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Research Article

New Čebyšev Type Inequalities and Applications for Functions of Self-Adjoint Operators on Complex Hilbert Spaces

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Several new error bounds for the Čebyšev functional under various assumptions are proved. Applications for functions of self-adjoint operators on complex Hilbert spaces are provided as well.

1. Introduction

In recent years the approximation problem of the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ via the famous Čebyšev functional

$$\mathcal{T}(f,g) = \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \tag{1}$$

increasingly became essential. In 1882, Čebyšev [1] derived an interesting result involving two absolutely continuous functions whose first derivatives are continuous and bounded and is given by

$$|\mathcal{T}(f,g)| \le \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty, \tag{2}$$

and the constant $1/12$ is the best possible.

In 1935, Grüss [2] proved another result for two integrable mappings f, g such that $\phi \le f(x) \le \Phi$ and $\gamma \le f(x) \le \Gamma$; the inequality

$$|\mathcal{T}(f,g)| \le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma) \tag{3}$$

holds, and the constant $1/4$ is the best possible.

In [3, p 302] Beesack et al. have proved the following Čebyšev inequality for absolutely continuous functions whose first derivatives belong to L_p spaces:

$$|\mathcal{T}(f,g)| \le \frac{b-a}{4}\left(\frac{2^p-1}{p(p+1)}\right)^{1/p}\left(\frac{2^q-1}{q(q+1)}\right)^{1/q}\|f'\|_p\|g'\|_q, \tag{4}$$

where $\|h\|_p := \left(\int_a^b |h(t)|^p dt\right)^{1/p}, \forall p > 1$, and $(1/p)+(1/q) = 1$.

For the constant

$$\omega(p,q) := \frac{1}{4}\left(\frac{2^p-1}{p(p+1)}\right)^{1/p}\left(\frac{2^q-1}{q(q+1)}\right)^{1/q} \tag{5}$$

we have

$$\frac{1}{8} \le \omega(p,q) \le \frac{1}{4} \tag{6}$$

for all $q = p/(p-1)$, $p > 1$. Furthermore, we have the following particular cases in (4).

(1) If $p = q = 2$, we have

$$|\mathcal{T}(f,g)| \le \frac{b-a}{8}\|f'\|_2\|g'\|_2. \tag{7}$$

(2) If $q \to \infty$, we have

$$|\mathcal{T}(f,g)| \le \frac{b-a}{4}\|f'\|_1\|g'\|_\infty. \tag{8}$$

In 1970, Ostrowski [4] has proved the following combination of the Čebyšev and Grüss results:

$$|\mathcal{T}(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{9}$$

where g is absolutely continuous with $g' \in L_\infty[a,b]$ and f is Lebesgue integrable on $[a,b]$ and satisfying $m \le f(t) \le M$, for all $t \in [a,b]$. The constant $1/8$ is the best possible.

In 1973, Lupaş [5] has improved Beesack et al. inequality (7), as follows:

$$|\mathcal{T}(f,g)| \le \frac{(b-a)}{\pi^2}\|f'\|_2\|g'\|_2, \tag{10}$$

provided that f, g are two absolutely continuous functions on $[a,b]$ with $f', g' \in L_2[a,b]$, where $\|h'\|_2 := \left(\int_a^b |h'(t)|^2 dt\right)^{1/2}$. The constant $1/\pi^2$ is the best possible.

More recently, and using the identity ([3], page 246),

$$\mathcal{T}(f,g) = \frac{1}{b-a}\int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right] \times \left[g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right] dt. \tag{11}$$

Dragomir [6] has proved the following inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$; then*

$$|\mathcal{T}(f,g)| \le \frac{1}{2}\bigvee_a^b(f)\cdot\frac{1}{b-a}\int_a^b\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt, \tag{12}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$. The constant $1/2$ is best possible in (12).

Another result when both functions are of bounded variation was considered in the same paper [6], as follows.

Theorem 2. *If $f, g : [a,b] \to \mathbb{C}$ are of bounded variation on $[a,b]$, then*

$$|\mathcal{T}(f,g)| \le \frac{1}{4}\bigvee_a^b(f)\cdot\bigvee_a^b(g). \tag{13}$$

The constant $1/4$ is best possible in (13).

Many authors have studied the functional (1) and, therefore, several bounds under various assumptions have been obtained; for more new results and generalizations the reader may refer to [6–21].

On other hand and in order to study the difference between two Riemann integral means, Barnett et al. [22] have proved the following estimates.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function with the property that $f' \in L_\infty[a,b]$; that is,*

$$\|f'\|_\infty := \operatorname{ess}\sup_{t\in[a,b]}|f'(t)|. \tag{14}$$

Then for $a \le c < d \le b$, we have the inequality

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(s)\,ds\right| &\le \left[\frac{1}{4} + \left(\frac{(a+b)/2-(c+d)/2}{(b-a)-(d-c)}\right)^2\right] \\ &\quad\times[(b-a)-(d-c)]\|f'\|_\infty \\ &\le \frac{1}{2}[(b-a)-(d-c)]\|f'\|_\infty.\end{aligned} \tag{15}$$

The constant $1/4$ in the first inequality and $1/2$ in the second inequality are the best possible.

After that, Cerone and Dragomir [23] have obtained the following three results as well.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping. Then for $a \le c < d \le b$, we have the inequalities*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(s)\,ds\right| \le \begin{cases} \dfrac{(b-a)}{(q+1)^{1/q}}\left[1+\left(\dfrac{\rho}{1+\rho}\right)^q\right]^{1/q} \times\left[\nu^{q+1}+\lambda^{q+1}\right]^{1/q}\|f'\|_p, & f' \in L_p[a,b], \\ & 1 \le p < \infty, \\ & \dfrac{1}{p}+\dfrac{1}{q} = 1; \\ \dfrac{1}{2}[1-\rho+|\nu-\lambda|]\|f'\|_1, & f' \in L_1[a,b], \end{cases} \tag{16}$$

where $(b-a)\nu = c-a$, $(b-a)\rho = d-c$ and $(b-a)\lambda = b-d$. Both inequalities in (16) *are sharp.*

Theorem 5. *Assume that the mapping $f : [a,b] \to \mathbb{R}$ is of r-H-Hölder type on $[a,b]$. Then for $a \le c < d \le b$, we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(s)\,ds\right| \le H\frac{(c-a)^{r+1}+(b-d)^{r+1}}{(r+1)[(b-a)-(d-c)]}. \tag{17}$$

Inequality (17) *is best possible in the sense that we cannot put in the right-hand side a constant less than* 1.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. The following bounds hold:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(s)\,ds\right| \le \begin{cases} \left[\dfrac{b-a-(d-c)}{2} + \left|\dfrac{c+d}{2} - \dfrac{a+b}{2}\right|\right]\dfrac{\bigvee_a^b(f)}{b-a}; \\ L\dfrac{(c-a)^2+(b-d)^2}{2[(b-a)-(d-c)]}; \\ \qquad \text{if } f \text{ is } L\text{-Lipschitzian} \\ \left(\dfrac{b-d}{b-a}\right)f(b) - \left(\dfrac{c-a}{b-a}\right)f(a) \\ \quad + \left[\dfrac{c+d-(a+b)}{b-a}\right]f(s_0); \\ \qquad \text{if } f \text{ is monotonic nondecreasing,} \end{cases} \tag{18}$$

where $s_0 = (cb-ad)/((b-a)-(d-c)) \in [c,d]$.

In this paper by utilising amongst others the inequalities from Theorems 3–6, several new bounds for the *Čebyšev functional* $\mathcal{T}(f,g)$ are provided.

The inequalities (15)–(18) are used in an essential way to obtain new error bounds for the $\mathcal{T}(f,g)$, which gives a significant application for these inequalities. Applications for functions of self-adjoint operators on complex Hilbert Spaces are provided as well.

2. The Case When f Is of Bounded Variation

We may start with the following result.

Theorem 7. *Let $f,g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is absolutely continuous on $[a,b]$; then*

$$|\mathcal{T}(f,g)| \le \bigvee_a^b(f)\cdot\begin{cases} \dfrac{1}{8}(b-a)\|g'\|_\infty, & g' \in L_\infty[a,b]; \\ \dfrac{1}{2}\left(\dfrac{1+(1/2^q)}{2(q+1)}\right)^{1/q}\|g'\|_p, & g' \in L_p[a,b]; \\ \dfrac{1}{8}\|g'\|_1, & g' \in L_1[a,b], \end{cases} \tag{19}$$

where $\|\cdot\|_p$ are the usual Lebesgue norms; that is,

$$\|h\|_p := \left(\int_a^b |h(t)|^p dt\right)^{1/p}, \quad \text{for } p \ge 1, \qquad \|h\|_\infty := \operatorname*{ess\,sup}_{t\in[a,b]} |h(t)|. \tag{20}$$

Proof. Using integration by parts, we have

$$\mathcal{T}(f,g) = -\frac{1}{b-a}\int_a^b\left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right)df(t). \tag{21}$$

It is known that for a continuous function $p : [a,b] \to \mathbb{R}$ and a function $\nu : [a,b] \to \mathbb{R}$ of bounded variation, the Riemann-Stieltjes integral $\int_a^b p(t)d\nu(t)$ exists and one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \le \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(\nu). \tag{22}$$

As f is of bounded variation on $[a,b]$, by (22) we have

$$\begin{aligned} &\left|-\frac{1}{b-a}\int_a^b\left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right)df(t)\right| \\ &\quad\le \frac{1}{b-a}\sup\left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|\cdot\bigvee_a^b(f) \\ &\quad= \frac{1}{b-a}\sup_{t\in[a,b]}\left|(t-a)\left[\frac{1}{t-a}\int_a^t g(s)\,ds \right.\right. \\ &\qquad\qquad \left.\left. -\frac{1}{b-a}\int_a^b g(s)\,ds\right]\right|\bigvee_a^b(f). \end{aligned} \tag{23}$$

In the inequality (15), setting $d = t$ and $c = a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \frac{1}{2}(b-t)\|g'\|_\infty. \tag{24}$$

Substituting (24) into (23), we get

$$\begin{aligned} |\mathcal{T}(f,g)| &\le \frac{1}{b-a}\cdot\frac{1}{2}\|g'\|_\infty\sup_{t\in[a,b]}\{(t-a)(b-t)\}\bigvee_a^b(f) \\ &= \frac{1}{8}(b-a)\|g'\|_\infty\bigvee_a^b(f), \end{aligned} \tag{25}$$

since $\sup_{t\in[a,b]}\{(t-a)(b-t)\}$, occurs at $t = (a+b)/2$, therefore, $\sup_{t\in[a,b]}\{(t-a)(b-t)\} = (1/4)(b-a)^2$, which proves the first inequality in (19).

In the inequality (16), setting $d = t$ and $c = a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \begin{cases} \dfrac{(b-t)^{1+(1/q)}}{(q+1)^{1/q}(b-a)^{1/q}} \\ \quad\times\left[1+\left(\dfrac{t-a}{b+t-2a}\right)^q\right]^{1/q}\|g'\|_p, & g' \in L_p[a,b], \\ \dfrac{1}{2}\dfrac{b-t}{b-a}\|g'\|_1, & g' \in L_1[a,b]. \end{cases} \tag{26}$$

Substituting (26) into (23), we get

$$\begin{aligned}
&|\mathcal{T}(f,g)| \\
&\quad\le \frac{1}{b-a}\bigvee_a^b(f) \\
&\qquad\times \sup_{t\in[a,b]}(t-a) \\
&\qquad\cdot\begin{cases}\dfrac{(b-t)^{1+(1/q)}}{(q+1)^{1/q}(b-a)^{1/q}} \\ \quad\times\left[1+\left(\dfrac{t-a}{b+t-2a}\right)^q\right]^{1/q}\|g'\|_p, & g'\in L_p[a,b], \\ \dfrac{1}{2}\dfrac{b-t}{b-a}\|g'\|_1, & g'\in L_1[a,b]\end{cases} \\
&\quad\le \bigvee_a^b(f)\times\begin{cases}\dfrac{[1+(1/2)^q]^{1/q}}{2^{1+1/q}(1+q)^{1/q}}\|g'\|_p, & g'\in L_p[a,b]; \\ \dfrac{1}{8}\|g'\|_1, & g'\in L_1[a,b],\end{cases}
\end{aligned} \tag{27}$$

where $p>1$ and $(1/p)+(1/q)=1$, which proves the second and the third inequalities in (19). □

Another result when g is of r-H-Hölder type is as follows.

Theorem 8. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is of r-H-Hölder type on $[a,b]$; then*

$$|\mathcal{T}(f,g)|\le\frac{Hr^r}{(r+1)^{r+1}}(b-a)^r\bigvee_a^b(f). \tag{28}$$

Proof. As f is of bounded variation and g is of r-H-Hölder type on $[a,b]$, by (23) and using (17) we have

$$\begin{aligned}
&|\mathcal{T}(f,g)| \\
&\quad\le \frac{1}{b-a}\sup_{t\in[a,b]}\left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du\right.\right. \\
&\qquad\qquad\left.\left.-\frac{1}{b-a}\int_a^b g(u)\,du\right]\right|\bigvee_a^b(f) \\
&\quad\le \frac{1}{b-a}\frac{H}{r+1}\sup_{t\in[a,b]}(t-a)(b-t)^r\bigvee_a^b(f) \\
&\quad= H\frac{r^r}{(r+1)^{r+1}}(b-a)^r\bigvee_a^b(f),
\end{aligned} \tag{29}$$

since $\sup_{t\in[a,b]}(t-a)(b-t)^r=(r^r/(r+1)^{r+1})(b-a)^{r+1}$, which completes the proof. □

Theorem 9. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$; then*

$$|\mathcal{T}(f,g)|\le\frac{1}{6}(b-a)[g(b)-g(a)]\bigvee_a^b(f). \tag{30}$$

Proof. As f and g is of bounded variation on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$, by (22) and using (23) we have

$$\begin{aligned}
&|\mathcal{T}(f,g)| \\
&\quad\le \frac{1}{b-a}\int_a^b\left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du\right.\right. \\
&\qquad\qquad\left.\left.-\frac{1}{b-a}\int_a^b g(u)\,du\right]\right|dt\bigvee_a^b(f).
\end{aligned} \tag{31}$$

In the third part of inequality (18), setting $d=t$ and $c=a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\le\frac{b-t}{b-a}[g(b)-g(a)]. \tag{32}$$

Substituting (32) into (31), we get

$$\begin{aligned}
|\mathcal{T}(f,g)| &\le \frac{1}{b-a}\cdot\frac{g(b)-g(a)}{b-a}\int_a^b(t-a)(b-t)\,dt\bigvee_a^b(f) \\
&= \frac{1}{6}(b-a)[g(b)-g(a)]\bigvee_a^b(f),
\end{aligned} \tag{33}$$

and thus the proof is finished. □

3. The Case When f Is Lipschitzian

In this section, we give some new bounds when f is L-Lipschitzian.

Theorem 10. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is L-Lipschitzian on $[a,b]$ and g is an absolutely continuous on $[a,b]$; then*

$$\begin{aligned}
&|\mathcal{T}(f,g)| \\
&\quad\le L\begin{cases}\dfrac{1}{12}(b-a)^2\|g'\|_\infty, & g'\in L_\infty[a,b]; \\ \dfrac{q^2(1+(1/2^q))^{1/q}}{(2q+1)(3q+1)(q+1)^{1/q}} \\ \quad\times(b-a)^2\|g'\|_p, & g'\in L_p[a,b]; \\ \dfrac{1}{12}(b-a)\|g'\|_1, & g'\in L_1[a,b],\end{cases}
\end{aligned} \tag{34}$$

where $p>1$ and $(1/p)+(1/q)=1$.

Proof. Using the fact that for a Riemann integrable function $p:[c,d]\to\mathbb{R}$ and L-Lipschitzian function $\nu:[c,d]\to\mathbb{R}$, one has the inequality

$$\left|\int_c^d p(t)\,d\nu(t)\right|\le L\int_c^d|p(t)|\,dt. \tag{35}$$

As f' is L-Lipschitzian on $[a,b]$, by (35) we have

$$\begin{aligned}|\mathcal{T}(f,g)| &\le \frac{L}{b-a}\int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| dt \\ &\le \frac{1}{2(b-a)}L\|g'\|_\infty \int_a^b (t-a)(b-t)\,dt \\ &= \frac{1}{12}L(b-a)^2\|g'\|_\infty,\end{aligned} \tag{36}$$

where for the last inequality we used the inequality (15), with $d=t$ and $a=c$, (see (24)).

In the inequality (16), setting $d=t$ and $a=c$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \begin{cases} \dfrac{(b-t)^{1+(1/q)}}{(q+1)^{1/q}(b-a)^{1/q}} \times\left[1+\left(\dfrac{t-a}{b+t-2a}\right)^q\right]^{1/q}\|g'\|_p & g'\in L_p[a,b], \\ & 1\le p<\infty, \\ & \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{1}{2}\dfrac{b-t}{b-a}\|g'\|_1, & g'\in L_1[a,b]. \end{cases} \tag{37}$$

Substituting (37) into (36), we get

$$\begin{aligned}|\mathcal{T}(f,g)| &\le \frac{L}{b-a}\begin{cases} \dfrac{\|g'\|_p}{(q+1)^{1/q}(b-a)^{1/q}} \times\displaystyle\int_a^b (t-a)(b-t)^{1+(1/q)} \times\left[1+\left(\dfrac{t-a}{b+t-2a}\right)^q\right]^{1/q} dt, & g'\in L_p[a,b], \\ \dfrac{1}{2}\dfrac{\|g'\|_1}{b-a}\displaystyle\int_a^b (t-a)(b-t)\,dt, & g'\in L_1[a,b], \end{cases} \\ &\le \frac{L}{b-a}\begin{cases} \dfrac{\|g'\|_p}{(q+1)^{1/q}(b-a)^{1/q}} \times\sup\limits_{t\in[a,b]}\left[1+\left(\dfrac{t-a}{b+t-2a}\right)^q\right]^{1/q} \times\displaystyle\int_a^b (t-a)(b-t)^{1+(1/q)}dt, & g'\in L_p[a,b], \\ \dfrac{1}{12}(b-a)^2\|g'\|_1, & g'\in L_1[a,b], \end{cases} \\ &= L\begin{cases} \dfrac{\|g'\|_p}{(q+1)^{1/q}}\left[1+\dfrac{1}{2^q}\right]^{1/q} \times(b-a)^2\dfrac{q^2}{(2q+1)(3q+1)}, & g'\in L_p[a,b], \\ \dfrac{1}{12}(b-a)\|g'\|_1, & g'\in L_1[a,b], \end{cases}\end{aligned} \tag{38}$$

where $\sup_{t\in[a,b]}[1+((t-a)/(b+t-2a))^q]^{1/q} = (1+(1/2^q))^{1/q}$, and

$$\begin{aligned}\int_a^b (t-a)(b-t)^{1+(1/q)}dt &= (b-a)^{3+(1/q)}\int_0^1 (1-t)\,t^{1+(1/q)}dt \\ &= (b-a)^{3+(1/q)}\frac{q^2}{(2q+1)(3q+1)},\end{aligned} \tag{39}$$

which proves the second and the third inequalities in (34). □

Theorem 11. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is L-Lipschitzian on $[a,b]$ and g is of r-H-Hölder type on $[a,b]$; then*

$$|\mathcal{T}(f,g)| \le LH\frac{(b-a)^{r+1}}{(r+1)^2(r+2)}. \tag{40}$$

Proof. As f is L-Lipschitzian and g is of r-H-Hölder type on $[a,b]$, by (36) and using (17) we have

$$\begin{aligned}|\mathcal{T}(f,g)| &\le \frac{L}{b-a}\int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| dt \\ &\le \frac{LH}{(r+1)(b-a)}\int_a^b (t-a)(b-t)^r dt \\ &= LH\frac{(b-a)^{r+1}}{(r+1)^2(r+2)},\end{aligned} \tag{41}$$

where, for the last inequality, a simple calculation yields that

$$\int_a^b (t-a)(b-t)^r dt = (b-a)^{r+2}\int_0^1 (1-t)t^r dt = \frac{(b-a)^{r+2}}{(r+1)(r+2)}, \tag{42}$$

which completes the proof. □

Corollary 12. *In Theorem 11, if g is M-Lipschitzian on $[a,b]$, then we have*

$$|\mathcal{T}(f,g)| \le \frac{1}{12}LM(b-a)^2. \tag{43}$$

Theorem 13. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$; then*

$$|\mathcal{T}(f,g)| \le \frac{1}{6}L(b-a)^2[g(b)-g(a)]. \tag{44}$$

Proof. As f is L-Lipschitzian on $[a,b]$ and g is monotonic nondecreasing on $[a,b]$, by (35) and using (36) we have

$$|\mathcal{T}(f,g)| \le \frac{L}{b-a}\int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| dt. \tag{45}$$

In the third part of inequality (18), setting $d=t$ and $c=a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \frac{b-t}{b-a}[g(b)-g(a)]. \tag{46}$$

Substituting (46) into (45), we get

$$|\mathcal{T}(f,g)| \le \frac{L}{b-a}\frac{g(b)-g(a)}{b-a}\int_a^b (t-a)(b-t)\,dt = \frac{1}{6}L(b-a)[g(b)-g(a)], \tag{47}$$

which completes the proof. □

4. More Inequalities

In this section we give other related results.

Theorem 14. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f and g are of bounded variation on $[a,b]$; then*

$$|\mathcal{T}(f,g)| \le \frac{1}{4}\bigvee_a^b(g)\bigvee_a^b(f). \tag{48}$$

The constant $1/4$ is the best possible.

Proof. As f and g is of bounded variation on $[a,b]$, by (22) and using (23), we have

$$|\mathcal{T}(f,g)| \le \frac{1}{b-a}\sup_{t\in[a,b]}\left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right|\bigvee_a^b(f). \tag{49}$$

In the first inequality of (18), setting $d=t$ and $c=a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \frac{b-t}{b-a}\bigvee_a^b(g). \tag{50}$$

Substituting (50) into (49), we get

$$|\mathcal{T}(f,g)| \le \frac{1}{(b-a)^2}\sup_{t\in[a,b]}\{(t-a)(b-t)\}\bigvee_a^b(g)\bigvee_a^b(f) = \frac{1}{4}\bigvee_a^b(g)\bigvee_a^b(f), \tag{51}$$

which proves the inequality. The sharpness case trivially holds by taking $f(t)=g(t)=\operatorname{sgn}(t-((a+b)/2))$, which completes the proof. □

When the integrator is of bounded variation we have the following.

Theorem 15. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f is L-Lipschitzian on $[a,b]$ and g is of bounded variation on $[a,b]$; then*

$$|\mathcal{T}(f,g)| \le \frac{1}{6}L(b-a)\bigvee_a^b(g). \tag{52}$$

Proof. As f is L-Lipschitzian on $[a,b]$ and g is of bounded variation on $[a,b]$, by (22) and using (23), we have

$$|\mathcal{T}(f,g)| \le \frac{L}{b-a}\int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| dt. \tag{53}$$

In the second inequality of (18), setting $d=t$ and $c=a$, we get

$$\left|\frac{1}{t-a}\int_a^t g(s)\,ds - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \frac{b-t}{b-a}\bigvee_a^b(g). \tag{54}$$

Substituting (54) into (53), we get

$$|\mathcal{T}(f,g)| \le \frac{L}{(b-a)^2}\int_a^b (t-a)(b-t)\,dt\bigvee_a^b(g) = \frac{1}{6}L(b-a)\bigvee_a^b(g), \tag{55}$$

and the proof is completed. □

When both functions are Lipschitzian we have the following.

Theorem 16. *Let $f, g : [a,b] \to \mathbb{R}$ be, respectively, such that f and g are L_1- and L_2-Lipschitzian on $[a,b]$; then*

$$|\mathcal{T}(f,g)| \le \frac{1}{12} L_1 L_2 (b-a)^2. \tag{56}$$

The constant $1/12$ is the best possible.

Proof. As f and g are L_1- and L_2-Lipschitzian on $[a,b]$, respectively, by (35) and using (36), we have

$$|\mathcal{T}(f,g)| \le \frac{L_1}{b-a} \int_a^b \left| (t-a) \left[\frac{1}{t-a} \int_a^t g(u)\,du - \frac{1}{b-a} \int_a^b g(u)\,du \right] \right| dt. \tag{57}$$

In the second inequality of (18), setting $d = t$ and $c = a$, we get

$$\left| \frac{1}{t-a} \int_a^t g(s)\,ds - \frac{1}{b-a} \int_a^b g(s)\,ds \right| \le \frac{1}{2} L_2 (b-t). \tag{58}$$

Substituting (58) into (57), we get

$$\begin{aligned} |\mathcal{T}(f,g)| &\le \frac{1}{2(b-a)} L_1 L_2 \int_a^b (t-a)(b-t)\,dt \\ &= \frac{1}{12} L_1 L_2 (b-a)^2, \end{aligned} \tag{59}$$

which proves the inequality. The sharpness case trivially holds by taking $f(t) = g(t) = t$, which completes the proof. □

Remark 17. Let g be as in Theorems 7–16. By applying the same techniques used in the corresponding proofs of each theorem, we may obtain several inequalities for monotonic nondecreasing integrator f using the fact that for a monotonic nondecreasing function $\nu : [a,b] \to \mathbb{R}$ and continuous function $p : [a,b] \to \mathbb{R}$, one has the inequality

$$\left| \int_a^b p(t)\,d\nu(t) \right| \le \int_a^b |p(t)|\,d\nu(t). \tag{60}$$

We leave the details to the interested reader.

5. Applications for Self-Adjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be self-adjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows:

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases} \tag{61}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{62}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded self-adjoint operators in Hilbert spaces; see for instance [24, page 256].

Theorem 18 (Spectral Representation Theorem). *Let A be a bonded self-adjoint operator on the Hilbert space H and let $m = \min\{\lambda \mid \lambda \in \mathrm{Sp}(A)\} =: \min \mathrm{Sp}(A)$ and $M = \max\{\lambda \mid \lambda \in \mathrm{Sp}(A)\} =: \max \mathrm{Sp}(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties:*

(a) *$E_\lambda \le E_{\lambda'}$ for $\lambda \le \lambda'$;*

(b) *$E_{m-0} = 0$, $E_M = I$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$;*

(c) *we have the representation*

$$A = \int_{m-0}^{M} \lambda\,dE_\lambda. \tag{63}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left(E_{\lambda_k} - E_{\lambda_{k-1}} \right) \right\| \le \varepsilon \tag{64}$$

whenever

$$\begin{gathered} \lambda_0 < m = \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \le \delta \quad \text{for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \quad \text{for } 1 \le k \le n \end{gathered} \tag{65}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\,dE_\lambda, \tag{66}$$

where the integral is of Riemann-Stieltjes type.

Corollary 19. *With the assumptions of Theorem 18 for A, E_λ, and φ we have the representations*

$$\begin{aligned} \varphi(A)x &= \int_{m-0}^{M} \varphi(\lambda)\,dE_\lambda x \quad \forall x \in H, \\ \langle \varphi(A)x, y \rangle &= \int_{m-0}^{M} \varphi(\lambda)\,d\langle E_\lambda x, y \rangle \quad \forall x, y \in H. \end{aligned} \tag{67}$$

In particular,

$$\langle \varphi(A)x, x \rangle = \int_{m-0}^{M} \varphi(\lambda)\,d\langle E_\lambda x, x \rangle \quad \forall x \in H. \tag{68}$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2 d\|E_\lambda x\|^2 \quad \forall x \in H. \tag{69}$$

We recall the following result (see [25]) that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$.

Theorem 20. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded self-adjoint operator A and let $m = \min Sp(A)$ and $M = \max Sp(A)$. Then for any $x, y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\langle E_{(\cdot)}x, y\rangle\right)\right]^2 \le \left\langle\left(E_\beta - E_\alpha\right)x, x\right\rangle\left\langle\left(E_\beta - E_\alpha\right)y, y\right\rangle, \tag{70}$$

where $\bigvee_\alpha^\beta(\langle E_{(\cdot)}x, y\rangle)$ denotes the total variation of the function $\langle E_{(\cdot)}x, y\rangle$ on $[\alpha, \beta]$.

Remark 21. For $\alpha = m - \varepsilon$ with $\varepsilon > 0$ and $\beta = M$ we get from (70) the inequality

$$\begin{aligned}&\bigvee_{m-\varepsilon}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right)\\ &\quad\le \left\langle\left(1_H - E_{m-\varepsilon}\right)x, x\right\rangle^{1/2}\left\langle\left(1_H - E_{m-\varepsilon}\right)y, y\right\rangle^{1/2}\end{aligned} \tag{71}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right) \le \|x\|\,\|y\|, \tag{72}$$

where $\bigvee_{m-0}^M(\langle E_{(\cdot)}x, y\rangle)$ denotes the limit $\lim_{\varepsilon\to 0+}[\bigvee_{m-\varepsilon}^M(\langle E_{(\cdot)}x, y\rangle)]$.

The inequality (72) was also proved in the recent monographs [26, 27] and will be utilized in the following.

After these preparations we can state and prove the following trapezoidal type inequality for functions of self-adjoint operators on Hilbert spaces.

Theorem 22. *Let A be a bonded self-adjoint operator on the Hilbert space H and let $m = \min\{\lambda \mid \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. If $f : [m, M] \to \mathbb{C}$ is such that its derivative f is of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned}&\left|\left\langle\left[f(A) - \frac{f(m)\left(M1_H - A\right) + f(M)\left(A - m1_H\right)}{M - m}\right]x, y\right\rangle\right|\\ &\quad\le \frac{1}{4}(M - m)\bigvee_{m}^{M}(f)\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right)\\ &\quad\le \frac{1}{4}(M - m)\bigvee_{m}^{M}(f)\,\|x\|\,\|y\|\end{aligned} \tag{73}$$

for any $x, y \in H$.

Proof. Utilising the inequality (48) for the function of bounded variation $g(\lambda) = \langle E_\lambda x, y\rangle$ and the continuous function f we have

$$\begin{aligned}&\left|\int_{m-0}^{M} f(\lambda)\,d\langle E_\lambda x, y\rangle - f(m)\frac{\int_{m-0}^{M}\langle E_\lambda x, y\rangle\,d\lambda}{M - m}\right.\\ &\quad\left. - f(M)\left(\langle x, y\rangle - \frac{\int_{m-0}^{M}\langle E_\lambda x, y\rangle\,d\lambda}{M - m}\right)\right|\\ &\quad\le \frac{1}{4}(M - m)\bigvee_{m}^{M}(f)\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right)\\ &\quad\le \frac{1}{4}(M - m)\bigvee_{m}^{M}(f)\,\|x\|\,\|y\|\end{aligned} \tag{74}$$

for any $x, y \in H$.

By the Spectral Representation Theorem we have

$$\begin{aligned}\int_{m-0}^{M} f(\lambda)\,d\langle E_\lambda x, y\rangle &= \langle f(A)x, y\rangle,\\ \int_{m-0}^{M}\langle E_\lambda x, y\rangle\,d\lambda &= \langle E_\lambda x, y\rangle\,\lambda\big|_{m-0}^{M} - \int_{m-0}^{M}\lambda\,d\langle E_\lambda x, y\rangle\\ &= M\langle x, y\rangle - \langle Ax, y\rangle\\ &= \langle\left(M1_H - A\right)x, y\rangle\end{aligned} \tag{75}$$

for any $x, y \in H$.

Substituting these values into (74) we deduce after simple calculations the desired result (73). □

The above inequality (73) can be utilized for different particular functions of interest in Operator Theory, such as the power, logarithmic, and exponential functions.

If we take $f(t) = t^p$ with $p \ge 1$, then for any positive operator A with $\mathrm{Sp}(A) \subset [m, M] \subset [0, \infty)$ we have the inequality

$$\begin{aligned}&\left|\left\langle\left[A^p - \frac{m^p\left(M1_H - A\right) + M^p\left(A - m1_H\right)}{M - m}\right]x, y\right\rangle\right|\\ &\quad\le \frac{1}{4}(M - m)\left(M^p - m^p\right)\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x, y\rangle\right)\\ &\quad\le \frac{1}{4}(M - m)\left(M^p - m^p\right)\|x\|\,\|y\|\end{aligned} \tag{76}$$

for any $x, y \in H$.

If we take the function $f(t) = \ln t$, then for any positive definite operator A with $\mathrm{Sp}(A) \subset [m, M] \subset (0, \infty)$ we have the inequality

$$\begin{aligned}\left|\left\langle\left[\ln A - \frac{\ln m\left(M1_H - A\right) + \ln M\left(A - m1_H\right)}{M - m}\right]x, y\right\rangle\right| &\le \ln\left(\frac{M}{m}\right)^{(M-m)/4} \bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\le \ln\left(\frac{M}{m}\right)^{(M-m)/4} \|x\|\,\|y\|\end{aligned} \tag{77}$$

for any $x, y \in H$.

Finally, if we take $f(t) = \exp(t)$, then we have for any self-adjoint operator A with $\mathrm{Sp}(A) \subset [m, M] \subset \mathbb{R}$ the inequality

$$\begin{aligned}\Bigg|\Bigg\langle\Bigg[&\exp(A) \\ &- \frac{\exp(m)\left(M1_H - A\right) + \exp(M)\left(A - m1_H\right)}{M - m}\Bigg]x, y\Bigg\rangle\Bigg| \\ &\le \frac{1}{4}(M - m)\left(\exp M - \exp m\right) \bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\le \frac{1}{4}(M - m)\left(\exp M - \exp m\right)\|x\|\,\|y\|\end{aligned} \tag{78}$$

for any $x, y \in H$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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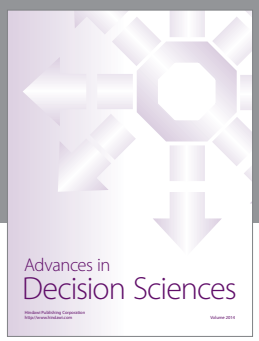

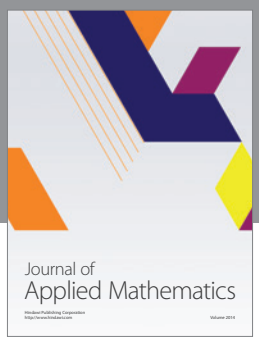

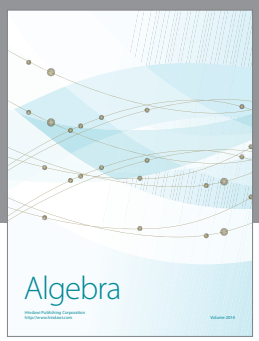

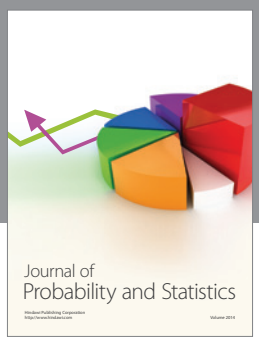

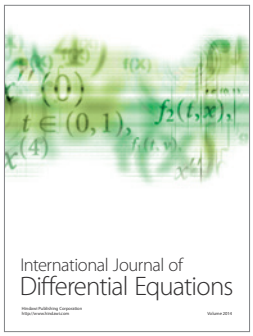

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