}$ quasicontinuous of u . Let $F : W_0^{1,p(x)}(\Omega) \rightarrow [0, \infty]$ and $u \rightarrow F(u) = \int_{\Omega} \tilde{u}^+ d\mu$.

F is convex and lower semicontinuous on $W_0^{1,p(x)}(\Omega)$ indeed; let $u_n \rightarrow u$ in $W_0^{1,p(x)}(\Omega)$; therefore, $u_n \rightarrow u$ a.e. in Ω , and hence $\tilde{u}_n^+ \rightarrow \tilde{u}^+$ a.e. in Ω . Then by Fatou's lemma, we get

$$\begin{aligned} F(u) &= \int_{\Omega} \tilde{u}^+ d\mu = \int_{\Omega} \liminf_n \tilde{u}_n^+ d\mu \\ &\leq \liminf_n \int_{\Omega} \tilde{u}_n^+ d\mu \leq \liminf_n F(u_n). \end{aligned} \quad (18)$$

According to the separability of $W^{-1,p'(x)}(\Omega)$, there exist $\lambda_n \in W^{-1,p'(x)}(\Omega)$ and $a_n \in \mathbb{R}$ such that $F(u) = \sup_n (\langle \lambda_n, u \rangle + a_n)$, and we have the following:

$$\begin{aligned} tF(u) &= F(tu) \geq (t\langle \lambda_n, u \rangle + a_n), \quad \forall t > 0, \forall n, \\ F(u) &\geq \langle \lambda_n, u \rangle + \frac{a_n}{t}, \quad \forall t > 0; \end{aligned} \quad (19)$$

hence, consider the following:

$$F(u) \geq \langle \lambda_n, u \rangle, \quad \forall u \in W_0^{1,p(x)}(\Omega). \quad (20)$$

Since $F(0) = \sup_n a_n = 0$, then $a_n \leq 0$ and we have the following:

$$F(u) \geq \sup_n \langle \lambda_n, u \rangle \geq \sup_n \langle \lambda_n, u \rangle + a_n = F(u). \quad (21)$$

We conclude that $F(u) = \sup_n \langle \lambda_n, u \rangle$.

Let $\varphi \in C_c^\infty(\Omega)$; we have the following:

$$\begin{aligned} \langle \lambda_n, \varphi \rangle &\leq \int_{\Omega} \varphi^+ d\mu \leq \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}, \\ \langle \lambda_n, -\varphi \rangle &\leq \int_{\Omega} (-\varphi)^+ d\mu = - \int_{\Omega} (\varphi)^+ d\mu \\ &\leq \int_{\Omega} (\varphi)^+ d\mu \leq \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}, \end{aligned} \quad (22)$$

which implies that

$$-\|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \leq \langle \lambda_n, \varphi \rangle. \quad (23)$$

Hence,

$$|\langle \lambda_n, \varphi \rangle| \leq \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}, \quad (24)$$

and then $\lambda_n \in W^{-1,p'(x)}(\Omega) \cap M_b(\Omega)$.

Since $F(-\varphi) = 0$, $\forall \varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$, then $0 \leq \langle \lambda_n, \varphi \rangle = \int_{\Omega} \varphi d\lambda_n$. Hence,

$$\|\lambda_n\|_{M_b(\Omega)} \leq \|\mu\|_{M_b(\Omega)}. \quad (25)$$

Let $\gamma = \sum_{n=1}^{\infty} \lambda_n / 2^n (\|\lambda_n\|_{W^{-1,p'(x)}} + 1)$; we get that γ is absolutely convergent in $W^{-1,p'(x)}(\Omega)$. For all $\varphi \in C_c^\infty(\Omega)$, we have the following:

$$\begin{aligned} |\langle \gamma, \varphi \rangle| &= \left| \sum_{n=1}^{\infty} \frac{\langle \lambda_n, \varphi \rangle}{2^n (\|\lambda_n\|_{W^{-1,p'(x)}} + 1)} \right| \\ &\leq \sum_{n=1}^{\infty} \frac{(\|\lambda_n\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)})}{2^n} \\ &\leq \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \\ &\leq \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}. \end{aligned} \quad (26)$$

This shows that $\gamma \in W^{-1,p'(x)}(\Omega)$. Let A be a Borel Ω , such that $\gamma(A) = 0$; we have the following:

$$\sum_{n=1}^{\infty} \frac{\lambda_n(A)}{2^n (\|\lambda_n\|_{W^{-1,p'(x)}} + 1)} = 0. \quad (27)$$

And we deduce that $\lambda_n(A) = 0$, for all $n \geq 1$, from which $\lambda_n \ll \gamma$. Using the theorem of Radon-Nikodym, then there exists $f_n \in L^1(\Omega, d\gamma)$ such that, for all $A \subset \Omega$, we have $\lambda_n(A) = \int_A f_n d\gamma$, and therefore $\lambda_n = f_n \gamma$.

Let $\varphi \in C_c^\infty(\Omega)$ and $\varphi \geq 0$; we have, for any Borel B and for all n ,

$$\begin{aligned} F(\varphi) &= \int_{\Omega} \varphi d\mu \geq \sup_n \int_{\Omega} \varphi d\lambda_n = \sup_n \int_{\Omega} \varphi f_n d\gamma, \\ \text{then } f_n \gamma &\leq \mu, \quad \int_B f_n d\gamma \leq \mu(B). \end{aligned} \quad (28)$$

Therefore, $\int_B \sup(f_1, f_2, \dots, f_k) d\gamma \leq \mu(B)$, for all $k \geq 1$.

For $k \rightarrow \infty$, we obtain the following, by monotone convergence theorem:

$$\int_B f d\gamma \leq \mu(B), \quad \forall B \subset \Omega, \text{ where } f = \sup_n f_n. \quad (29)$$

Remark that

$$\begin{aligned} \int_{\Omega} \varphi d\mu &= \sup_n \int_{\Omega} \varphi f_n d\gamma \leq \int_{\Omega} \varphi f d\gamma \leq \int_{\Omega} \varphi d\mu, \\ \forall \varphi &\in C_c^\infty(\Omega); \end{aligned} \quad (30)$$

hence $\mu = f\gamma$, and since $\mu(\Omega) < \infty$, then $f \in L^1(\Omega, d\gamma)$. $\square$

Theorem 7. Let $p \in C_+(\overline{\Omega})$ and let $\mu \in M_b(\Omega)$. Then,

$$\mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega) \iff \mu \in M_b^{p(x)}(\Omega). \quad (31)$$

Proof. Let $\mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$, and show that $\mu \in M_b^{p(x)}(\Omega)$. Indeed, we have $\mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$ and then there exist $f \in L^1(\Omega)$ and $(g_\beta)_\beta \in L^{p'(x)}(\Omega)$ such that

$$\mu = f + \sum_{|\beta| \leq 1} D^\beta g_\beta. \quad (32)$$

Let $B \subset \Omega$ such that $\text{cap}_{p(x)}(B, \Omega) = 0$, and we prove that $\mu(B) = 0$. Indeed, for all $\epsilon > 0$, there exists $u_\epsilon \in C_0^\infty(\Omega)$: $u_\epsilon = 1$ in B , $u_\epsilon \geq 0$ and $\int_\Omega |\nabla u_\epsilon|^{p(x)} dx < \epsilon$.

Let $(f_n)_n \in L^{p'(x)}(\Omega)$, such that $f_n \rightarrow f$ in $L^1(\Omega)$; consider the sequence $\mu_n = f_n + \sum_{|\beta|=1} D^\beta g_\beta$; we have $\mu_n \rightarrow \mu$ in $M_b(\Omega)$. Indeed

$$\mu_n(B) = \int_B d\mu_n = \int_B \left(f_n + \sum_{|\beta|=1} D^\beta g_\beta \right) u_\epsilon dx. \quad (33)$$

We obtain the following:

$$|\mu_n(B)| \leq \int_B |f_n| u_\epsilon dx + \int_B |(g_\beta) \cdot \nabla u_\epsilon| dx. \quad (34)$$

The Poincaré inequality leads to

$$|\mu_n(B)| \leq C \left(\|f_n\|_{L^{p'(x)}(\Omega)} \|u_\epsilon\|_{L^{p(x)}(\Omega)} + \|g_\beta\|_{L^{p(x)}(\Omega)} \|\nabla u_\epsilon\|_{L^{p(x)}(\Omega)} \right) \leq C\epsilon. \quad (35)$$

Since $\mu_n \rightarrow \mu$ in $M_b(\Omega)$, then $\mu_n(B) \rightarrow \mu(B)$, and we deduce that $\mu(B) = 0$.

Conversely, let $\mu \in M_b^{p(x)}(\Omega)$, and we can always assume that $\mu \geq 0$. And, by Theorem 6, we deduce that $\mu = h\gamma$ where $h \in L^1(\Omega, \gamma)$, $\gamma \in W^{-1,p'(x)}(\Omega)$, $\gamma \geq 0$, and $h \geq 0$. Let $(K_n)_n$ be an increasing sequence of compact Ω such that $\cup_{n=1}^\infty K_n = \Omega$ and let $\mu_n^{(1)} = T_n(h\chi_{K_n})\gamma \in W^{-1,p'(x)}(\Omega)$, $(\mu_n^{(1)})_n$ be an increasing positive sequence in $W^{-1,p'(x)}(\Omega)$, with compact supports in Ω . Let

$$\begin{aligned} \mu_0 &= \mu_0^{(1)}, \\ \mu_n &= \mu_n^{(1)} - \mu_{n-1}^{(1)}. \end{aligned} \quad (36)$$

Then, $\mu = \sum_{n=1}^\infty \mu_n$ converges strongly in $M_b(\Omega)$ and $\sum_{n=1}^\infty \|\mu_n\|_{M_b(\Omega)} < \infty$.

Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $\rho(x) \geq 0$ a.e., and let $\int_{\mathbb{R}^N} \rho(x) dx = 1$. Let $\rho_n(x) = n^N \rho(nx)$. We have $\mu_n * \rho_m \rightarrow \rho_n$ in $W^{-1,p'(x)}(\Omega)$ as $m \rightarrow \infty$ and $\mu_n * \rho_m \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, for m is large enough.

Choose $m = m_n$ such that $\|\mu_n * \rho_{m_n} - \mu_n\|_{W^{-1,p'(x)}(\Omega)} \leq 2^{-n}$. Then $\mu_n = f_n + g_n$, where

$$\begin{aligned} f_n &= \mu_n * \rho_{m_n}, \\ g_n &= \mu_n - \mu_n * \rho_{m_n}. \end{aligned} \quad (37)$$

We have $\sum g_n$ that converges in $W^{-1,p'(x)}(\Omega)$ and then $g = \sum_{n=1}^\infty g_n \in W^{-1,p'(x)}(\Omega)$.

Moreover, $\|f_n\|_{L^1(\Omega)} = \|\mu_n * \rho_{m_n}\|_{L^1(\Omega)} \leq \|\mu_n\|_{M_b(\Omega)}$ and then $\sum f_n$ is absolutely convergent in $L^1(\Omega)$. Then let $f = \sum f_n \in L^1(\Omega)$; we conclude that

$$\mu = \sum_{n=1}^\infty \mu_n = \sum_{n=1}^\infty f_n + \sum_{n=1}^\infty g_n = f + g. \quad (38)$$

□

Definition 8. Let μ be a Radon measure and let E be a Borel set of Ω .

- (i) The restriction of μ to E is the measure $\lambda = \mu|_E$ defined by

$$\lambda(B) = \mu|_E(B) = \mu(E \cap B) \quad (39)$$

for any Borel set B of Ω .

- (ii) The measure λ is concentrated on E if $\lambda = \lambda|_E$.

Proposition 9. Let $\mu \in M_b(\Omega)$ and let $p \in \mathcal{C}_+(\overline{\Omega})$ with $p_+ \leq N$.

Then μ uniquely decomposes as $\mu = \mu_0 + \lambda$, where

$$\begin{aligned} \mu_0 &\in M_b^{p(x)}(\Omega), \\ \lambda &= \mu|_E, \quad \text{cap}_{p(x)}(E, \Omega) = 0. \end{aligned} \quad (40)$$

Proof. We assume that μ is positive (if not, consider μ^+ and μ^-). We first show the uniqueness. Assume that $\mu = \mu_0 + \lambda = \mu'_0 + \lambda'$, where $(\mu_0, \mu'_0) \in (M_b^{p(x)}(\Omega))^2$ and $\lambda = \mu|_E$ with $\text{cap}_{p(x)}(E, \Omega) = 0$ and $\lambda' = \mu|_{E'}$ with $\text{cap}_{p(x)}(E', \Omega) = 0$. We have $\mu_0 - \mu'_0 = \lambda' - \lambda$. Let B be a Borel subset of Ω ; if $\text{cap}_{p(x)}(B, \Omega) = 0$, then $(\mu_0 - \mu'_0)(B) = 0$, and therefore $\mu_0 = \mu'_0$ and $\lambda = \lambda'$; if we consider $\text{cap}_{p(x)}(B, \Omega) \neq 0$ and since $\lambda' - \lambda$ is focused on $E \cap E'$ such that $\text{cap}_{p(x)}(E \cap E', \Omega) = 0$, then we have

$$\begin{aligned} (\lambda - \lambda')(B) &= (\lambda - \lambda')(B \cap E \cap E') \\ &= (\mu'_0 - \mu_0)(B \cap E \cap E') \\ &\leq (\mu'_0 - \mu_0)(E \cap E') = 0. \end{aligned} \quad (41)$$

Therefore, $\mu_0 = \mu'_0$, which proves the uniqueness.

Now we show the existence of μ . Let

$$\begin{aligned} \alpha &= \sup \{ \mu(A) : A \text{ be a Borel subset of } \Omega \\ &\text{and } \text{cap}_{p(x)}(A, \Omega) = 0 \} < \infty; \end{aligned} \quad (42)$$

there exists a sequence $(A_n)_n$ such that A_n is a Borel subset of Ω , $\lim_{n \rightarrow \infty} \mu(A_n) = \alpha$, and $\text{cap}_{p(x)}(A_n, \Omega) = 0$. Let $E = \cup_n A_n$ be a Borel subset of Ω ; we have $\mu(E) = \alpha$ and $\text{cap}_{p(x)}(E, \Omega) = 0$.

For $\lambda = \mu|_E$ and $\mu_0 = \mu - \lambda$, there exist $E \in B(\Omega)$: $\text{cap}_{p(x)}(E, \Omega) = 0$ and $\lambda = \mu|_E$.

It remains to show that $\mu_0 \in M_b^{p(x)}(\Omega)$. By contradiction, assume that A is a Borel subset of Ω such that $\text{cap}_{p(x)}(A, \Omega) = 0$ and $0 < \mu_0(A) = \mu(A) - \lambda(A) = \mu(A) - \mu(A \cap E) = \mu(A \setminus E)$. We have $\text{cap}_{p(x)}(A \cup E, \Omega) = 0$ and $\mu(A \cup E) = \mu(E) + \mu(A \setminus E) = \alpha + \mu(A \setminus E) > \alpha$, which contradicts the definition of α . □

Lemma 10. Let λ be a positive measure in $M_b(\Omega)$ concentrated on a set E such that $\text{cap}_{p(x)}(E, \Omega) = 0$. Then $\forall \delta > 0$, $\exists \psi_\delta \in C_0^\infty(\Omega)$, and $0 \leq \psi_\delta \leq 1$, such that

$$\begin{aligned} \int_{\Omega} |\nabla \psi_\delta|^{p(x)} dx &= \varepsilon_\delta \longrightarrow_{\delta \rightarrow 0^+} 0, \\ \int_{\Omega} (1 - \psi_\delta) d\lambda &= \varepsilon_\delta \longrightarrow_{\delta \rightarrow 0^+} 0. \end{aligned} \quad (43)$$

Proof. Let $\lambda \in M_b(\Omega)$; then for any $\delta > 0$, there exists K_δ which is a compact set in E with $\lambda(E \setminus K_\delta) \leq \delta$.

Moreover, $\text{cap}_{p(x)}(E, \Omega) = 0$; hence, $\text{cap}_{p(x)}(K_\delta, \Omega) = 0$, and then there exists $\psi_\delta \in C_0^\infty(\Omega)$ such that $\psi_\delta = 1$ on a neighborhood of K_δ , $0 \leq \psi_\delta \leq 1$, and $\int_{\Omega} |\nabla \psi_\delta|^{p(x)} dx = \varepsilon_\delta$ converges to 0 as δ goes to 0^+ .

We have $0 \leq \int_{\Omega} (1 - \psi_\delta) d\lambda = \int_E (1 - \psi_\delta) d\lambda + \int_{\Omega \setminus E} (1 - \psi_\delta) d\lambda$, as λ is concentrated on E from which $\int_{\Omega \setminus E} (1 - \psi_\delta) d\lambda = 0$; therefore $\int_{\Omega} (1 - \psi_\delta) d\lambda = \int_{E \setminus K_\delta} (1 - \psi_\delta) d\lambda + \int_{K_\delta} (1 - \psi_\delta) d\lambda$, and since $\psi_\delta = 1$ on a neighborhood of K_δ , we have then the following:

$$0 \leq \int_{\Omega} (1 - \psi_\delta) d\lambda \leq \lambda(E \setminus K_\delta) \leq \delta. \quad (44)$$

□

Remark 11. If E is compact, we can choose $K_\delta = E$, in view of Lemma 10, and we can deduce that ψ_δ converges to 0 strongly in $W_0^{1,p(x)}(\Omega)$ as δ goes to 0^+ .

4. Essential Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$, ($N \geq 2$) and $p \in \mathcal{C}_+(\overline{\Omega})$. We consider a Leray-Lions operator a from $W_0^{1,p(x)}(\Omega)$ to dual $W^{-1,p'(x)}(\Omega)$ defined by

$$Au = -\text{div}(a(x, \nabla u)), \quad (45)$$

where $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory operator, satisfying the following conditions:

$$|a(x, \xi)| \leq l(x) + \beta |\xi|^{p(x)-1}, \quad (46)$$

$$a(x, \xi) \xi \geq \alpha |\xi|^{p(x)}, \quad (47)$$

$$[a(x, \xi) - a(x, \bar{\xi})] (\xi - \bar{\xi}) > 0, \quad \forall \xi \neq \bar{\xi}, \quad (48)$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$, where $l(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\beta > 0$, $\alpha > 0$.

The nonlinear term $g(x, s, \xi)$ satisfies the following:

$$|g(x, s, \xi)| \leq b(|s|) (d(x) + |\xi|^{p(x)}), \quad (49)$$

where $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous growth function and $d : \Omega \rightarrow \mathbb{R}^+$ with $c \in L^1(\Omega)$. Consider the following:

$$g(x, s, \xi) \text{sgn}(s) \geq \rho |\xi|^{p(x)}, \quad (50)$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$, and, for $|s| \geq \sigma$, $s \in \mathbb{R}$, where $\sigma, \rho > 0$. We consider the following problem:

$$\begin{aligned} -\text{div}(a(x, \nabla u)) + g(x, u, \nabla u) &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (51)$$

where $\mu \in M_b(\Omega)$.

5. Approximate Problem

5.1. Existence of a Solution of the Approximate Problem. Consider the following approximate problem:

$$\begin{aligned} u_n &\in W_0^{1,p(x)}(\Omega), \\ \int_{\Omega} a(x, \nabla u_n) \nabla v dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) v dx &= \int_{\Omega} F \nabla v dx, \quad \forall v \in W_0^{1,p(x)}(\Omega), \end{aligned} \quad (52)$$

where $g_n(x, s, \xi) = g(x, s, \xi) / (1 + (1/n)|g(x, s, \xi)|)$.

Define the operators A and G_n of $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p'(x)}(\Omega)$ by

$$\begin{aligned} \langle Au, v \rangle &= \int_{\Omega} a(x, \nabla u) \nabla v dx, \\ \langle G_n u, v \rangle &= \int_{\Omega} g_n(x, u, \nabla u) v dx. \end{aligned} \quad (53)$$

Lemma 12. The operator G_n is bounded.

Proof. For all u, v in $W_0^{1,p(x)}(\Omega)$, and by the Hölder inequality,

$$\begin{aligned} |\langle G_n u, v \rangle| &= \left| \int_{\Omega} g_n(x, u, \nabla u) v dx \right| \\ &\leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|g_n(x, u, \nabla u)\|_{p'(x)} \|v\|_{p(x)} \\ &\leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \left(\int_{\Omega} \|g_n(x, u, \nabla u)\|_{p'(x)}^{p'(x)} dx \right)^{1/\gamma} \|v\|_{p(x)} \\ &\leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) (\text{meas}(\Omega))^{1/\gamma} n^{p'_+} \|v\|_{p(x)}, \end{aligned} \quad (54)$$

where

$$\gamma = \begin{cases} p'_- & \text{if } \|g_n(x, u, \nabla u)\|_{p'(x)} \geq 1, \\ p'_+ & \text{if } \|g_n(x, u, \nabla u)\|_{p'(x)} < 1, \end{cases} \quad (55)$$

from which $|\int_{\Omega} g_n(x, u, \nabla u) v dx| \leq C_n \|v\|_{p(x)}$, for all n fixed. □

Lemma 13. The operator $T_n = A + G_n : W_0^{1,p(x)}(\Omega) \rightarrow W^{-1,p'(x)}(\Omega)$ is pseudomonotone and coercive in the following sense: $|\langle T_n v, v \rangle| / \|v\|_{1,p(x)} \rightarrow +\infty$ when $\|v\|_{1,p(x)} \rightarrow +\infty$.

Proof. Let $u, v \in W_0^{1,p(x)}(\Omega)$; we have

$$\begin{aligned}
& |\langle Au, v \rangle| \\
&= \left| \int_{\Omega} a(x, \nabla u) \nabla v \, dx \right| \leq \left(\frac{1}{p'} + \frac{1}{p} \right) \\
&\quad \times \|a(x, \nabla u)\|_{p'(x)} \|\nabla v\|_{p(x)} \\
&\leq \left(\frac{1}{p'} + \frac{1}{p} \right) \left(\int_{\Omega} |a(x, \nabla u)|^{p'(x)} \, dx \right)^{1/\gamma'} \|\nabla v\|_{p(x)} \\
&\leq \left(\frac{1}{p'} + \frac{1}{p} \right) \left[\int_{\Omega} 2^{p'_+} \left(l^{p'(x)}(x) + |\nabla u|^{(p(x)-1)p'(x)} \right) \, dx \right]^{1/\gamma'} \\
&\quad \times \|\nabla v\|_{p(x)} \\
&\leq \left(\frac{1}{p'} + \frac{1}{p} \right) 2^{p'_+/ \gamma'} \left[\int_{\Omega} \left(l^{p'(x)}(x) + |\nabla u|^{p(x)} \right) \, dx \right]^{1/\gamma'} \\
&\quad \times \|\nabla v\|_{p(x)} \\
&\leq \left(\frac{1}{p'} + \frac{1}{p} \right) 2^{p'_+/ \gamma'} \left[\left(\int_{\Omega} l^{p'(x)}(x) \, dx \right)^{1/\gamma'} \right. \\
&\quad \left. + \left(\int_{\Omega} |\nabla u|^{p(x)} \, dx \right)^{1/\gamma'} \right] \|\nabla v\|_{p(x)} \\
&\leq C_1 \left(C + |\nabla u|^{\gamma_0/\gamma'} \right) \|\nabla v\|_{p(x)}, \tag{56}
\end{aligned}$$

where

$$\begin{aligned}
\gamma' &= \begin{cases} p'_+ & \text{if } \|a(x, \nabla u)\|_{p'(x)} \geq 1, \\ p'_- & \text{if } \|a(x, \nabla u)\|_{p'(x)} < 1, \end{cases} \\
\gamma_0 &= \begin{cases} p_+ & \text{if } \|\nabla u\|_{p(x)} \geq 1, \\ p_- & \text{if } \|\nabla u\|_{p(x)} < 1. \end{cases} \tag{57}
\end{aligned}$$

Hence, operator A is bounded and, by Lemma 12, G_n is bounded; then it follows that T_n is bounded.

We show that T_n is pseudomonotone: let $(u_k)_k$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\begin{aligned}
u_k &\rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega), \\
T_n u_k &\rightharpoonup \chi \quad \text{weakly in } W^{-1,p'(x)}(\Omega), \\
\limsup_k \langle T_n u_k, u_k \rangle &\leq \langle \chi, u \rangle. \tag{58}
\end{aligned}$$

We prove that $\chi = T_n u$ and $\langle T_n u_k, u_k \rangle$ converges to $\langle \chi, u \rangle$ as k goes to $+\infty$. Indeed, we get that $T_n u_k$ weakly converges to χ

in $W^{-1,p'(x)}(\Omega)$ as k tends to $+\infty$; then $\langle T_n u_k, v \rangle$ converges to $\langle \chi, v \rangle$, for all $v \in W_0^{1,p(x)}(\Omega)$, as k tends to $+\infty$, and we obtain

$$\begin{aligned}
\langle \chi, v \rangle &= \lim_{k \rightarrow \infty} \left(\int_{\Omega} a(x, \nabla u_k) \nabla v \, dx \right. \\
&\quad \left. + \int_{\Omega} g_n(x, u_k, \nabla u_k) v \, dx \right), \tag{59} \\
&\quad \forall v \in W_0^{1,p(x)}(\Omega).
\end{aligned}$$

According to (46) and (49), we conclude that there exist $\varphi \in (L^{p'(x)}(\Omega))^N$ and $\psi_n \in L^{p'(x)}(\Omega)$ such that $a(x, \nabla u_k)$ converges to φ weakly in $(L^{p'(x)}(\Omega))^N$ and $g_n(x, u_k, \nabla u_k)$ converges to ψ_n weakly in $L^{p'(x)}(\Omega)$. Hence $\langle \chi, v \rangle = \int_{\Omega} \varphi \nabla v \, dx + \int_{\Omega} \psi_n v \, dx$. We have the following:

$$\begin{aligned}
\limsup_k \langle T_n u_k, u_k \rangle &= \limsup_k \int_{\Omega} a(x, \nabla u_k) \nabla u_k \, dx \\
&\quad + \int_{\Omega} g_n(x, u_k, \nabla u_k) u_k \, dx \tag{60} \\
&\leq \int_{\Omega} \varphi \nabla v \, dx + \int_{\Omega} \psi_n v \, dx.
\end{aligned}$$

Hence,

$$\limsup_k \int_{\Omega} a(x, \nabla u_k) \nabla u_k \, dx \leq \int_{\Omega} \varphi \nabla v \, dx. \tag{61}$$

In view of (48), we have $\int_{\Omega} (a(x, \nabla u_k) - a(x, \nabla u))(\nabla u_k - \nabla u) > 0$, and then

$$\begin{aligned}
\int_{\Omega} a(x, \nabla u_k) \nabla u_k \, dx &> - \int_{\Omega} a(x, \nabla u) \nabla u \\
&\quad + \int_{\Omega} a(x, \nabla u) \nabla u_k \, dx \tag{62} \\
&\quad + \int_{\Omega} a(x, \nabla u_k) \nabla u \, dx.
\end{aligned}$$

We deduce that

$$\liminf_k \int_{\Omega} a(x, \nabla u_k) \nabla u \, dx \geq \int_{\Omega} \varphi \nabla u \, dx. \tag{63}$$

In view of (61) we obtain the following:

$$\lim_k \int_{\Omega} a(x, \nabla u_k) \nabla u_k \, dx = \int_{\Omega} \varphi \nabla u \, dx. \tag{64}$$

Finally,

$$\lim_k \langle T_n u_k, u_k \rangle = \langle \chi, u \rangle. \tag{65}$$

On the other hand, we use (64) and we deduce the following:

$$\begin{aligned}
& \lim_k \int_{\Omega} (a(x, \nabla u_k) - a(x, \nabla u)) (\nabla u_k - \nabla u) \\
&= \lim_k \int_{\Omega} a(x, \nabla u_k) \nabla u_k dx \\
&\quad - \int_{\Omega} a(x, \nabla u_k) \nabla u dx - \int_{\Omega} a(x, \nabla u) \nabla u_k dx \\
&\quad + \int_{\Omega} a(x, \nabla u) \nabla u dx = 0.
\end{aligned} \tag{66}$$

So, in view of [9], we conclude that u_k converges to u a.e. in Ω and strongly in $W_0^{1,p(x)}(\Omega)$; hence $a(x, \nabla u_k)$ converges to $a(x, \nabla u)$ weakly in $(L^{p'(x)}(\Omega))^N$ and $g_n(x, u_k, \nabla u_k)$ converges to $g_n(x, u, \nabla u)$ weakly in $L^{p'(x)}(\Omega)$. Since $T_n u_k$ converges weakly to χ , then

$$\begin{aligned}
& \lim_k \left(\int_{\Omega} a(x, \nabla u_k) \nabla v dx + \int_{\Omega} g_n(x, u_k, \nabla u_k) v dx \right) \\
&= \langle \chi, v \rangle, \\
&\langle T_n u, v \rangle = \langle \chi, v \rangle, \quad \forall v \in W_0^{1,p(x)}(\Omega);
\end{aligned} \tag{67}$$

hence $\chi = T_n u$, and we conclude that the operator T_n is pseudomonotone.

We show that T_n is coercive and we have the following:

$$\begin{aligned}
\langle T_n v, v \rangle &= \int_{\Omega} a(x, \nabla v) \nabla v dx + \int_{\Omega} g_n(x, v, \nabla v) v dx \\
&\geq \alpha \int_{\Omega} |\nabla v|^{p(x)} dx + \int_{\Omega} g_n(x, v, \nabla v) \operatorname{sgn}(v) |v| dx \\
&\geq \alpha \int_{\Omega} |\nabla v|^{p(x)} dx + \rho \int_{\Omega} |\nabla v|^{p(x)} |v| dx \\
&\geq \alpha \int_{\Omega} |\nabla v|^{p(x)} dx
\end{aligned} \tag{68}$$

from which

$$\begin{aligned}
& \lim_{\|v\|_{W_0^{1,p(x)}(\Omega)} \rightarrow \infty} \left(\frac{\langle T_n v, v \rangle}{\|v\|_{W_0^{1,p(x)}(\Omega)}} \right) \\
&\geq \lim_{\|v\|_{W_0^{1,p(x)}(\Omega)} \rightarrow \infty} \frac{\alpha \|v\|_{W_0^{1,p(x)}(\Omega)}^{\gamma}}{\|v\|_{W_0^{1,p(x)}(\Omega)}},
\end{aligned} \tag{69}$$

where

$$\gamma = \begin{cases} p^+ & \text{if } \|v\|_{1,p(x)} \leq 1 \\ p^- & \text{if } \|v\|_{1,p(x)} > 1 \end{cases} \tag{70}$$

and $\gamma > 1$, which implies that

$$\lim_{\|v\|_{W_0^{1,p(x)}(\Omega)} \rightarrow \infty} \alpha \|v\|_{W_0^{1,p(x)}(\Omega)}^{\gamma-1} = +\infty; \tag{71}$$

hence T_n is coercive.

According to [10], there exists $u_n \in W_0^{1,p(x)}(\Omega)$ solution of (45). $\square$

5.2. Estimation of the Solution Approached. Consider the following approximate problem:

$$\begin{aligned}
Au_n + g_n(x, u_n, \nabla u_n) &= -\operatorname{div}(F) \quad \text{in } \Omega, \\
u_n &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{72}$$

where $g_n(x, s, \xi) = g(x, s, \xi)/(1 + (1/n)|g(x, s, \xi)|)$ and $F \in (L^{s(x)}(\Omega))^N$ with $s(\cdot) > N/(p(\cdot) - 1)$.

Lemma 14. *The sequence $(u_n)_n$ is bounded in $L^\infty(\Omega)$.*

Proof. Choose $v = G_k(u_n) = u_n - T_k(u_n)$ as test function in (72); we obtain the following:

$$\begin{aligned}
& \int_{\Omega} a(x, \nabla u_n) \nabla G_k(u_n) dx \\
&\quad + \int_{\Omega} g_n(x, u_n, \nabla u_n) G_k(u_n) dx \\
&= \int_{\Omega} F \nabla G_k(u_n) dx.
\end{aligned} \tag{73}$$

For $|s| \geq k \geq \sigma$, we have $G_k(s) = s - k \operatorname{sign}(s)$. Therefore, for $|s| \geq k \geq \sigma$, we have $\int_{\Omega} g_n(x, u_n, \nabla u_n) G_k(u_n) dx \geq 0$, and hence

$$\int_{\Omega} a(x, \nabla u_n) \nabla G_k(u_n) dx \leq \int_{\Omega} F \nabla G_k(u_n) dx. \tag{74}$$

$\square$

Assertion 1. Let $A_{k,n} = \{x \in \Omega / |u_n(x)| \geq k\}$. We have

$$\frac{\alpha}{2} \int_{\Omega} |\nabla G_k(u_n)|^{p(x)} dx \leq C \int_{A_{k,n}} |F|^{p'(x)} dx. \tag{75}$$

Indeed

$$\begin{aligned}
& \int_{\Omega} a(x, \nabla u_n) \nabla G_k(u_n) dx \leq \int_{\Omega} F \nabla G_k(u_n) dx \\
&\leq C \int_{A_{k,n}} |F|^{p'(x)} dx \\
&\quad + \frac{\alpha}{2} \int_{A_{k,n}} |\nabla G_k(u_n)|^{p(x)} dx.
\end{aligned} \tag{76}$$

According to $\nabla G_k(u_n) = \nabla u_n$ of $A_{k,n}$ and to (47), we obtain Assertion 1.

Assertion 2. We have the following:

$$\left(\int_{\Omega} |G_k(u_n)|^{p^*(x)} dx \right)^{C_0} \leq C (\operatorname{meas}(A_{k,n}))^{C_2} \tag{77}$$

with $p^*(x) = Np(x)/(N - p(x))$, $p(x) < N$, and C, C_0 , and C_2 are positive constants.

Proof. In fact, we have $F \in (L^{s(x)}(\Omega))^N$ with $s(\cdot) > N/(p(\cdot) - 1)$, and since $N/(p(\cdot) - 1) > p'(\cdot)$, then $|F| \in L^{p'(x)}(\Omega)$. According to the inequality of Hölder, we obtain

$$\begin{aligned} \int_{A_{k,n}} |F|^{p'(x)} dx &\leq \left(\frac{1}{(s/p')^-} + \frac{1}{(s/(s-p'))^-} \right) \\ &\quad \times \| |F|^{p'(x)} \|_{s(x)/p'(x)} \| 1_{A_{k,n}} \|_{s(x)/(s(x)-p'(x))} \\ &\leq C \left(\int_{\Omega} |F|^{s(x)} dx \right)^{1/\gamma_1} \left(\int_{\Omega} 1_{A_{k,n}} dx \right)^{1/\gamma_2}, \end{aligned} \quad (78)$$

and therefore

$$\int_{A_{k,n}} |F|^{p'(x)} dx \leq C \left(\int_{\Omega} |F|^{s(x)} dx \right)^{1/\gamma_1} (\text{meas}(A_{k,n}))^{1/\gamma_2}, \quad (79)$$

where

$$\begin{aligned} \gamma_1 &= \begin{cases} \left(\frac{s}{p'} \right)^- & \text{if } \| |F|^{p'(x)} \|_{s(x)/p'(x)} \geq 1, \\ \left(\frac{s}{p'} \right)^+ & \text{if } \| |F|^{p'(x)} \|_{s(x)/p'(x)} < 1, \end{cases} \\ \gamma_2 &= \begin{cases} \left(\frac{s}{s-p'} \right)^- & \text{if } \| 1_{A_{k,n}} \|_{s(x)/(s(x)-p'(x))} \geq 1, \\ \left(\frac{s}{p'} \right)^+ & \text{if } \| 1_{A_{k,n}} \|_{s(x)/(s(x)-p'(x))} < 1, \end{cases} \end{aligned} \quad (80)$$

since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$; hence $\|G_k(u_n)\|_{p^*(x)} \leq C \|\nabla G_k(u_n)\|_{p(x)}$, and therefore

$$\left(\int_{\Omega} |G_k(u_n)|^{p^*(x)} dx \right)^{1/\gamma_3} \leq C \left(\int_{\Omega} |\nabla G_k(u_n)|^{p(x)} dx \right)^{1/\gamma_4}, \quad (81)$$

where

$$\begin{aligned} \gamma_3 &= \begin{cases} (p^*)^- & \text{if } \|G_k(u_n)\|_{p^*(x)} \geq 1, \\ (p^*)^+ & \text{if } \|G_k(u_n)\|_{p^*(x)} < 1, \end{cases} \\ \gamma_4 &= \begin{cases} p^- & \text{if } \|\nabla G_k(u_n)\|_{p(x)} \geq 1, \\ p^+ & \text{if } \|\nabla G_k(u_n)\|_{p(x)} < 1; \end{cases} \end{aligned} \quad (82)$$

therefore

$$\int_{\Omega} |\nabla G_k(u_n)|^{p(x)} dx \geq C \left(\int_{\Omega} |G_k(u_n)|^{p^*(x)} dx \right)^{\gamma_4/\gamma_3}. \quad (83)$$

According to Assertion 1, (46), and (47), we obtain Assertion 2. $\square$

Assertion 3. We have, for all n ,

$$\|u_n\|_{L^\infty(\Omega)} \leq C_1. \quad (84)$$

Proof. Choose $h > k + 1$, and in $A_{h,n} = \{x \in \Omega : |u_n(x)| \geq h\}$ we have: $|G_k(u_n)| \geq h - k$. Hence, in view of Assertion 2, we obtain

$$\begin{aligned} \int_{A_{h,n}} (h-k)^{p^*(x)} dx &\leq \int_{\Omega} |G_k(u_n)|^{p(x)} dx \\ &\leq C (\text{meas}(A_{k,n}))^{\gamma_3/\gamma_2\gamma_4} \end{aligned} \quad (85)$$

and then $(h-k)^{p^*} (\text{meas}(A_{k,n})) \leq C (\text{meas}(A_{k,n}))^{\gamma_3/\gamma_2\gamma_4}$, for all $h > k > \sigma$, which gives

$$\text{meas}(A_{k,n}) \leq \frac{C}{(h-k)^{p^*}} (\text{meas}(A_{k,n}))^{\gamma_3/\gamma_2\gamma_4}. \quad (86)$$

By a suitable choice of $s(\cdot)$, we have $\gamma_3/\gamma_2\gamma_4 > 1$. Now, we use the result of Stampacchia [11]; then there exists a constant M (independent of n) such that $\text{meas}(A_{k,n}) = 0$, for all $k \geq \sigma + M$; then $\|u_n\|_{L^\infty(\Omega)} \leq \sigma + M$, and we obtain Assertion 3. $\square$

We prove the following theorem.

Theorem 15. *There exists a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ and u such that u_n converges to u strongly in $W_0^{1,p(x)}(\Omega)$ and u is a weak solution of the following problem:*

$$\begin{aligned} Au + g(x, u, \nabla u) &= -\text{div}(F) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (87)$$

Proof. We need the following Lemmas.

Lemma 16. *Let c and d be two positive constants, for $\lambda = c^2/4d^2$, and let the function $\varphi_\lambda(t) = te^{\lambda t^2}$. Then one has $d\varphi'_\lambda(t) - c|\varphi_\lambda(t)| \geq d/2$.*

Proof. We have $\varphi'_\lambda(t) = e^{\lambda t^2} + 2\lambda t^2 e^{\lambda t^2}$; then

$$\begin{aligned} d\varphi'_\lambda(t) - c|\varphi_\lambda(t)| &= e^{\lambda t^2} (2\lambda t^2 - c|t| + d) \geq e^{\lambda t^2} \frac{d}{2}, \\ &\quad \forall t \in \mathbb{R}. \end{aligned} \quad (88)$$

Lemma 17. *There exists a positive constant C such that, for all n ,*

$$\|u_n\|_{W_0^{1,p(x)}(\Omega)} \leq C. \quad (89)$$

Proof. In view of Assertion 2, we have $\|u_n\|_{L^\infty(\Omega)} \leq C_1$, and we denote that $C_2 = b(C_1)$.

Let $v_n = \varphi_\lambda(u_n) \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, where $\lambda = C_2/(\alpha - (1/p^-))$ (with $\alpha > 1$).

Choose v_n as test function in (52), and it follows that

$$\begin{aligned} &\int_{\Omega} a(x, \nabla u_n) \nabla u_n \varphi'_\lambda(u_n) dx \\ &\quad + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_\lambda(u_n) dx \\ &= \int_{\Omega} F \nabla u_n \varphi'_\lambda(u_n) dx. \end{aligned} \quad (90)$$

Hence, consider

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla u_n \varphi'_\lambda(u_n) dx \\ &= \int_{\Omega} F \nabla u_n \varphi'_\lambda(u_n) dx \\ & \quad - \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_\lambda(u_n) dx; \end{aligned} \quad (91)$$

this implies that

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla u_n \varphi'_\lambda(u_n) dx \\ & \leq \int_{\Omega} |F| |\nabla u_n| \varphi'_\lambda(u_n) dx \\ & \quad + \int_{\Omega} |g_n(x, u_n, \nabla u_n)| |\varphi_\lambda(u_n)| dx, \end{aligned} \quad (92)$$

and we have

$$\begin{aligned} & \alpha \int_{\Omega} |\nabla u_n|^{p(x)} \varphi'_\lambda(u_n) dx \\ & \leq \int_{\Omega} \frac{|F|^{p'(x)}}{p'(x)} \varphi'_\lambda(u_n) dx + \int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} \varphi'_\lambda(u_n) dx \\ & \quad + \int_{\Omega} b(|u_n|) (|\nabla u_n|^{p(x)} + d(x)) |\varphi_\lambda(u_n)| dx; \end{aligned} \quad (93)$$

this implies that

$$\begin{aligned} & \left(\alpha - \frac{1}{p_-} \right) \int_{\Omega} |\nabla u_n|^{p(x)} \varphi'_\lambda(u_n) dx \\ & \quad - \int_{\Omega} C_2 |\nabla u_n|^{p(x)} |\varphi_\lambda(u_n)| dx \\ & \leq \frac{M_0}{p'_-} \int_{\Omega} |F|^{p'(x)} dx + C_2 \int_{\Omega} d(x) |\varphi_\lambda(u_n)| dx. \end{aligned} \quad (94)$$

Hence,

$$\begin{aligned} & \int_{\Omega} |\nabla u_n|^{p(x)} \left[\left(\alpha - \frac{1}{p_-} \right) \varphi'_\lambda(u_n) - C_2 |\varphi_\lambda(u_n)| \right] dx \\ & \leq \frac{M_0}{p'_-} \int_{\Omega} |F|^{p'(x)} dx + C_2 M_1 \int_{\Omega} d(x) dx. \end{aligned} \quad (95)$$

In view of Lemma 16 we have the following:

$$\left(\alpha - \frac{1}{p_-} \right) \varphi'_\lambda(u_n) - C_2 |\varphi_\lambda(u_n)| \geq \frac{(\alpha - (1/p_-))}{2}. \quad (96)$$

Hence,

$$\begin{aligned} & \frac{1}{2} \left(\alpha - \frac{1}{p_-} \right) \int_{\Omega} |\nabla u_n|^{p(x)} dx \\ & \leq \frac{M_0}{p'_-} \int_{\Omega} |F|^{p'(x)} dx + C_2 M_1 \int_{\Omega} d(x) dx. \end{aligned} \quad (97)$$

Therefore, there exists a positive constant C such that

$$\|u_n\|_{W_0^{1,p(x)}(\Omega)} \leq C. \quad (98)$$

There exists a subsequence still denoted by $(u_n)_n$ and u such that

$$\begin{aligned} & u_n \text{ converges to } u \text{ a.e. in } \Omega, \\ & \text{weakly in } W_0^{1,p(x)}(\Omega) \text{ and in } L^\infty(\Omega) \text{ weakly-}^*. \end{aligned} \quad (99)$$

Lemma 18. *One has the following:*

$$\lim_{n \rightarrow \infty} \int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \nabla (u_n - u) dx = 0, \quad (100)$$

and u_n converges to u strongly in $W_0^{1,p(x)}(\Omega)$ as n tends to $+\infty$.

Proof. We choose $u_n - u$ as a function test in the approximate problem (99) and we obtain

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla (u_n - u) dx \\ & \quad + \int_{\Omega} g_n(x, u_n, \nabla u_n) (u_n - u) dx \\ &= \int_{\Omega} F \nabla (u_n - u) dx. \end{aligned} \quad (101)$$

This implies that

$$\begin{aligned} & \int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \nabla (u_n - u) dx \\ &= - \int_{\Omega} g_n(x, u_n, \nabla u_n) (u_n - u) dx \\ & \quad - \int_{\Omega} a(x, \nabla u) \nabla (u_n - u) dx + \int_{\Omega} F \nabla (u_n - u) dx. \end{aligned} \quad (102)$$

Since $F \in L^{p'(x)}(\Omega)$, $a(x, \nabla u) \in (L^{p'(x)}(\Omega))^N$, and u_n converges to u a.e. in Ω , weakly in $W_0^{1,p(x)}(\Omega)$, and in $L^\infty(\Omega)$ weakly-*, then

$$\begin{aligned} & \int_{\Omega} F \nabla (u_n - u) dx \longrightarrow 0, \\ & \int_{\Omega} a(x, \nabla u) \nabla (u_n - u) dx \longrightarrow 0. \end{aligned} \quad (103)$$

We prove that $\int_{\Omega} g_n(x, u_n, \nabla u_n)(u_n - u)dx \rightarrow 0$. Indeed, we have the following:

$$\begin{aligned}
 & \left| \int_{\Omega} g_n(x, u_n, \nabla u_n)(u_n - u)dx \right| \\
 & \leq \int_{\Omega} b(|u_n|) \left(d(x) + |\nabla u_n|^{p(x)} \right) |u_n - u| dx \\
 & \leq \int_{\Omega} b(c) |\nabla u_n|^{p(x)} |u_n - u| dx \\
 & \quad + \int_{\Omega} b(c) d(x) |u_n - u| dx \\
 & \leq b(c) \int_{\Omega} |\nabla u_n|^{p(x)} |u_n - u| dx \\
 & \quad + \int_{\Omega} b(c) d(x) |u_n - u| dx.
 \end{aligned} \tag{104}$$

Since $d(x) \in L^1(\Omega)$, $|\nabla u_n|^{p(x)}$ is bounded in $L^1(\Omega)$, and $u_n \rightharpoonup u$ in $L^\infty(\Omega)$ weakly-*, then

$$\begin{aligned}
 & \int_{\Omega} |\nabla u_n|^{p(x)} |u_n - u| dx \rightarrow 0, \\
 & \int_{\Omega} b(c) d(x) |u_n - u| dx \rightarrow 0.
 \end{aligned} \tag{105}$$

Therefore, $\int_{\Omega} g_n(x, u_n, \nabla u_n)(u_n - u)dx \rightarrow_{n \rightarrow \infty} 0$. Finally, we obtain

$$\int_{\Omega} (a(x, \nabla u_n) - a(x, \nabla u)) \nabla (u_n - u) dx \rightarrow 0. \tag{106}$$

We conclude, by [9], that u_n converges to u strongly in $W_0^{1,p(x)}(\Omega)$ as n tends to $+\infty$. $\square$

6. Results of Existence

We state the following two theorems.

Theorem 19. Let $f \in L^\infty(\Omega)$ and let $F \in (L^{s(x)}(\Omega))^N$ with $s(\cdot) > N/(p(\cdot) - 1)$.

Then there exists a solution $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ of the following problem:

$$\begin{aligned}
 Au + g(x, u, \nabla u) &= f - \operatorname{div}(F) \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial\Omega,
 \end{aligned} \tag{107}$$

in the sense that

$$\begin{aligned}
 & \int_{\Omega} a(x, \nabla u) \nabla v dx + \int_{\Omega} g(x, u, \nabla u) v dx \\
 &= \int_{\Omega} f v dx + \int_{\Omega} F \nabla v dx, \\
 & \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega).
 \end{aligned} \tag{108}$$

Proof. Without loss of generality, we can assume that $f = 0$. Let $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, and we have

$$\int_{\Omega} a(x, \nabla u_n) \nabla v dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) v dx = \int_{\Omega} F \nabla v dx. \tag{109}$$

By Lemma 18, we have $\nabla u_n \rightarrow \nabla u$ strongly in $L^{p(x)}(\Omega)$, and then $\nabla u_n \rightarrow \nabla u$ a.e. in Ω . We use (46) and (49) and passing to the limit, we obtain the following:

$$\begin{aligned}
 & \int_{\Omega} a(x, \nabla u) \nabla v dx + \int_{\Omega} g(x, u, \nabla u) v dx = \int_{\Omega} F \nabla v dx, \\
 & \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega).
 \end{aligned} \tag{110}$$

$\square$

Theorem 20. Let $\mu \in M_b(\Omega)$. Under the assumptions (45)–(50), there exists a solution u of problem (51) in the sense that $u \in W_0^{1,p(x)}(\Omega)$, $g(x, u, \nabla u) \in L^1(\Omega)$, and

$$\begin{aligned}
 & \int_{\Omega} a(x, \nabla u) \nabla v dx + \int_{\Omega} g(x, u, \nabla u) v dx = \int_{\Omega} v d\mu, \\
 & \forall v \in C_0^\infty(\Omega) \text{ iff } \mu \in M_b^{p(x)}(\Omega).
 \end{aligned} \tag{111}$$

Proof. Suppose there exists a solution u of problem (51), with $u \in W_0^{1,p(x)}(\Omega)$ and $g(x, u, \nabla u) \in L^1(\Omega)$. According to (46), we have $a(x, \nabla u) \in (L^{p'(x)}(\Omega))^N$; then $-\operatorname{div}(a(x, \nabla u)) \in W^{-1,p'(x)}(\Omega)$, and $\mu = g(x, u, \nabla u) - \operatorname{div}(a(x, \nabla u)) = f - \operatorname{div}(a(x, \nabla u))$ in $L^1(\Omega) + W^{-1,p'(x)}(\Omega)$; with $f = g(x, u, \nabla u)$ and by Theorem 7, we have $\mu \in M_b^{p(x)}(\Omega)$.

Conversely, if $\mu \in M_b^{p(x)}(\Omega)$, by Theorem 7, $\mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$, and then $\mu = f - \operatorname{div}(F)$, where $f \in L^1(\Omega)$ and $F \in (L^{p'(x)}(\Omega))^N$. Let $(f_n)_n$ be a sequence in $L^\infty(\Omega)$ such that f_n converges to f strongly in $L^1(\Omega)$ and let $(F_n)_n$ be a sequence in $(L^\infty(\Omega))^N$ such that F_n converges to F strongly in $(L^{p'(x)}(\Omega))^N$. By Theorem 19, there exists a solution u_n of

$$\begin{aligned}
 Au_n + g(x, u_n, \nabla u_n) &= f_n - \operatorname{div}(F_n) \quad \text{in } \Omega, \\
 u_n &= 0 \quad \text{on } \partial\Omega,
 \end{aligned} \tag{112}$$

such that $u_n \in W_0^{1,p(x)}(\Omega)$, $g(x, u_n, \nabla u_n) \in L^1(\Omega)$, and, for all $v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}
 & \int_{\Omega} a(x, \nabla u_n) \nabla v dx + \int_{\Omega} g(x, u_n, \nabla u_n) v dx \\
 &= \int_{\Omega} f_n v dx + \int_{\Omega} F_n \nabla v dx.
 \end{aligned} \tag{113}$$

We choose $v = \varphi(T_\sigma(u_n))$, where φ is given by Lemma 16 and σ is given by (50); with $d = \alpha/2$ and $b = b(\sigma)$, where $b(s)$ is given by (49), we obtain

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla \varphi(T_\sigma(u_n)) dx \\ & + \int_{\Omega} g(x, u_n, \nabla u_n) \varphi(T_\sigma(u_n)) dx \\ & = \int_{\Omega} f_n \varphi(T_\sigma(u_n)) dx + \int_{\Omega} F_n \nabla \varphi(T_\sigma(u_n)) dx, \end{aligned} \quad (114)$$

which implies that

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \varphi'_\sigma \nabla T_\sigma(u_n) dx + \int_{\Omega} g(x, u_n, \nabla u_n) \varphi_\sigma dx \\ & = \int_{\Omega} f_n \varphi_\sigma dx + \int_{\Omega} F_n \varphi'_\sigma \nabla T_\sigma(u_n) dx, \end{aligned} \quad (115)$$

where $\varphi_\sigma = \varphi(T_\sigma(u_n))$, et $\varphi'_\sigma = \varphi'(T_\sigma(u_n))$.

According to (47) and Young's inequality, we obtain

$$\begin{aligned} & \alpha \int_{\Omega} |\nabla T_\sigma(u_n)|^{p(x)} \varphi'_\sigma dx + \int_{\Omega} g(x, u_n, \nabla u_n) \varphi_\sigma dx \\ & \leq \varphi(\sigma) \int_{\Omega} |f_n| dx + \varphi'(\sigma) \int_{\Omega} |F_n|^{p'(x)} dx \\ & + \frac{\alpha}{2} \int_{\Omega} |\nabla T_\sigma(u_n)|^{p(x)} \varphi'_\sigma dx. \end{aligned} \quad (116)$$

Hence, consider

$$\begin{aligned} & \frac{\alpha}{2} \int_{\Omega} |\nabla T_\sigma(u_n)|^{p(x)} \varphi'_\sigma dx + \int_{\Omega} g(x, u_n, \nabla u_n) \varphi_\sigma dx \\ & \leq C(\varphi(\sigma) + \varphi'(\sigma)). \end{aligned} \quad (117)$$

We can write the following:

$$\begin{aligned} & \int_{\Omega} g(x, u_n, \nabla u_n) \varphi_\sigma dx \\ & = \int_{\{|u_n| < \sigma\}} g(x, u_n, \nabla u_n) \varphi_\sigma dx \\ & + \int_{\{|u_n| \geq \sigma\}} g(x, u_n, \nabla u_n) \varphi_\sigma dx. \end{aligned} \quad (118)$$

Using (49), we get

$$\begin{aligned} & \int_{\{|u_n| < \sigma\}} g(x, u_n, \nabla u_n) \varphi_\sigma dx \\ & \leq \int_{\{|u_n| < \sigma\}} \varphi_\sigma b(T_\sigma(u_n)) (|\nabla T_\sigma(u_n)|^{p(x)} + d(x)) dx \end{aligned} \quad (119)$$

and using (50), we obtain

$$\begin{aligned} & \int_{\{|u_n| \geq \sigma\}} g(x, u_n, \nabla u_n) \varphi_\sigma dx \\ & \geq \rho \varphi(\sigma) \int_{\{|u_n| \geq \sigma\}} |\nabla u_n|^{p(x)} dx. \end{aligned} \quad (120)$$

Now we use (119) and (120) in (117) and we write the following:

$$\begin{aligned} & \int_{\Omega} |\nabla T_\sigma(u_n)|^{p(x)} \left[\frac{\alpha}{2} \varphi'_\sigma - b(\sigma) |\varphi(\sigma)| \right] \\ & + \rho \varphi(\sigma) \int_{\{|u_n| \geq \sigma\}} |\nabla u_n|^{p(x)} dx \leq C. \end{aligned} \quad (121)$$

In view of Lemma 16, we have $(\alpha/2)\varphi'_\sigma - b(\sigma)|\varphi(\sigma)| \geq \alpha/4$, and then

$$\int_{\Omega} |\nabla T_\sigma(u_n)|^{p(x)} dx + \int_{\{|u_n| \geq \sigma\}} |\nabla u_n|^{p(x)} dx \leq C. \quad (122)$$

Hence, there is $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \text{ converges to } u \text{ a.e. in } \Omega \text{ and weakly in } W_0^{1,p(x)}(\Omega). \quad (123)$$

□

Let us show that we have u_n converging to u strongly in $W_0^{1,p(x)}(\Omega)$. We start by proving that

$$\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|u_n| \geq k\}} |\nabla u_n|^{p(x)} dx = 0. \quad (124)$$

Indeed, for $k \geq 2$, choose $v = T_1(u_n - T_{k-1}(u_n)) \equiv \psi_{k-1}(u_n)$ as a test function in (113), and using (47) and Young's inequality, we get

$$\begin{aligned} & \alpha \int_{\Omega} |\nabla \psi_{k-1}(u_n)|^{p(x)} dx + \int_{\Omega} g(x, u_n, \nabla u_n) \psi_{k-1}(u_n) dx \\ & \leq \int_{\{|u_n| \geq k-1\}} |f_n| dx + \int_{\{k-1 \leq u_n \leq k\}} |F_n|^{p'(x)} dx \\ & + \frac{\alpha}{2} \int_{\Omega} |\nabla \psi_{k-1}(u_n)|^{p(x)} dx. \end{aligned} \quad (125)$$

We have $\psi_{k-1}(s)$, $s \geq 0$, for all $s \in \mathbb{R}$, and $\psi_{k-1}(s) = 0$, for all $|s| \leq k-1$; hence,

$$|g(x, u_n, \nabla u_n)| \chi_{\{u_n \geq k\}} \leq g(x, u_n, \nabla u_n) \psi_{k-1}(u_n). \quad (126)$$

Therefore,

$$\begin{aligned} & \int_{\{|u_n| \geq k\}} |g(x, u_n, \nabla u_n)| dx \\ & \leq \int_{\{|u_n| \geq k\}} |f_n| dx + \int_{\{k-1 \leq u_n \leq k\}} |F_n|^{p'(x)} dx, \end{aligned} \quad (127)$$

and, by (125), we have

$$\begin{aligned} & \frac{\alpha}{2} \int_{\Omega} |\nabla \psi_{k-1}(u_n)|^{p(x)} dx \\ & \leq \int_{\{|u_n| \geq k\}} |f_n| dx + \int_{\{k-1 \leq u_n \leq k\}} |F_n|^{p'(x)} dx. \end{aligned} \quad (128)$$

Since $(u_n)_n$ is bounded in $L^1(\Omega)$, then $\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \text{meas}\{x \in \Omega/k - 1 \leq |u_n|\} = 0$. We use (127) and (128) and since f_n strongly converges in $L^1(\Omega)$ and $|F_n|$ strongly converges in $L^{p'(x)}(\Omega)$, we conclude that

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{|u_n| \geq k\}} |g(x, u_n, \nabla u_n)| dx &= 0, \\ \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla \psi_{k-1}(u_n)|^{p(x)} dx &= 0. \end{aligned} \quad (129)$$

Now we show that $\forall k \geq 0$ and

$$T_k(u_n) \text{ converges to } T_k(u) \text{ strongly in } W_0^{1,p(x)}(\Omega). \quad (130)$$

Indeed, let $k \geq 1$ be fixed, and choose $v = \varphi(T_k(u_n) - T_k(u))$ as test function in (113) where φ is the function of Lemma 16 with $c = 1$ and $d = b(k)/\alpha$, and we obtain

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \varphi'(T_k(u_n) - T_k(u)) \nabla(T_k(u_n) - T_k(u)) dx \\ & + \int_{\Omega} g(x, u_n, \nabla u_n) \varphi(T_k(u_n) - T_k(u)) dx \\ & = \int_{\Omega} f_n \varphi(T_k(u_n) - T_k(u)) dx \\ & + \int_{\Omega} F_n \varphi'(T_k(u_n) - T_k(u)) \nabla(T_k(u_n) - T_k(u)) dx. \end{aligned} \quad (131)$$

To simplify writing, we put $\varphi'_n = \varphi'(T_k(u_n) - T_k(u))$ and $\varphi_n = \varphi(T_k(u_n) - T_k(u))$.

We have u_n which converges to u a.e. in Ω , φ_n which converges to $\varphi(0) = 0$ in $L^\infty(\Omega)$ weakly-*, and φ'_n which converges to $\varphi'(0) = 1$, and since f_n strongly converges to f in $L^1(\Omega)$, we obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n \varphi(T_k(u_n) - T_k(u)) dx = 0. \quad (132)$$

Since $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega)$ and $(F_n)_n$ converges to F strongly in $(L^{p'(x)}(\Omega))^N$, we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\Omega} F_n \varphi'(T_k(u_n) - T_k(u)) \\ \times \nabla(T_k(u_n) - T_k(u)) dx &= 0. \end{aligned} \quad (133)$$

Write the first term on the left-hand side of (131) as

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \varphi'(T_k(u_n) - T_k(u)) \nabla(T_k(u_n) - T_k(u)) dx \\ & = \int_{\Omega} a(x, \nabla T_k(u_n)) \varphi'_n \nabla(T_k(u_n) - T_k(u)) dx \\ & + \int_{\Omega} a(x, \nabla G_k(u_n)) \varphi'_n \nabla(T_k(u_n) - T_k(u)) dx, \end{aligned} \quad (134)$$

where $G_k(s) = s - T_k(s)$. Since $a(x, \nabla G_k(u_n))$ is bounded in $(L^{p'(x)}(\Omega))^N$ and $\nabla T_k(u) \chi_{\{|u_n| \geq k\}}$ converges to zero in $(L^\infty(\Omega))$ weakly-* as n goes to $+\infty$, then

$$\lim_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla G_k(u_n)) \varphi'_n \nabla(T_k(u_n) - T_k(u)) dx = 0. \quad (135)$$

The first term on the right-hand side of (134) is

$$\begin{aligned} & \int_{\Omega} a(x, \nabla T_k(u_n)) \varphi'_n \nabla(T_k(u_n) - T_k(u)) dx \\ & = \int_{\Omega} (a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))) \\ & \quad \times \varphi'_n \nabla(T_k(u_n) - T_k(u)) dx \\ & + \int_{\Omega} a(x, \nabla T_k(u)) \varphi'_n \nabla(T_k(u_n) - T_k(u)) dx. \end{aligned} \quad (136)$$

Since $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $(L^{p'(x)}(\Omega))^N$ and $a(x, \nabla T_k(u)) \in (L^{p(x)}(\Omega))^N$, then

$$\lim_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla T_k(u)) \varphi'_n \nabla(T_k(u_n) - T_k(u)) dx = 0. \quad (137)$$

We can write that

$$\begin{aligned} & \int_{\Omega} g(x, u_n, \nabla u_n) \varphi(T_k(u_n) - T_k(u)) dx \\ & = \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi(T_k(u_n) - T_k(u)) dx \\ & + \int_{\{|u_n| > k\}} g(x, u_n, \nabla u_n) \varphi(T_k(u_n) - T_k(u)) dx. \end{aligned} \quad (138)$$

In view of (50), we have

$$\int_{\{|u_n| > k\}} g(x, u_n, \nabla u_n) \varphi(T_k(u_n) - T_k(u)) dx \geq 0. \quad (139)$$

On the other hand, we have

$$\begin{aligned}
& \left| \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n) \varphi(T_k(u_n) - T_k(u)) dx \right| \\
& \leq b(k) \int_{\Omega} d(x) |\varphi_n| dx \\
& \quad + b(k) \int_{\Omega} |\nabla T_k(u_n)|^{p(x)} |\varphi_n| dx, \\
& \leq b(k) \int_{\Omega} d(x) |\varphi_n| dx + \frac{b(k)}{\alpha} \\
& \quad \times \left[\int_{\Omega} (a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))) \right. \\
& \quad \times \nabla(T_k(u_n) - T_k(u)) |\varphi_n| dx \\
& \quad + \int_{\Omega} |a(x, \nabla T_k(u_n)) \nabla T_k(u)| |\varphi_n| dx \\
& \quad \left. + \int_{\Omega} a(x, \nabla T_k(u)) \nabla(T_k(u_n) - T_k(u)) |\varphi_n| dx \right].
\end{aligned} \tag{140}$$

Since $d(x) \in L^1(\Omega)$, φ_n converges to 0 a.e. in Ω and in $L^\infty(\Omega)$ weakly-*, the sequence $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega)$, and $a(x, \nabla T_k(u)) \in (L^{p'(x)}(\Omega))^N$, then

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{\Omega} d(x) |\varphi_n| dx = 0, \\
& \lim_{n \rightarrow \infty} \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u) |\varphi_n| dx = 0.
\end{aligned} \tag{141}$$

Now since $a(x, \nabla T_k(u_n))$ is bounded in $(L^{p'(x)}(\Omega))^N$ and $\nabla T_k(u) \varphi_n$ converges to zero strongly in $(L^{p(x)}(\Omega))^N$, then we have

$$\begin{aligned}
& \int_{\Omega} |a(x, \nabla T_k(u_n)) \nabla T_k(u)| |\varphi_n| dx \\
& \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|a(x, \nabla T_k(u_n))\|_{p'(x)} \\
& \quad \times \|\nabla T_k(u) |\varphi_n|\|_{p(x)} \\
& \leq C \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|\nabla T_k(u) |\varphi_n|\|_{p(x)} \\
& \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \left(\int_{\Omega} |\nabla T_k(u) |\varphi_n||^{p(x)} dx \right)^{1/\gamma},
\end{aligned} \tag{142}$$

where

$$\gamma = \begin{cases} p^- & \text{if } \|\nabla T_k(u_n) |\varphi_n|\|_{p(x)} \geq 1, \\ p^+ & \text{if } \|\nabla T_k(u_n) |\varphi_n|\|_{p(x)} < 1. \end{cases} \tag{143}$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |a(x, \nabla T_k(u_n)) \nabla T_k(u)| |\varphi_n| dx = 0. \tag{144}$$

Now using (131) to (141), we conclude that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{\Omega} [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \\
& \quad \times \nabla(T_k(u_n) - T_k(u)) \left(\varphi'_n - \frac{b(k)}{\alpha} |\varphi_n| \right) dx = 0,
\end{aligned} \tag{145}$$

and in view of [9], we have (130).

Now we prove that

$$u_n \text{ converges to } u \text{ strongly in } W_0^{1,p(x)}(\Omega). \tag{146}$$

Indeed, let E be a measurable subset of Ω , and then consider

$$\begin{aligned}
\int_E |\nabla u_n|^{p(x)} dx &= \int_{E \cap \{|u_n| \leq k\}} |\nabla u_n|^{p(x)} dx \\
&+ \int_{E \cap \{|u_n| > k\}} |\nabla u_n|^{p(x)} dx.
\end{aligned} \tag{147}$$

We have on the one hand

$$\int_{E \cap \{|u_n| > k\}} |\nabla u_n|^{p(x)} dx \leq \int_{\{|u_n| > k\}} |\nabla u_n|^{p(x)} dx, \tag{148}$$

and since

$$\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|u_n| > k\}} |\nabla u_n|^{p(x)} dx = 0, \tag{149}$$

then, for $\varepsilon > 0$, there exists $k \geq \sigma$ such that

$$\int_{E \cap \{|u_n| > k\}} |\nabla u_n|^{p(x)} dx \leq \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}. \tag{150}$$

Furthermore, $\int_{E \cap \{|u_n| \leq k\}} |\nabla u_n|^{p(x)} dx \leq \int_E |\nabla T_k(u_n)|^{p(x)} dx$, and since $T_k(u_n)$ converges to $T_k(u)$ strongly in $W_0^{1,p(x)}(\Omega)$, then there exists $\delta > 0$ such that, for $\text{mes}(E) < \delta$,

$$\int_{E \cap \{|u_n| \leq k\}} |\nabla u_n|^{p(x)} dx \leq \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}. \tag{151}$$

In view of (150) and (151) we deduce that $(|\nabla u_n|^{p(x)})_n$ is equintegrable. Since ∇u_n converges to ∇u a.e., then by the Vitali theorem we obtain (146).

Let us show that

$$g(x, u_n, \nabla u_n) \text{ converges to } g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{152}$$

Since $g(x, u_n, \nabla u_n)$ converges to $g(x, u, \nabla u)$, a.e. in Ω , it suffices to show that the sequence $(g(x, u_n, \nabla u_n))_n$ is equintegrable. Let E be a measurable subset of Ω ; we have

$$\begin{aligned}
\int_E |g(x, u_n, \nabla u_n)| dx &\leq \int_E |g(x, u_n, \nabla T_k(u_n))| dx \\
&+ \int_{\{|u_n| > k\}} |g(x, u_n, \nabla u_n)| dx.
\end{aligned} \tag{153}$$

Since $\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|u_n| > k\}} |g(x, u_n, \nabla u_n)| dx = 0$, then there exists $k \geq \sigma$ such that

$$\int_{E \cap \{|u_n| > k\}} |g(x, u_n, \nabla u_n)| dx \leq \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}. \quad (154)$$

In view of (49), we have

$$\begin{aligned} & \int_{E \cap \{|u_n| \leq k\}} |g(x, u_n, \nabla u_n)| dx \\ & \leq b(k) \int_E (d(x) + |\nabla T_k(u_n)|^{p(x)}) dx, \end{aligned} \quad (155)$$

since $d \in L^1(\Omega)$ and $T_k(u_n)$ converges to $T_k(u)$ strongly in $W_0^{1,p(x)}(\Omega)$; therefore, there exists $\delta > 0$ such that $\text{meas}(E) < \delta$ and $\int_{E \cap \{|u_n| \leq k\}} |g(x, u_n, \nabla u_n)| dx \leq \varepsilon/2$, $\forall n \in \mathbb{N}$, and this shows that $(g(x, u_n, \nabla u_n))_n$ is equintegrable.

Now, let u_n be a solution of the approximate problem; then we have

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla v dx + \int_{\Omega} g(x, u_n, \nabla u_n) v dx \\ & = \int_{\Omega} f_n v dx + \int_{\Omega} F_n \nabla v dx, \end{aligned} \quad (156)$$

$$\forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega).$$

Letting n tend to $+\infty$, we conclude that u satisfies the following:

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u) \nabla v dx + \int_{\Omega} g(x, u, \nabla u) v dx = \int_{\Omega} v d\mu, \\ & \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega). \end{aligned} \quad (157)$$

7. A Result of Nonexistence

Theorem 21. Let $\lambda \in M_b(\Omega)$ be positive and concentrated in a subset E of Ω such that $\text{cap}_{p(\cdot)}(E) = 0$. Let $(f_n)_n$ be a sequence of positive functions in $L^1(\Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n \varphi dx = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C^0(\overline{\Omega}). \quad (158)$$

Assume (49), (50), and that g satisfies $g(x, s, \xi) \geq 0$, a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, and $\forall \xi \in \mathbb{R}^N$.

Let u_n be solution of

$$\begin{aligned} & Au_n + g(x, u_n, \nabla u) = f_n \quad \text{in } \Omega, \\ & u_n = 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (159)$$

in the sense that

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u_n) \nabla v dx + \int_{\Omega} g(x, u_n, \nabla u) v dx = \int_{\Omega} f_n v dx, \\ & \forall v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega). \end{aligned} \quad (160)$$

Then, there exists $k > 0$, such that $T_k(u_n)$ converges to 0 strongly in $W_0^{1,p(x)}(\Omega)$, the sequence u_n converges to 0 weakly in $W_0^{1,p(x)}(\Omega)$, and

$$\lim_{n \rightarrow \infty} \int_{\Omega} g(x, u_n, \nabla u) \varphi dx = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C_0^1(\Omega). \quad (161)$$

Proof. Consider the following.

Step 1. In view of the proof of Theorem 19 and the assumptions made about a and g , we have the sequence $(u_n)_n$ which is bounded in $W_0^{1,p(x)}(\Omega)$ and then there exists a subsequence, still denoted by $(u_n)_n$, such that u_n converges to u weakly in $W_0^{1,p(x)}(\Omega)$ and a.e. in Ω . By (46), there exists $G \in (L^{p'(x)}(\Omega))^N$ such that $a(x, \nabla u_n)$ converges to G weakly in $(L^{p'(x)}(\Omega))^N$.

The function b is continuous and let $k > 0$ such that $b(k)k \leq \alpha/2$; choose $v = (k - T_k(u_n))\psi_\delta \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ as test function in (160), where ψ_δ is given by Lemma 16, and then we obtain

$$\begin{aligned} & - \int_{\Omega} (x, \nabla T_k(u_n)) \nabla T_k(u_n) \psi_\delta dx \\ & + \int_{\Omega} a(x, \nabla(u_n)) \nabla \psi_\delta (k - T_k(u_n)) dx \\ & \times \int_{\Omega} g(x, u_n, \nabla u) (k - T_k(u_n)) \psi_\delta dx \\ & = \int_{\Omega} f_n \psi_\delta (k - T_k(u_n)) dx. \end{aligned} \quad (162)$$

Since $\nabla \psi_\delta (k - T_k(u_n))$ converges to $\nabla \psi_\delta (k - T_k(u))$ strongly in $(L^{p(x)}(\Omega))^N$ as n tends to ∞ and $\nabla \psi_\delta (k - T_k(u))$ converges to zero strongly in $(L^{p(x)}(\Omega))^N$ as δ tends to 0, then

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{\Omega} a(x, \nabla(u_n)) \nabla \psi_\delta (k - T_k(u_n)) dx \\ & = \int_{\Omega} G \nabla \psi_\delta (k - T_k(u)) dx = 0, \end{aligned} \quad (163)$$

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{\Omega} f_n \psi_\delta (k - T_k(u_n)) dx = 0.$$

In view of (49) we have $k - T_k(u_n) = 0$, for $u_n > k$, since $u_n \geq 0$, and we have

$$\begin{aligned} & \left| \int_{\Omega} g(x, u_n, \nabla u) (k - T_k(u_n)) \psi_\delta dx \right| \\ & \leq \int_{(0 \leq u_n \leq k)} b(u_n) (k - T_k(u_n)) \psi_\delta \\ & \quad \times (d(x) + |\nabla u_n|^{p(x)}) dx. \\ & \leq \int_{\Omega} b(k) k \psi_\delta(x) d(x) dx + \int_{\Omega} b(k) k c |\nabla u_n|^{p(x)} dx, \\ & \leq \frac{\alpha}{2} \int_{\Omega} d(x) \psi_\delta(x) dx + \frac{\alpha}{2} \int_{\Omega} |\nabla T_k(u_n)|^{p(x)} dx. \end{aligned} \quad (164)$$

Since ψ_δ converges to 0 weakly in $L^\infty(\Omega)^*$ and $d \in L^1(\Omega)$, then

$$\lim_{\delta \rightarrow 0} \int_{\Omega} d(x) \psi_\delta(x) dx = 0. \quad (165)$$

We use (47) and (163)–(165) in (162); we conclude that

$$\lim_{\delta \rightarrow 0} \int_{\Omega} |\nabla T_k(u_n)|^{p(x)} \psi_\delta(x) dx = 0. \quad (166)$$

Step 2. Choose $v = T_k(u_n)(1 - \psi_\delta) \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ as test function in (160); we have the following:

$$\begin{aligned} & \int_{\Omega} a(x, \nabla T_k(u_n)) \nabla T_k(u_n) (1 - \psi_\delta) dx \\ & - \int_{\Omega} a(x, \nabla u_n) \nabla \psi_\delta T_k(u_n) dx \\ & + \int_{\Omega} g(x, u_n, \nabla u) T_k(u_n) (1 - \psi_\delta) dx \\ & = \int_{\Omega} f_n T_k(u_n) (1 - \psi_\delta) dx. \end{aligned} \quad (167)$$

We pass to the limit as n tends to $+\infty$ and δ tends to zero in (167), and using the same arguments of Step 1, we conclude that

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \alpha \int_{\Omega} |\nabla T_k(u_n)|^{p(x)} (1 - \psi_\delta(x)) dx = 0. \quad (168)$$

Using Step 1, we deduce that $\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla T_k(u_n)|^{p(x)} dx = 0$, and we have $T_k(u_n)$ which converges to 0 strongly in $W_0^{1,p(x)}(\Omega)$. Since the limit is independent of the choice of the subsequence, the sequence $(T_k(u_n))_n$ converges strongly in $W_0^{1,p(x)}(\Omega)$; hence u_n converges to $u = 0$ weakly in $W_0^{1,p(x)}(\Omega)$ and ∇u_n converges to 0 a.e. in Ω . Then, we obtain $G = 0$.

Now, let $\varphi \in C_0^1(\Omega)$, and we obtain

$$\int_{\Omega} a(x, \nabla u_n) \nabla \varphi dx + \int_{\Omega} g(x, u_n, \nabla u_n) \varphi dx = \int_{\Omega} f_n \varphi dx. \quad (169)$$

Passing to the limit, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} a(x, \nabla u_n) \nabla \varphi dx &= \int_{\Omega} G \nabla \varphi dx = 0, \\ \lim_{n \rightarrow \infty} \int_{\Omega} f_n \varphi dx &= \lim_{n \rightarrow \infty} \int_{\Omega} \varphi d\mu. \end{aligned} \quad (170)$$

Therefore, consider

$$\lim_{n \rightarrow \infty} \int_{\Omega} g(x, u_n, \nabla u_n) \varphi dx = \int_{\Omega} \varphi d\mu. \quad (171)$$

The proof of Theorem 21 is complete. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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