Research Article

New Čebyšev Type Inequalities and Applications for Functions of Self-Adjoint Operators on Complex Hilbert Spaces

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Several new error bounds for the Čebyšev functional under various assumptions are proved. Applications for functions of self-adjoint operators on complex Hilbert spaces are provided as well.

1. Introduction

In recent years the approximation problem of the Riemann-Stieltjes integral \[ \int_{a}^{b} f(t) du(t) \] via the famous Čebyšev functional \[ T(f, g) = \frac{1}{b-a} \int_{a}^{b} f(t) g(t) dt - \frac{1}{b-a} \int_{a}^{b} f(t) dt \cdot \frac{1}{b-a} \int_{a}^{b} g(t) dt \] increasingly became essential. In 1882, Čebyšev [1] derived an interesting result involving two absolutely continuous functions whose first derivatives are continuous and bounded and is given by

\[ |T(f, g)| \leq \frac{1}{12} (b-a)^2 \| f' \|_{\infty} \| g' \|_{\infty}, \] (2)

and the constant 1/12 is the best possible.

In 1935, Grüss [2] proved another result for two integrable mappings \( f, g \) such that \( \phi \leq f(x) \leq \Phi \) and \( \gamma \leq f(x) \leq \Gamma \); the inequality

\[ |T(f, g)| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma) \] (3)

holds, and the constant 1/4 is the best possible.

In [3, p 302] Beesack et al. have proved the following Čebyšev inequality for absolutely continuous functions whose first derivatives belong to \( L_p \) spaces:

\[ |T(f, g)| \leq \frac{b-a}{4} \left( \frac{2^p - 1}{p(p+1)} \right)^{1/p} \left( \frac{2^q - 1}{q(q+1)} \right)^{1/q} \| f' \|_p \| g' \|_q, \] (4)

where \( \| h \|_p := \left( \int_a^b |h(t)|^p dt \right)^{1/p}, \forall p > 1, \text{ and } (1/p) + (1/q) = 1. \)

For the constant

\[ \omega(p, q) := \frac{1}{4} \left( \frac{2^p - 1}{p(p+1)} \right)^{1/p} \left( \frac{2^q - 1}{q(q+1)} \right)^{1/q} \] (5)

we have

\[ \frac{1}{8} \leq \omega(p, q) \leq \frac{1}{4} \] (6)

for all \( q = p/(p-1), p > 1. \) Furthermore, we have the following particular cases in (4).

(1) If \( p = q = 2, \) we have

\[ |T(f, g)| \leq \frac{b-a}{8} \| f' \|_2 \| g' \|_2. \] (7)

(2) If \( q \to \infty, \) we have

\[ |T(f, g)| \leq \frac{b-a}{4} \| f' \|_1 \| g' \|_{\infty}. \] (8)
In 1970, Ostrowski [4] has proved the following combination of the Čebyshev and Grüss results:

\[
|\mathcal{S}(f, g)| \leq \frac{1}{8} (b - a) (M - m) \|g'\|_\infty, \tag{9}
\]

where \( g \) is absolutely continuous with \( g' \in L_{\infty}[a,b] \) and \( f \) is Lebesgue integrable on \([a,b]\) and satisfying \( m \leq f(t) \leq M \), for all \( t \in [a,b] \). The constant \( 1/8 \) is the best possible.

In 1973, Lupas¸ [5] has improved Beesack et al. inequality (7), as follows:

\[
|\mathcal{S}(f, g)| \leq \frac{(b - a)}{\pi^2} \|f'\|_2 \|g'\|_2, \tag{10}
\]

provided that \( f, g \) are two absolutely continuous functions on \([a,b]\) with \( f', g' \in L_2[a,b] \), where \( \|h'\|_2 := \left(\int_a^b |h'(t)|^2 \, dt\right)^{1/2} \).

The constant \( 1/\pi^2 \) is the best possible.

More recently, and using the identity (13), page 246,

\[
\mathcal{S}(f, g) = \frac{1}{b - a} \int_a^b \left[f(t) - \frac{f(a) + f(b)}{2}\right] g(t) - \frac{1}{b - a} \int_a^b g(s) \, ds \, dt. \tag{11}
\]

Dragomir [6] has proved the following inequality.

**Theorem 1.** Let \( f : [a, b] \to \mathbb{C} \) be of bounded variation on \([a,b]\) and \( g : [a, b] \to \mathbb{C} \) a Lebesgue integrable function on \([a,b]\); then

\[
|\mathcal{S}(f, g)| \leq \frac{1}{2} \sqrt{V_a^b(f)} \cdot \int_a^b [g(t) - \frac{1}{b - a} \int_a^b g(s) \, ds] \, dt, \tag{12}
\]

where \( V_a^b(f) \) denotes the total variation of \( f \) on the interval \([a,b]\). The constant \( 1/2 \) is best possible in (12).

Another result when both functions are of bounded variation was considered in the same paper [6], as follows.

**Theorem 2.** If \( f, g : [a, b] \to \mathbb{C} \) are of bounded variation on \([a,b]\), then

\[
|\mathcal{S}(f, g)| \leq \frac{1}{4} \sqrt{V_a^b(f)} \cdot \sqrt{V_a^b(g)}. \tag{13}
\]

The constant \( 1/4 \) is best possible in (13).

Many authors have studied the functional (1) and, therefore, several bounds under various assumptions have been obtained; for more new results and generalizations the reader may refer to [6–21].

On other hand and in order to study the difference between two Riemann integral means, Barnett et al. [22] have proved the following estimates.

**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous function with the property that \( f' \in L_{\infty}[a,b] \); that is,

\[
\|f'\|_{\infty} := \text{ess sup}_{t \in [a,b]} |f'(t)|. \tag{14}
\]

Then for \( a \leq c < d \leq b \), we have the inequality

\[
\left|\int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(s) \, ds\right| \leq \frac{1}{4} + \left[\frac{(a + b)/2 - (c + d)/2}{(b - a) - (d - c)}\right]^2 \times [(b - a) - (d - c)] \|f'|_{\infty} \tag{15}
\]

The constant \( 1/4 \) in the first inequality and \( 1/2 \) in the second inequality are the best possible.

After that, Cerone and Dragomir [23] have obtained the following three results as well.

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be an absolutely continuous mapping. Then for \( a \leq c < d \leq b \), we have the inequalities

\[
\left|\int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(s) \, ds\right| \leq \left\{
\begin{array}{ll}
\frac{(b - a)}{(q + 1)^{1/q}} \left[1 + \left(\frac{\rho}{1 + \rho}\right)^q\right]^{1/q} & \\
\times [\lambda^{q+1} + \lambda^{q+1}]^{1/q} \|f'\|_p, & f' \in L_p [a,b], \\\n1 \leq p < \infty, & 1 + \frac{1}{p} = 1; \\
\frac{1}{2} [1 - \rho + |\rho - \lambda|] \|f'\|_1, & f' \in L_1 [a,b],
\end{array}
\right. \tag{16}
\]

where \((b - a)\rho = c - a, (b - a)\rho = d - c \) and \((b - a)\lambda = b - d). Both inequalities in (16) are sharp.

**Theorem 5.** Assume that the mapping \( f : [a, b] \to \mathbb{R} \) is of \( r\)-Hölder type on \([a,b]\). Then for \( a \leq c < d \leq b \), we have the inequality

\[
\left|\int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(s) \, ds\right| \leq H \left(\frac{(c - a)^{r+1} + (b - d)^{r+1}}{(r + 1) [(b-a) - (d-c)]}\right), \tag{17}
\]

Inequality (17) is best possible in the sense that we cannot put in the right-hand side a constant less than 1.
Theorem 6. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a mapping of bounded variation on \([a, b]\). The following bounds hold:

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d g(s) \, ds \right| \leq \left| \frac{b-a}{(b-a)-(d-c)} + \frac{c+d-a+b}{2} \right| \frac{\mathcal{V}(f)}{b-a};
\]

\[
\leq \left[ \frac{b-a-(d-c)}{2} + \frac{c+d-a+b}{2} \right] \frac{\mathcal{V}(f)}{b-a};
\]

\[
\frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{b-a} \int_a^b g(s) \, ds \in L_p[a, b],
\]

\[
\frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{b-a} \int_a^b g(s) \, ds \in L_1[a, b].
\]

(18)

where \( s_0 = (cb - ad)/(b-a)-(d-c) \) \in \([c, d]\).

In this paper by utilising amongst others the inequalities from Theorems 3–6, several new bounds for the Čebyšev functional \( \mathcal{T}(f, g) \) are provided.

The inequalities (15)–(18) are used in an essential way to obtain new error bounds for the \( \mathcal{T}(f, g) \), which gives a significant application for these inequalities. Applications for functions of self-adjoint operators on complex Hilbert Spaces are provided as well.

2. The Case When \( f \) Is of Bounded Variation

We may start with the following result.

Theorem 7. Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be such that \( f \) is of bounded variation on \([a, b]\) and \( g \) is absolutely continuous on \([a, b]\); then

\[
\| \mathcal{T}(f, g) \| \leq \frac{1}{8} \left( (b-a) \| g' \|_{\infty} \right)^{1/2},
\]

\[
\leq \left| \frac{b-a}{8} \int_a^b |h(t)| \, dt \right|^{1/2}, \quad \text{for} \ p \geq 1,
\]

\[
\| h \|_{p} := \left( \int_a^b |h(t)|^p \, dt \right)^{1/p}, \quad \text{for} \ p \geq 1,
\]

(19)

where \( \| \cdot \|_p \) are the usual Lebesgue norms; that is,

\[
\| h \|_{\infty} := \text{ess sup}_{t \in [a, b]} |h(t)|.
\]

Proof. Using integration by parts, we have

\[
\mathcal{T}(f, g) = -\frac{1}{b-a} \int_a^b \left( \int_a^t g(s) \, ds - \frac{t-a}{b-a} \int_a^b g(s) \, ds \right) df(t).
\]

(21)

It is known that for a continuous function \( p : [a, b] \rightarrow \mathbb{R} \) and a function \( \nu : [a, b] \rightarrow \mathbb{R} \) of bounded variation, the Riemann–Stieltjes integral \( \int_a^b p(t) \, d\nu(t) \) exists and one has the inequality

\[
\left| \int_a^b p(t) \, d\nu(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \nu(b) \nu(a).
\]

(22)

As \( f \) is of bounded variation on \([a, b]\), by (22) we have

\[
\left| \frac{1}{b-a} \int_a^b \left( \int_a^t g(s) \, ds - \frac{t-a}{b-a} \int_a^b g(s) \, ds \right) df(t) \right| \leq \frac{1}{b-a} \sup_{t \in [a, b]} \left| \int_a^t g(s) \, ds - \frac{t-a}{b-a} \int_a^b g(s) \, ds \right| \nu(f)
\]

\[
= \frac{1}{b-a} \sup_{t \in [a, b]} \left( \int_a^t g(s) \, ds - \frac{t-a}{b-a} \int_a^b g(s) \, ds \right) \nu(f).
\]

(23)

In the inequality (15), setting \( d = t \) and \( c = a \), we get

\[
\frac{1}{t-a} \int_a^t g(s) \, ds - \frac{1}{b-a} \int_a^b g(s) \, ds \leq \frac{1}{2} (b-t) \| g' \|_{\infty}.
\]

(24)

Substituting (24) into (23), we get

\[
\| \mathcal{T}(f, g) \| \leq \frac{1}{b-a} \cdot \frac{1}{2} \| g' \|_{\infty} \sup_{t \in [a, b]} |(t-a)(b-t)| \nu(f)
\]

\[
= \frac{1}{8} (b-a) \| g' \|_{\infty} \nu(f),
\]

(25)

since \( \sup_{t \in [a, b]} |(t-a)(b-t)| = (1/4)(b-a)^2 \), which proves the first inequality in (19).

In the inequality (16), setting \( d = t \) and \( c = a \), we get

\[
\frac{1}{t-a} \int_a^t g(s) \, ds - \frac{1}{b-a} \int_a^b g(s) \, ds \leq \frac{1}{2} (b-t) \| g' \|_{\infty}.
\]

(26)
Substituting (26) into (23), we get
\[ |\mathcal{F}(f, g)| \leq \frac{1}{b-a} \int_a^b (f) \times \sup_{t \in [a,b]} (t-a) \times \left( \frac{(b-t)^{1+(1/q)}}{(q+1)^{1/q}(b-a)^{1/q}} \times \left[ 1 + \left( \frac{t-a}{b-t-a} \right)^q \right]^{1/q} \right) \|g\|_p, \quad g' \in L_p[a,b], \frac{1}{2} \leq t \leq b-a, g' \in L_1[a,b] \]
(27)
where \( p > 1 \) and \((1/p) + (1/q) = 1\), which proves the second and the third inequalities in (19).

Another result when \( g \) is of \( r-H \)-Hölder type is as follows.

**Theorem 8.** Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is of bounded variation on \([a, b]\) and \( g \) is of \( r-H \)-Hölder type on \([a, b]\); then
\[ |\mathcal{F}(f, g)| \leq \frac{Hr^r}{(r+1)^{r+1}} (b-a)^r \mathcal{L}(f). \tag{28} \]

**Proof.** As \( f \) is of bounded variation and \( g \) is of \( r-H \)-Hölder type on \([a, b]\), by (23) and using (17) we have
\[ |\mathcal{F}(f, g)| \leq \frac{1}{b-a} \sup_{t \in [a,b]} \left[ (t-a) \left( \frac{1}{t-a} \int_a^t g(u) \, du \right) \right] - \frac{1}{b-a} \int_a^b g(u) \, du \] \[ \leq \frac{1}{b-a} \int_a^b g(u) \, du \mathcal{F}(f), \]
(29)
since \( \sup_{t \in [a,b]} (t-a) = (r/r + 1)(b-a)^{r+1} \), which completes the proof.

**Theorem 9.** Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is of bounded variation on \([a, b]\) and \( g \) is monotonic nondecreasing on \([a, b]\); then
\[ |\mathcal{F}(f, g)| \leq \frac{1}{6} (b-a) \left[ g(b) - g(a) \right] \mathcal{L}(f). \tag{30} \]

**Proof.** As \( f \) and \( g \) is of bounded variation on \([a, b]\) and \( g \) is monotonic nondecreasing on \([a, b]\), by (22) and using (23) we have
\[ |\mathcal{F}(f, g)| \leq \frac{1}{b-a} \int_a^b (t-a) \left( \frac{1}{t-a} \int_a^t g(u) \, du \right) - \frac{1}{b-a} \int_a^b g(u) \, du \mathcal{F}(f). \tag{31} \]
In the third part of inequality (18), setting \( d = t \) and \( c = a \), we get
\[ \left| \frac{1}{t-a} \int_a^t g(s) \, ds - \frac{1}{b-a} \int_a^b g(s) \, ds \right| \leq \frac{b-t}{b-a} \left[ g(b) - g(a) \right]. \tag{32} \]
Substituting (32) into (31), we get
\[ |\mathcal{F}(f, g)| \leq \frac{1}{b-a} \cdot \frac{g(b) - g(a)}{b-a} \int_a^b (t-a) \mathcal{L}(f) \]
(33)
and thus the proof is finished.

**3. The Case When \( f \) Is Lipschitzian**

In this section, we give some new bounds when \( f \) is \( L \)-Lipschitzian.

**Theorem 10.** Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is \( L \)-Lipschitzian on \([a, b]\) and \( g \) is an absolutely continuous on \([a, b]\); then
\[ |\mathcal{F}(f, g)| \leq \frac{1}{12} (b-a)^2 \left\| g' \right\|_{L^2(a,b)}, \quad g' \in L^2(a,b); \]
\[ \leq L \cdot \left( \frac{1}{12} (b-a)^2 \left\| g' \right\|_{L^2(a,b)}, \quad g' \in L^2(a,b); \right\| \frac{q^2(1+(1/2))^{1/4}}{(q+1)^{1/4}} \times (b-a)^2 \left\| g' \right\|_{L^2(a,b)}, \quad g' \in L^2(a,b); \right\| \frac{1}{12} (b-a)^2 \left\| g' \right\|_{L^2(a,b)}, \quad g' \in L^2(a,b); \] \[ = L \cdot \left( \frac{1}{12} (b-a)^2 \left\| g' \right\|_{L^2(a,b)}, \quad g' \in L^2(a,b); \right\| \frac{q^2(1+(1/2))^{1/4}}{(q+1)^{1/4}} \times (b-a)^2 \left\| g' \right\|_{L^2(a,b)}, \quad g' \in L^2(a,b); \] \[ \frac{1}{12} (b-a)^2 \left\| g' \right\|_{L^2(a,b)}, \quad g' \in L^2(a,b); \]
(34)
where \( p > 1 \) and \((1/p) + (1/q) = 1\).

**Proof.** Using the fact that for a Riemann integrable function \( p : [c, d] \to \mathbb{R} \) and \( L \)-Lipschitzian function \( v : [c, d] \to \mathbb{R} \), one has the inequality
\[ \left| \int_c^d p(t) \, dt \right| \leq L \int_c^d |p(t)| \, dt. \tag{35} \]
As \( f' \) is \( L \)-Lipschitzian on \([a, b]\), by (35) we have

\[
|\mathcal{F}(f, g)| \leq \frac{L}{b-a} \int_a^b (t-a) \left[ \frac{1}{t-a} \int_a^t g(u) \, du \right] dt \\
- \frac{1}{b-a} \int_a^b g(u) \, du \\
\leq \frac{1}{2(b-a)} L \|g'\|_{\infty} \int_a^b (t-a) (b-t) \, dt \\
= \frac{1}{12} L(b-a)^2 \|g'\|_{\infty},
\]

(36)

where for the last inequality we used the inequality (15), with \( d = t \) and \( a = c \), (see (24)).

In the inequality (16), setting \( d = t \) and \( a = c \), we get

\[
\left| \frac{1}{t-a} \int_a^t g(s) \, ds - \frac{1}{b-a} \int_a^b g(s) \, ds \right| \\
\leq \left\{ \frac{(b-t)^{1+1/(1,q)}}{(q+1)^{1/q}(b-a)^{1/q}} \times \left[ 1 + \left( \frac{t-a}{b+t-2a} \right)^q \right]^{1/q} \|g'\|_p \right\} g' \in L_p[a, b], \\
1 \leq p < \infty, \\
\frac{1}{p} + \frac{1}{q} = 1; \\
\frac{1}{b-a} \left\| g' \right\|_1, \\
g' \in L_1[a, b].
\]

(37)

Substituting (37) into (36), we get

\[
|\mathcal{F}(f, g)| \leq \frac{L}{b-a} \left\{ \frac{\|g'\|_p}{(q+1)^{1/q}(b-a)^{1/q}} \times \left[ 1 + \left( \frac{t-a}{b+t-2a} \right)^q \right]^{1/q} \right\} g' \in L_p[a, b], \\
\frac{1}{b-a} \left\| g' \right\|_1, \\
g' \in L_1[a, b].
\]

\[
= \frac{L}{(b-a)^2} \left\| g' \right\|_1, \\
g' \in L_1[a, b],
\]

(38)

where \( \sup_{t \in[a,b]} [1 + ((t-a)/(b+t-2a))^q]^{1/q} \) \( = (1+(1/2^q))^{1/q} \), and

\[
\int_a^b (t-a) (b-t)^{1+1/(1,q)} \, dt \\
= (b-a)^{3+1/(q)} \int_0^1 (1-t) t^{1+1/(1,q)} \, dt \\
= (b-a)^{3+1/(q)} \frac{q^2}{(2q+1)(3q+1)} \\
\]

(39)

which proves the second and the third inequalities in (34).

\[\square\]

**Theorem 11.** Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is \( L \)-Lipschitzian on \([a, b]\) and \( g \) is of \( r \)-\( H \)-Hölder type on \([a, b]\); then

\[
|\mathcal{F}(f, g)| \leq LH \frac{(b-a)^{r+1}}{(r+1)^2 (r+2)}.
\]

(40)

**Proof.** As \( f \) is \( L \)-Lipschitzian and \( g \) is of \( r \)-\( H \)-Hölder type on \([a, b]\), by (36) and using (17) we have

\[
|\mathcal{F}(f, g)| \\
\leq \frac{L}{b-a} \int_a^b (t-a) \left[ \frac{1}{t-a} \int_a^t g(u) \, du \right] dt \\
- \frac{1}{b-a} \int_a^b g(u) \, du \right\} dt \\
\leq \frac{LH}{(r+1)^2 (r+2)} \int_a^b (t-a) (b-t) \, dt \\
= LH \frac{(b-a)^{r+1}}{(r+1)^2 (r+2)},
\]

(41)
where, for the last inequality, a simple calculation yields that
\[
\int_a^b (t-a)(b-t)\,dt = (b-a)^2\int_0^1 (1-t)\,dt = \frac{(b-a)^2}{(r+1)(r+2)}
\] (42)
which completes the proof.

**Corollary 12.** In Theorem 11, if \(g\) is \(M\)-Lipschitzian on \([a, b]\), then we have
\[
|\mathcal{T}(f, g)| \leq \frac{1}{12}LM(b-a)^2.
\] (43)

**Theorem 13.** Let \(f, g : [a, b] \rightarrow \mathbb{R}\) be such that \(f\) is of bounded variation on \([a, b]\) and \(g\) is monotonic nondecreasing on \([a, b]\); then
\[
|\mathcal{T}(f, g)| \leq \frac{1}{6}L(b-a)\left[ g(b) - g(a) \right].
\] (44)

**Proof.** As \(f\) is \(L\)-Lipschitzian on \([a, b]\) and \(g\) is of bounded variation on \([a, b]\), by (35) and using (36) we have
\[
|\mathcal{T}(f, g)| \leq \frac{L}{b-a} \int_a^b (t-a) \left[ \frac{1}{t-a} \int_a^t g(u)\,du - \frac{1}{b-a} \int_a^b g(u)\,du \right] dt.
\] (45)
In the third part of inequality (18), setting \(d = t\) and \(c = a\), we get
\[
\left| \int_a^t g(s)\,ds - \frac{1}{b-a} \int_a^b g(s)\,ds \right| \leq \frac{b-t}{b-a} \left[ g(b) - g(a) \right].
\] (46)
Substituting (46) into (45), we get
\[
|\mathcal{T}(f, g)| \leq \frac{L}{b-a} \int_a^b \left[ (t-a)(b-t) \right] dt = \frac{1}{6}L(b-a) \left[ g(b) - g(a) \right],
\] (47)
which completes the proof.

**4. More Inequalities**

In this section we give other related results.

**Theorem 14.** Let \(f, g : [a, b] \rightarrow \mathbb{R}\) be such that \(f\) and \(g\) are of bounded variation on \([a, b]\); then
\[
|\mathcal{T}(f, g)| \leq \frac{1}{4} \int_a^b g(u)\,du \int_a^b f(u)\,du.
\] (48)
The constant \(1/4\) is the best possible.

**Proof.** As \(f\) and \(g\) is of bounded variation on \([a, b]\), by (22) and using (23), we have
\[
|\mathcal{T}(f, g)|
\leq \frac{1}{b-a} \sup_{a \leq s \leq b} \left| t-a \left[ \frac{1}{t-a} \int_a^t g(u)\,du \right] - \frac{1}{b-a} \int_a^b g(u)\,du \right|^\bigvee (f).
\] (49)
In the first inequality of (18), setting \(d = t\) and \(c = a\), we get
\[
\left| \int_a^t g(s)\,ds - \frac{1}{b-a} \int_a^b g(s)\,ds \right| \leq \frac{b-t}{b-a} \int_a^b (g(s)).
\] (50)
Substituting (50) into (49), we get
\[
|\mathcal{T}(f, g)| \leq \frac{1}{(b-a)^2} \int_a^b \left[ (t-a)(b-t) \right] \left[ g(b) - g(a) \right],
\] (51)
which proves the inequality. The sharpness case trivially holds by taking \(f(t) = g(t) = \text{sgn}(t - (a + b)/2)\), which completes the proof.

When the integrator is of bounded variation we have the following.

**Theorem 15.** Let \(f, g : [a, b] \rightarrow \mathbb{R}\) be such that \(f\) is \(L\)-Lipschitzian on \([a, b]\) and \(g\) is of bounded variation on \([a, b]\); then
\[
|\mathcal{T}(f, g)| \leq \frac{1}{6}L(b-a) \int_a^b (g) \,du.
\] (52)

**Proof.** As \(f\) is \(L\)-Lipschitzian on \([a, b]\) and \(g\) is of bounded variation on \([a, b]\), by (22) and using (23), we have
\[
|\mathcal{T}(f, g)| \leq \frac{L}{b-a} \int_a^b \left[ (t-a)(b-t) \right] dt.
\] (53)
In the second inequality of (18), setting \(d = t\) and \(c = a\), we get
\[
\left| \int_a^t g(s)\,ds - \frac{1}{b-a} \int_a^b g(s)\,ds \right| \leq \frac{b-t}{b-a} \int_a^b (g(s)).
\] (54)
Substituting (54) into (53), we get
\[
|\mathcal{T}(f, g)| \leq \frac{L}{(b-a)^2} \int_a^b \left( t-a \right) \left( b-t \right) \int_a^b (g).
\] (55)
and the proof is completed.
When both functions are Lipschitzian we have the following.

**Theorem 16.** Let \( f, g : [a, b] \to \mathbb{R} \) be, respectively, such that \( f \) and \( g \) are \( L_1 \)- and \( L_2 \)-Lipschitzian on \([a, b]; \) then

\[
|\mathcal{F}(f, g)| \leq \frac{1}{12} L_1 L_2 (b - a)^2. \tag{56}
\]

The constant 1/12 is the best possible.

**Proof.** As \( f \) and \( g \) are \( L_1 \)- and \( L_2 \)-Lipschitzian on \([a, b]; \) respectively, by (35) and using (36), we have

\[
|\mathcal{F}(f, g)| \leq \frac{L_1}{b - a} \int_a^b |g(u) - g(u)| \, du \tag{57}
\]

In the second inequality of (18), setting \( d = t \) and \( c = a, \) we get

\[
\left| \frac{1}{t - a} \int_a^t g(s) \, ds - \frac{1}{b - a} \int_a^b g(s) \, ds \right| \leq \frac{1}{2} L_2 (b - t). \tag{58}
\]

Substituting (58) into (57), we get

\[
|\mathcal{F}(f, g)| \leq \frac{1}{2(b - a)} L_1 L_2 \int_a^b (t - a) (b - t) \, dt \tag{59}
\]

\[
= \frac{1}{12} L_1 L_2 (b - a)^2,
\]

which proves the inequality. The sharpness case trivially holds by taking \( f(t) = g(t) = t, \) which completes the proof. \( \square \)

**Remark 17.** Let \( g \) be as in Theorems 7–16. By applying the same techniques used in the corresponding proofs of each theorem, we may obtain several inequalities for monotonic nondecreasing integrator \( f \) using the fact that for a monotonic nondecreasing function \( v : [a, b] \to \mathbb{R} \) and continuous function \( p : [a, b] \to \mathbb{R}, \) one has the inequality

\[
\left| \int_a^b p(t) \, dv(t) \right| \leq \int_a^b |p(t)| \, dv(t). \tag{60}
\]

We leave the details to the interested reader.

## 5. Applications for Self-Adjoint Operators

We denote by \( \mathcal{B}(H) \) the Banach algebra of all bounded linear operators on a complex Hilbert space \((H; \langle \cdot, \cdot \rangle). \) Let \( A \in \mathcal{B}(H) \) be self-adjoint and let \( \varphi_\lambda \) be defined for all \( \lambda \in \mathbb{R} \) as follows:

\[
\varphi_\lambda(s) := \begin{cases} 
1, & \text{for } -\infty < s \leq \lambda, \\
0, & \text{for } \lambda < s < +\infty.
\end{cases} \tag{61}
\]

Then for every \( \lambda \in \mathbb{R} \) the operator

\[
E_\lambda := \varphi_\lambda(A)
\]

is a projection which reduces \( A. \)

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded self-adjoint operators in Hilbert spaces; see for instance [24, page 256].

**Theorem 18** (Spectral Representation Theorem). Let \( A \) be a bounded self-adjoint operator on the Hilbert space \( H \) and let \( m = \min \{|\lambda| : \lambda \in \text{Sp}(A)\} =: \min \text{Sp}(A) \) and \( M = \max \{|\lambda| : \lambda \in \text{Sp}(A)\} =: \max \text{Sp}(A). \) Then there exists a family of projections \( \{E_\lambda\}_{\lambda \in \mathbb{R}}, \) called the spectral family of \( A, \) with the following properties:

(a) \( E_\lambda \leq E_\lambda' \) for \( \lambda \leq \lambda'; \)

(b) \( E_{m-0} = 0, E_{M+0} = E_A \) for all \( \lambda \in \mathbb{R}; \)

(c) we have the representation

\[
A = \int_{m-0}^M \lambda \, dE_\lambda. \tag{63}
\]

More generally, for every continuous complex-valued function \( \varphi \) defined on \( \mathbb{R} \) and for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\left\| \varphi(A) - \sum_{k=1}^n \varphi\left(E_{\lambda_k} - E_{\lambda_{k-1}}\right) \right\| \leq \varepsilon \tag{64}
\]

whenever

\[
\lambda_0 < m = \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = M,
\]

\[
\lambda_k - \lambda_{k-1} \leq \delta \quad \text{for } 1 \leq k \leq n,
\]

\[
\lambda_k \in [\lambda_{k-1}, \lambda_k] \quad \text{for } 1 \leq k \leq n
\]

this means that

\[
\varphi(A) = \int_{m-0}^M \varphi(\lambda) \, dE_\lambda, \tag{66}
\]

where the integral is of Riemann-Stieltjes type.

**Corollary 19.** With the assumptions of Theorem 18 for \( A, E_\lambda, \) and \( \varphi \) we have the representations

\[
\varphi(A) x = \int_{m-0}^M \varphi(\lambda) \, dE_\lambda x \quad \forall x \in H, \tag{67}
\]

\[
\langle \varphi(A) x, y \rangle = \int_{m-0}^M \varphi(\lambda) \, d \langle E_\lambda x, y \rangle \quad \forall x, y \in H.
\]

In particular,

\[
\langle \varphi(A) x, x \rangle = \int_{m-0}^M \varphi(\lambda) \, d \langle E_\lambda x, x \rangle \quad \forall x \in H. \tag{68}
\]

Moreover, we have the equality

\[
\|\varphi(A) x\|^2 = \int_{m-0}^M \|\varphi(\lambda)\|^2 d\|E_\lambda x\|^2 \quad \forall x \in H. \tag{69}
\]
We recall the following result (see [25]) that provides an upper bound for the total variation of the function $R \ni \lambda \mapsto \langle E_\lambda, x, y \rangle \in C$ on an interval $[\alpha, \beta]$.

**Theorem 20.** Let $\{E_\lambda\}_{\lambda \in R}$ be the spectral family of the bounded self-adjoint operator $A$ and let $m = \min \text{Sp}(A)$ and $M = \max \text{Sp}(A)$. Then for any $x, y \in H$ and $\alpha < \beta$ we have the inequality

$$\left[ \sqrt{\sum_{n=0}^{\infty} \left( \langle E_\lambda, x, y \rangle \right)^2} \right] ^2 \leq \left( \langle E_{\beta} - E_{\alpha} \rangle x, x \rangle \langle E_{\beta} - E_{\alpha} \rangle y, y \rangle \right) ,$$

where $\sqrt[\beta]{\sum_{n=0}^{\infty} \left( \langle E_\lambda, x, y \rangle \right)^2}$ denotes the total variation of the function $\langle E_\lambda, x, y \rangle$ on $[\alpha, \beta]$.

**Remark 21.** For $\alpha = m - \varepsilon$ with $\varepsilon > 0$ and $\beta = M$ we get from (70) the inequality

$$\sum_{m=0}^{M} \langle E_\lambda x, y \rangle \leq \langle (1_H - E_{m-\varepsilon}) x, x \rangle^{1/2} \langle (1_H - E_{m-\varepsilon}) y, y \rangle^{1/2}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\sum_{m=0}^{M} \langle E_\lambda x, y \rangle \leq \|x\| \|y\| ,$$

(72)

where $\sum_{m=0}^{M} \langle E_\lambda x, y \rangle$ denotes the limit $\lim_{\varepsilon \to 0} \sum_{m=0}^{M} \langle E_\lambda x, y \rangle$.

The inequality (72) was also proved in the recent monographs [26, 27] and will be utilized in the following.

After these preparations we can state and prove the following trapezoidal type inequality for functions of self-adjoint operators on Hilbert spaces.

**Theorem 22.** Let $A$ be a bounded self-adjoint operator on the Hilbert space $H$ and let $m = \min \{ \lambda \mid \lambda \in \text{Sp}(A) \} =: \min \text{Sp}(A)$ and $M = \max \{ \lambda \mid \lambda \in \text{Sp}(A) \} =: \max \text{Sp}(A)$. If $f : [m, M] \to C$ is such that its derivative $f$ is of bounded variation on $[m, M]$, then we have the inequality

$$\left\langle \left\langle f \left( A \right) - \frac{f \left( m \right) \left( M1_H - A \right) + f \left( M \right) \left( A - m1_H \right)}{M - m} \right\rangle, x, y \right\rangle \leq \frac{1}{4} \left( M - m \right) \sum_{m=0}^{M} \left( \left\langle E_\lambda x, y \right\rangle \right) \leq \frac{1}{4} \left( M - m \right) \left\| x \right\| \left\| y \right\|$$

for any $x, y \in H$.

**Proof.** Utilising the inequality (48) for the function of bounded variation $g(A) = \langle E_\lambda x, y \rangle$ and the continuous function $f$ we have

$$\int_{m=0}^{M} f \left( \lambda \right) d \langle E_\lambda x, y \rangle = \langle f \left( A \right) x, y \rangle ,$$

$$\int_{m=0}^{M} \left( \langle E_\lambda x, y \rangle - \frac{f \left( m \right) \left( M1_H - A \right) + f \left( M \right) \left( A - m1_H \right)}{M - m} \right) d \langle E_\lambda x, y \rangle \leq \frac{1}{4} \left( M - m \right) \sum_{m=0}^{M} \left( \left\langle E_\lambda x, y \right\rangle \right) \leq \frac{1}{4} \left( M - m \right) \left\| x \right\| \left\| y \right\|$$

for any $x, y \in H$. The above inequality (73) can be utilized for different particular functions of interest in Operator Theory, such as the power, logarithmic, and exponential functions.

If we take $f(t) = t^p$ with $p \geq 1$, then for any positive operator $A$ with $\text{Sp}(A) \subset [m, M] \subset [0, \infty)$ we have the inequality

$$\int_{m=0}^{M} \left( A^p - \frac{m^p \left( M1_H - A \right) + M^p \left( A - m1_H \right)}{M - m} \right) d \langle E_\lambda x, y \rangle \leq \frac{1}{4} \left( M - m \right) \left( M^p - m^p \right) \sum_{m=0}^{M} \left( \left\langle E_\lambda x, y \right\rangle \right) \leq \frac{1}{4} \left( M - m \right) \left( M^p - m^p \right) \left\| x \right\| \left\| y \right\|$$

for any $x, y \in H$. 

(76)
If we take the function $f(t) = \ln t$, then for any positive definite operator $A$ with $\text{Sp}(A) \subset [m, M] \subset (0, \infty)$ we have the inequality

$$\left\langle \left[ \ln A - \frac{\ln m (M1_H - A) + \ln M (A - m1_H)}{M - m} \right] x, y \right\rangle$$

$$\leq \ln \left( \frac{M}{m} \right)^{(M-m)/4} M \sum_{m=0}^{M} \left\langle (E_\ell x, y) \right\rangle$$

$$\leq \ln \left( \frac{M}{m} \right)^{(M-m)/4} \|x\| \|y\|$$

(77)

for any $x, y \in H$.

Finally, if we take $f(t) = \exp(t)$, then we have for any self-adjoint operator $A$ with $\text{Sp}(A) \subset [m, M] \subset \mathbb{R}$ the inequality

$$\left\langle \left[ \exp(A) - \frac{\exp(m) (M1_H - A) + \exp(M) (A - m1_H)}{M - m} \right] x, y \right\rangle$$

$$\leq \frac{1}{4} (M - m) (\exp M - \exp m) \sum_{m=0}^{M} \left\langle (E_\ell x, y) \right\rangle$$

$$\leq \frac{1}{4} (M - m) (\exp M - \exp m) \|x\| \|y\|$$

(78)

for any $x, y \in H$.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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