Research Article

Existence of Multiple Solutions for a Quasilinear Biharmonic Equation

Wen-Wu Pan and Cheng-En Yu

Department of Science, Sichuan University of Science and Engineering, Zigong 643000, China

Correspondence should be addressed to Wen-Wu Pan; wenwupan@sohu.com

Received 18 May 2014; Accepted 20 August 2014; Published 29 October 2014

Academic Editor: George L. Karakostas

Copyright © 2014 W.-W. Pan and C.-E. Yu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Using three critical points theorems, we prove the existence of at least three solutions for a quasilinear biharmonic equation.

1. Introduction

In this paper, we show the existence of at least three weak solutions for the Navier boundary value problem

$$\begin{aligned}\Delta^2 u - \operatorname{div}(\nabla u) &= \lambda f(x, u) + g(u) \quad \text{in } \Omega, \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega,\end{aligned}\quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($4 > N \geq 1$) is a nonempty bounded open set with a sufficient smooth boundary $\partial\Omega$, $\lambda > 0$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function, and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L > 0$; that is,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2| \quad (2)$$

for every $t_1, t_2 \in \mathbb{R}$ and $g(0) = 0$.

Motivated by the fact that such problems are used to describe a large class of physical phenomena, many authors looked for existence and multiplicity of solutions for forth-order nonlinear equations. For an overview on this subject, we cite the papers [1–23]. For instance, when $N = 1$, in [22], Liu and Li, using Ricceri's three critical points theorem [24], established the existence of at least three weak solutions for the following problem:

$$\begin{aligned}u^{iv} + \alpha u'' + \beta u &= \lambda f(x, u) \quad x \in (a, b), \\ u(a) = u(b) = u''(a) &= u''(b) = 0,\end{aligned}\quad (3)$$

where α, β are real constants and λ is a positive parameter and $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a L^2 -Carathéodory function. Later, some authors generalized this type of equation (see [1–5, 10, 12, 15, 19]). When $N > 1$, Liu and Su [21] also used Ricceri's three critical points theorem [24] to established the existence of at least three weak solutions for the following problem:

$$\begin{aligned}\Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \lambda f(x, u) \quad \text{in } \Omega, \\ u = \Delta u &= 0 \quad \text{on } \partial\Omega,\end{aligned}\quad (4)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a nonempty bounded open set with a sufficient smooth boundary $\partial\Omega$, $p > \max\{1, N/2\}$, $\lambda > 0$, and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function. After that some authors used different critical point theorems to get one nontrivial, at least three, and infinitely many solutions (see [6–8, 16, 18]). Elliptic systems were also considered by [9, 11, 13, 14, 17, 20, 23].

The goal of the present paper is to establish some new criteria for (1) to have at least three weak solutions (Theorems 4 and 5). Our analysis is mainly based on recent critical point theorems that are contained in Theorems 2 and 3. In fact, employing rather different three critical points theorems, under different assumptions on the nonlinear term f , we obtain the exact collections of λ for which (1) admits at least three weak solutions in the space $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

A special case of our main results is the following theorem.

Theorem 1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$ and $g(0) = 0$ such that $L < 1/K^2 m(\Omega)$, where K is defined by (17). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and put $F(t) = \int_0^t f(\xi) d\xi$ for each $t \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d > 0$ and $F(\xi) \geq 0$ in $[0, d]$ and

$$\liminf_{\xi \rightarrow 0} \frac{F(\xi)}{\xi^2} = 0, \quad \limsup_{|\xi| \rightarrow +\infty} \frac{F(\xi)}{\xi^2} = 0. \quad (5)$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ the problem

$$\begin{aligned} \Delta^2 u - \operatorname{div}(\nabla u) &= \lambda f(u) + g(u) \quad \text{in } \Omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\Omega \end{aligned} \quad (6)$$

admits at least three weak solutions.

2. Preliminaries

First we here recall for the reader's convenience our main tools to prove the results. The first result has been obtained in [25] and the second one in [26].

Theorem 2 (see [25, Theorem 3.1]). Let X be a separable and reflexive real Banach space, $\Phi : X \rightarrow \mathbb{R}$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* , and $\Psi : X \rightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = \Psi(x_0) = 0$ and that

$$\lim_{\|x\| \rightarrow +\infty} (\Phi(x) - \lambda\Psi(x)) = +\infty \quad \forall \lambda \in [0, +\infty[. \quad (7)$$

Further, assume that there are $r > 0$, $x_1 \in X$, such that $r < \Phi(x_1)$ and

$$\sup_{x \in \Phi^{-1}([-\infty, r])^w} \Psi(x) < \frac{r}{r + \Phi(x_1)} \Psi(x_1); \quad (8)$$

here $\overline{\Phi^{-1}([-\infty, r])}^w$ denotes the closure of $\Phi^{-1}([-\infty, r])$ in the weak topology. Then, for each

$$\lambda \in \Lambda_1 := \left[\frac{\Phi(x_1)}{\Psi(x_1) - \sup_{x \in \Phi^{-1}([-\infty, r])^w} \Psi(x)}, \frac{r}{\sup_{x \in \Phi^{-1}([-\infty, r])^w} \Psi(x)} \right], \quad (9)$$

the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0 \quad (10)$$

has at least three solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subseteq \left[0, \frac{hr}{r(\Psi(x_1)/\Phi(x_1)) - \sup_{x \in \Phi^{-1}([-\infty, r])^w} \Psi(x)} \right] \quad (11)$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, (10) has at least three solutions in X whose norms are less than σ .

Theorem 3 (see [26, Theorem 3.6]). Let X be a reflexive real Banach space; let $\Phi : X \rightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable whose Gâteaux derivative admits a continuous inverse on X^* , and let $\Psi : X \rightarrow \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist $r \in \mathbb{R}$ and $u_1 \in X$ with $0 < r < \Phi(u_1)$, such that

$$\begin{aligned} \text{(A1)} \quad & \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) < r(\Psi(u_1)/\Phi(u_1)); \\ \text{(A2)} \quad & \text{for each } \lambda \in \Lambda_r :=]\Phi(u_1)/\Psi(u_1), r / \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)[\text{ the functional } \Phi - \lambda\Psi \text{ is coercive.} \end{aligned}$$

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X .

Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be an L^1 -Carathéodory function and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$; that is,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2| \quad (12)$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$.

Put

$$F(x, t) := \int_0^t f(x, \xi) d\xi, \quad G(t) := - \int_0^t g(\xi) d\xi \quad (13)$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. Denote

$$X := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega); \quad (14)$$

the usual norm in X is defined by

$$\|u\| := \left(\int_{\Omega} |\Delta u(x)|^2 + |\nabla u(x)|^2 dx \right)^{1/2}. \quad (15)$$

Note that X is a separable and reflexive real Banach space.

We say that a function $u \in X$ is a *weak solution* of problem (1) if

$$\begin{aligned} \int_{\Omega} \Delta u(x) \Delta v(x) dx + \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx \\ = \lambda \int_{\Omega} f(x, u(x)) v(x) dx - \int_{\Omega} g(u(x)) v(x) dx = 0, \end{aligned} \quad (16)$$

for all $v \in X$.

It is well known that $(X, \|\cdot\|)$ is embedded in $(C^0(\Omega), \|\cdot\|_{\infty})$ and

$$\|u\|_{\infty} \leq K \|u\| \quad (17)$$

for all $u \in X$. Since $N < 4$, one has $K < +\infty$.

Suppose that the Lipschitz constant $L > 0$ of the function g satisfies $L < 1/K^2 m(\Omega)$. For other basic notations and definitions, we refer the reader to [27–29].

3. Main Results

Our main results are the following theorems.

Theorem 4. Assume that there exist a function $w \in X$, a positive function $a \in L^1$, and two positive constants r and γ with $\gamma < p$ such that

$$(A1) \quad \|w\|^2 > 2r/(1 - K^2 Lm(\Omega));$$

$$(A2) \quad \int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx < r \left(\int_{\Omega} F(x, w(x)) dx / (r + ((1 + K^2 Lm(\Omega))/2) \|w\|^2) \right);$$

$$(A3) \quad F(x, t) \leq a(x)(1 + |t|^\gamma) \text{ for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}.$$

Then, for each λ in

$$\Lambda_1 := \left[\left(\left((1 + K^2 Lm(\Omega)) / 2 \right) \|w\|^2 \right) \times \left(\int_{\Omega} F(x, w(x)) dx - \int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \right)^{-1} \frac{r}{\int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx} \right], \quad (18)$$

problem (1) admits at least three weak solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subseteq \left[0, (hr) \left(\frac{2r}{(1 + K^2 Lm(\Omega)) \|w\|^2} \int_{\Omega} F(x, w(x)) dx - \int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \right)^{-1} \right] \quad (19)$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the problem (1) admits at least three weak solutions in X whose norms are less than σ .

Theorem 5. Assume that there exists a function $w \in X$ and a positive constant r such that

$$(B1) \quad \|w\|^2 > 2r/(1 - K^2 Lm(\Omega));$$

$$(B2) \quad \int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx / r < (2/(1 + K^2 Lm(\Omega))) \left(\int_{\Omega} F(x, w(x)) dx / \|w\|^2 \right);$$

$$(B3) \quad 2K^2/(1 - K^2 Lm(\Omega)) \limsup_{|t| \rightarrow +\infty} F(x, t)/t^2 < \left(\int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \right) / r.$$

Then, for each

$$\lambda \in \left[\frac{1 + K^2 Lm(\Omega)}{2} \frac{\|w\|^2}{\int_{\Omega} F(x, w(x)) dx}, \frac{r}{\int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx} \right], \quad (20)$$

problem (1) admits at least three weak solutions.

Let us give particular consequences of Theorems 4 and 5 for a fixed test function w . Now, fix $x^0 \in \Omega$ and pick γ with $\gamma > 0$ such that $B(x^0, \gamma) \subset \Omega$ where $B(x^0, \gamma)$ denotes the ball with center at x^0 and radius of γ . Put

$$Q = \int_{B(x^0, \gamma) \setminus B(x^0, \gamma/2)} \left| \frac{12}{\gamma^3} |x - x^0| l - \frac{24}{\gamma^2} l + \frac{9}{\gamma} \frac{l}{|x - x^0|} \right|^2 dx,$$

$$R = \frac{\pi^{N/2}}{\Gamma(N/2)} \int_{(\gamma/2)^2}^{\gamma^2} \left| \frac{12(N+1)}{\gamma^3} \sqrt{t} + \frac{9(N-1)}{\gamma} \frac{1}{\sqrt{t}} - \frac{24N}{\gamma^2} \right|^2 \times t^{N/2-1} dt, \quad (21)$$

$$\theta = K(R + Q)^{1/2}, \quad (22)$$

where $l = (\sum_{i=1}^N x_i^2)^{1/2}$, $|x - x^0| = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$, and $m(\Omega)$ denotes the volume of Ω .

Corollary 6. Assume that there exist a positive function $a \in L^1$ and three positive constants c , d , and γ with $c < \theta d$ and $\gamma < 2$ such that Assumption (A3) in Theorem 4 holds. Furthermore, suppose that

$$(A4) \quad F(x, t) \geq 0 \text{ for a.e. } x \in \Omega \setminus B(x^0, \gamma/2) \text{ and all } t \in [0, d];$$

$$(A5) \quad \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx < (1 - K^2 Lm(\Omega)) c^2 \left(\int_{B(x^0, \gamma/2)} F(x, d) dx / ((1 - K^2 Lm(\Omega)) c^2 + (1 + K^2 Lm(\Omega)) (\theta d)^2) \right).$$

Then, for each λ in

$$\Lambda'_1 := \left[\frac{1 + K^2 Lm(\Omega)}{2} \times (\theta d)^2 \left(2K^2 \left(\int_{B(x^0, \gamma/2)} F(x, d) dx - \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx \right) \right)^{-1}, \frac{1 - K^2 Lm(\Omega)}{2} \frac{c^2}{2K^2 \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx} \right], \quad (23)$$

problem (1) admits at least three weak solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda'_2 \subseteq \left[0, \frac{1 - K^2 Lm(\Omega)}{2} \times \frac{(h/2)(c/K)^2}{(c/\theta d)^2 \int_{B(x^0, \gamma/2)} F(x, d) dx - \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx} \right] \quad (24)$$

and a positive real number σ such that, for each $\lambda \in \Lambda'_2$, problem (1) admits at least three weak solutions in X whose norms are less than σ .

Proof. We claim that all the assumptions of Theorem 4 are fulfilled with w given by

$$w(x) = \begin{cases} 0 & \text{for } x \in \Omega \setminus B(x^0, \gamma), \\ d \left(\frac{4}{\gamma^3} |x - x^0|^3 - \frac{12}{\gamma^2} |x - x^0|^2 + \frac{9}{\gamma} |x - x^0| - 1 \right) & \text{for } x \in B(x^0, \gamma) \setminus B(x^0, \gamma/2), \\ d & \text{for } x \in B(x^0, \gamma/2), \end{cases} \quad (25)$$

and $r := ((1 - K^2 Lm(\Omega))/2)(c/K)^2$. It is easy to verify that $w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, and, in particular, one has

$$\|w\|^2 = (R + Q) d^2, \quad (26)$$

and consequently from (22) we see that

$$\|w\| = \frac{\theta d}{K}. \quad (27)$$

Thus, (A1) holds. Since $0 \leq w(x) \leq d$ for each $x \in \Omega$, the condition (A4) ensures that

$$\int_{\Omega \setminus B(x^0, \gamma)} F(x, w(x)) dx + \int_{B(x^0, \gamma) \setminus B(x^0, \gamma/2)} F(x, w(x)) dx \geq 0, \quad (28)$$

so, from (A5),

$$\begin{aligned} & \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx < (1 - K^2 Lm(\Omega)) c^2 \\ & \times \frac{\int_{B(x^0, \gamma/2)} F(x, d) dx}{(1 - K^2 Lm(\Omega)) c^2 + (1 + K^2 Lm(\Omega)) (\theta d)^2} \\ & = (1 - K^2 Lm(\Omega)) c^2 \frac{1}{2K^2} \\ & \times \left(\left(\int_{B(x^0, \gamma/2)} F(x, d) dx \right) \times \left(\frac{1}{2K^2} (1 - K^2 Lm(\Omega)) c^2 + \frac{1}{2K^2} (1 + K^2 Lm(\Omega)) (\theta d)^2 \right)^{-1} \right) \\ & = r \frac{\int_{\Omega} F(x, w(x)) dx}{r + ((1 + K^2 Lm(\Omega))/2) \|w\|^2}, \end{aligned} \quad (29)$$

and thus (A2) holds. Next notice that

$$\begin{aligned} & \frac{((1 + K^2 Lm(\Omega))/2) \|w\|^2}{\int_{\Omega} F(x, w(x)) dx - \int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1 - LK^2 m(\Omega))}} F(x, t) dx} \\ & \leq \frac{(1 + K^2 Lm(\Omega)) (\theta d)^2 / 2K^2}{\int_{B(x^0, \gamma/2)} F(x, d) dx - \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx} \\ & = \frac{r}{\int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1 - LK^2 m(\Omega))}} F(x, t) dx} \\ & = \frac{(1 - K^2 Lm(\Omega)) c^2}{2K^2 \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx}. \end{aligned} \quad (30)$$

In addition note that

$$\begin{aligned}
 & \frac{(1 + K^2 Lm(\Omega) (\theta d)^2) / 2K^2}{\int_{B(x^0, \gamma/2)} F(x, d) dx - \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx} \\
 & < \left(\frac{1 + K^2 Lm(\Omega) d(\theta)^2}{2K^2} \right) \\
 & \times \left(\left(\frac{(1 - K^2 Lm(\Omega)) c^2}{(1 - K^2 Lm(\Omega)) c^2 + (1 + K^2 Lm(\Omega)) (\theta d)^2} - 1 \right) \right. \\
 & \quad \left. \times \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx \right)^{-1} \\
 & = \frac{(1 - K^2 Lm(\Omega)) c^2}{2K^2 \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx}.
 \end{aligned} \tag{31}$$

Finally note that

$$\begin{aligned}
 (hr) & \left(\frac{2r}{(1 + K^2 Lm(\Omega)) \|w\|^2} \int_{\Omega} F(x, w(x)) dx \right. \\
 & \quad \left. - \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1 - K^2 Lm(\Omega))}} F(x, t) dx \right)^{-1} \\
 & \leq \frac{1 - K^2 Lm(\Omega)}{2} \\
 & \quad \times \left(\frac{h}{2} \left(\frac{c}{K} \right)^2 \right) \left(\left(\frac{c}{\theta d} \right)^2 \int_{B(x^0, \gamma/2)} F(x, d) dx \right. \\
 & \quad \left. - \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx \right)^{-1},
 \end{aligned} \tag{32}$$

and, taking into account that $\Lambda'_1 \subseteq \Lambda_1$ and $\Lambda_2 \subseteq \Lambda'_2$, we have the desired conclusion directly from Theorem 4. $\square$

Corollary 7. Assume that there exist two positive constants c and d with $c < \theta d$ such that the assumption (A4) in Corollary 6 holds. Furthermore, suppose that

$$(B4) \quad \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx / (1 - K^2 Lm(\Omega)) c^2 < \int_{B(x^0, \gamma/2)} F(x, d) dx / (1 + K^2 Lm(\Omega)) (\theta d)^2;$$

$$(B5) \quad \limsup_{|t| \rightarrow +\infty} F(x, t) / t^2 < \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx / c^2.$$

Then, for each

$$\lambda \in \left[\frac{1 + K^2 Lm(\Omega)}{2} \frac{1}{2K^2 \int_{B(x^0, \gamma/2)} F(x, d) dx}, \frac{1 - K^2 Lm(\Omega)}{2} \frac{c^2}{2K^2 \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx} \right], \tag{33}$$

problem (1) admits at least three weak solutions.

Proof. All the assumptions of Theorem 5 are fulfilled by choosing w as given in (25) and $r := ((1 - K^2 Lm(\Omega))/2)(c/K)^2$ and bearing in mind that

$$\|w\| = \frac{\theta d}{K} \tag{34}$$

and recalling

$$\begin{aligned}
 & \int_{\Omega \setminus B(x^0, \gamma)} F(x, w(x)) dx \\
 & + \int_{B(x^0, \gamma) \setminus B(x^0, \gamma/2)} F(x, w(x)) dx \geq 0.
 \end{aligned} \tag{35}$$

Hence, by applying Theorem 5, we have the conclusion. $\square$

Proof of Theorem 1. Fix $\lambda > \lambda^* := m(\Omega)(1 + K^2 Lm(\Omega))(\theta d)^2 / m(B(x^0, \gamma))F(d)$ for some $d > 0$. Since

$$\liminf_{\xi \rightarrow 0} \frac{F(\xi)}{\xi^2} = 0, \tag{36}$$

there is $\{c_m\}_{m \in \mathbb{N}} \subseteq]0, +\infty[$ such that $\lim_{m \rightarrow +\infty} c_m = 0$ and

$$\lim_{m \rightarrow +\infty} \frac{\sup_{|\xi| \leq c_m} F(\xi)}{c_m} = 0. \tag{37}$$

In fact, one has

$$\lim_{m \rightarrow +\infty} \frac{\sup_{|\xi| \leq c_m} F(\xi)}{c_m} = \lim_{m \rightarrow +\infty} \frac{F(\xi_{c_m})}{\xi_{c_m}^2} \cdot \frac{\xi_{c_m}^2}{c_m} = 0, \tag{38}$$

where $F(\xi_{c_m}) = \sup_{|\xi| \leq c_m} F(\xi)$. Hence, there is $\bar{c} > 0$ such that

$$\frac{\sup_{|\xi| \leq \bar{c}} F(\xi)}{\bar{c}^2} < \min \left\{ \frac{m(B(x^0, \gamma))(1 - K^2 Lm(\Omega))F(d)}{m(\Omega)(1 + K^2 Lm(\Omega))(\theta d)^2}, \frac{(1 - K^2 Lm(\Omega))}{\lambda} \right\} \tag{39}$$

and $\bar{c} < \theta d$. From Corollary 7 we have the desired conclusion. $\square$

4. Proofs

Proof of Theorem 4. Our aim is to apply Theorem 2 to our problem. To this end, for each $u \in X$, we let the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2} \|u\|^2 + \int_{\Omega} G(u(x)) dx, \tag{40}$$

$$\Psi(u) := \int_{\Omega} F(x, u(x)) dx,$$

and put

$$I_{\lambda}(u) := \Phi(u) - \lambda \Psi(u) \quad \forall u \in X. \tag{41}$$

The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\begin{aligned} \Phi'(u)(v) &= \int_{\Omega} \Delta u(x) \Delta v(x) dx \\ &+ \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx \\ &- \int_{\Omega} g(u(x)) v(x) dx, \end{aligned} \quad (42)$$

for every $v \in X$. Furthermore, the differential $\Phi' : X \rightarrow X^*$ is a Lipschitzian operator. Indeed, for any $u, v \in X$, there holds

$$\begin{aligned} &\|\Phi'(u) - \Phi'(v)\|_{X^*} \\ &= \sup_{\|w\| \leq 1} |(\Phi'(u) - \Phi'(v), w)| \\ &\leq \sup_{\|w\| \leq 1} |(u - v, w)| \\ &\quad + \sup_{\|w\| \leq 1} \int_{\Omega} |g(u(x)) - g(v(x))| |w(x)| dx \\ &\leq \sup_{\|w\| \leq 1} \|u - v\| \|w\| \\ &\quad + \sup_{\|w\| \leq 1} \left(\int_{\Omega} |g(u(x)) - g(v(x))|^2 \right)^{1/2} \\ &\quad \times \left(\int_{\Omega} |w(x)|^2 \right)^{1/2}. \end{aligned} \quad (43)$$

Recalling that g is Lipschitz continuous and the embedding $X \hookrightarrow L^2(\Omega)$ is compact, the claim is true. In particular, we derive that Φ is continuously differentiable. The inequality (17) yields for any $u, v \in X$ the estimate

$$\begin{aligned} &(\Phi'(u) - \Phi'(v), u - v) \\ &= (u - v, u - v) - \int_{\Omega} (g(u(x)) - g(v(x))) \\ &\quad \times (u(x) - v(x)) dx \\ &\geq (1 - K^2 L m(\Omega)) \|u - v\|^2. \end{aligned} \quad (44)$$

By the assumption $L < 1/K^2 m(\Omega)$, it turns out that Φ' is a strongly monotone operator. So, by applying Minty-Browder theorem [29, Theorem 26.A], $\Phi' : X \rightarrow X^*$ admits a Lipschitz continuous inverse. On the other hand, the fact that X is compactly embedded into $C^0(\Omega)$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable, and with compact derivative, whose Gateaux derivative at the point $u \in X$ is given by

$$\Psi'(u)(v) = \int_{\Omega} f(x, u(x)) v(x) dx, \quad (45)$$

for every $v \in X$. Note that the weak solutions of (1) are exactly the critical points of I_{λ} . Also, since g is Lipschitz continuous and satisfies $g(0) = 0$, we have from (17) that

$$\frac{1 - K^2 L m(\Omega)}{2} \|u\|^2 \leq \Phi(u) \leq \frac{1 + K^2 L m(\Omega)}{2} \|u\|^2, \quad (46)$$

for all $u \in X$, and so Φ is coercive.

Furthermore from (A3) for any fixed $\lambda \in [0, +\infty[$, using (46), taking (17) into account, we have

$$\begin{aligned} &\Phi(u) - \lambda \Psi(u) \\ &= \frac{1}{2} \|u\|^2 + \int_{\Omega} G(u(x)) dx - \lambda \int_{\Omega} F(x, u(x)) dx \\ &\geq \frac{1 - K^2 L m(\Omega)}{2} \|u\|^2 - \lambda \int_{\Omega} a(x) (1 + |u(x)|^{\gamma}) dx \\ &\geq \frac{1 - K^2 L m(\Omega)}{2} \|u\|^2 - \lambda \|a\|_{L^1(\Omega)} (m(\Omega) + K^{\gamma} \|u\|^{\gamma}), \end{aligned} \quad (47)$$

and so

$$\lim_{\|u\| \rightarrow +\infty} (\Phi(u) - \lambda \Psi(u)) = +\infty. \quad (48)$$

Also according to (A1) we achieve $\Phi(w) > r$. From the definition of Φ and by using (46) we have

$$\begin{aligned} &\Phi^{-1}([-\infty, r]) \\ &= \{u \in X : \Phi(u) < r\} \\ &\subseteq \left\{ u \in X : \|u\| < \sqrt{\frac{2r}{1 - LK^2 m(\Omega)}} \right\} \\ &\subseteq \left\{ u \in X : |u(x)| < K \sqrt{\frac{2r}{1 - LK^2 m(\Omega)}} \quad \forall x \in \Omega \right\}. \end{aligned} \quad (49)$$

So, we obtain

$$\sup_{u \in \overline{\Phi^{-1}([-\infty, r])}^w} \Psi(u) \leq \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1 - LK^2 m(\Omega))}} F(x, t) dx. \quad (50)$$

Therefore, from (A2) and (46), we have

$$\begin{aligned} &\sup_{u \in \overline{\Phi^{-1}([-\infty, r])}^w} \Psi(u) \\ &\leq \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1 - LK^2 m(\Omega))}} F(x, t) dx \\ &< \frac{r}{r + ((1 + K^2 L m(\Omega))/2) \|w\|^2} \int_{\Omega} F(x, w(x)) dx \\ &< \frac{r}{r + \Phi(w)} \Psi(w). \end{aligned} \quad (51)$$

Now, we can apply Theorem 2. Note that, for each $x \in \Omega$,

$$\begin{aligned} & \frac{\Phi(w)}{\Psi(w) - \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \\ & \leq \left(\frac{1 + K^2 Lm(\Omega)}{2} \|w\|^2 \right) \\ & \quad \times \left(\int_{\Omega} F(x, w(x)) dx \right. \\ & \quad \left. - \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \right)^{-1}, \\ & \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \\ & \geq \frac{r}{\int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx}. \end{aligned} \quad (52)$$

Note also that (A2) implies

$$\begin{aligned} & \frac{\left((1 + K^2 Lm(\Omega)) / 2 \right) \|w\|^2}{\int_{\Omega} F(x, w(x)) dx - \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx} \\ & < \left(\frac{1 + K^2 Lm(\Omega)}{2} \|w\|^2 \right) \\ & \quad \times \left(\left(\frac{r + \left((1 + K^2 Lm(\Omega)) / 2 \right) \|w\|^2}{r} - 1 \right) \right. \\ & \quad \left. \times \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \right)^{-1} \\ & = \frac{r}{\int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx}. \end{aligned} \quad (53)$$

Also,

$$\begin{aligned} & \frac{hr}{r(\Psi(w)/\Phi(w)) - \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)} \\ & \leq (hr) \left(\frac{2r}{(1 + K^2 Lm(\Omega)) \|w\|^2} \int_{\Omega} F(x, w(x)) dx \right. \\ & \quad \left. - \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \right)^{-1} = \rho. \end{aligned} \quad (54)$$

From (A2) it follows that

$$\begin{aligned} & \frac{2r}{(1 + K^2 Lm(\Omega)) \|w\|^2} \int_{\Omega} F(x, w(x)) dx \\ & - \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \\ & > \left(\frac{2r}{(1 + K^2 Lm(\Omega)) \|w\|^2} \right. \\ & \quad \left. - \frac{r}{r + \left((1 + K^2 Lm(\Omega)) / 2 \right) \|w\|^2} \right) \\ & \quad \times \int_{\Omega} F(x, w(x)) dx \\ & \geq \left(\frac{2r}{(1 + K^2 Lm(\Omega)) \|w\|^2} - \frac{2r}{(1 + K^2 Lm(\Omega)) \|w\|^2} \right) \\ & \quad \times \int_{\Omega} F(x, w(x)) dx = 0, \end{aligned} \quad (55)$$

since $\int_{\Omega} F(x, w(x)) dx \geq 0$ (note $F(x, 0) = 0$ so $\int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx \geq 0$ and now apply (A2)). Now with $x_0 = 0$ and $x_1 = w$ from Theorem 2 (note $\Psi(0) = 0$) it follows that, for each $\lambda \in \Lambda_1$, the problem (1) admits at least three weak solutions and there exist an open interval $\Lambda_2 \subseteq [0, \rho]$ and a real positive number σ such that, for each $\lambda \in \Lambda_2$, the problem (1) admits at least three weak solutions whose norms in X are less than σ . Thus, the conclusion is achieved. $\square$

Proof of Theorem 5. To apply Theorem 3 to our problem, we take the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ as given in the proof of Theorem 4. Let us prove that the functionals Φ and Ψ satisfy the conditions required in Theorem 3. The regularity assumptions on Φ and Ψ , as requested in Theorem 3, hold. According to (B1) we deduce $\Phi(w) > r$. From the definition of Φ we have

$$\begin{aligned} & \Phi^{-1}([-\infty, r]) \\ & \subseteq \left\{ u \in X : |u(x)| < K \sqrt{\frac{2r}{1 - LK^2m(\Omega)}} \quad \forall x \in \Omega \right\}, \end{aligned} \quad (56)$$

and it follows that

$$\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) \leq \int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx. \quad (57)$$

Therefore, due to assumption (B2), we have

$$\begin{aligned} & \frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} \\ & \leq \frac{\int_{\Omega} \sup_{|t| \leq K \sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx}{r} \end{aligned}$$

$$\begin{aligned}
&< \frac{2}{1 + K^2 Lm(\Omega)} \frac{\int_{\Omega} F(x, w(x)) dx}{\|w\|^p} \\
&\leq \frac{\Psi(w)}{\Phi(w)}.
\end{aligned} \tag{58}$$

Furthermore, from (B3) there exist two constants $\eta, \vartheta \in \mathbb{R}$ with

$$\eta < \frac{\int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx}{r} \tag{59}$$

such that

$$\frac{2K^2}{1 - K^2 Lm(\Omega)} F(x, t) \leq \eta t^2 + \vartheta \tag{60}$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$. Fix $u \in X$. Then

$$F(x, u(x)) \leq \frac{1 - K^2 Lm(\Omega)}{2K^2} (\eta |u(x)|^2 + \vartheta) \tag{61}$$

for all $x \in \Omega$. Now, to prove the coercivity of the functional $\Phi - \lambda\Psi$, first we assume that $\eta > 0$. So, for any fixed

$$\lambda \in \left[\frac{1 + K^2 Lm(\Omega)}{2} \frac{\|u\|^2}{\int_{\Omega} F(x, w(x)) dx}, \frac{r}{\int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx} \right], \tag{62}$$

using (61), we have

$$\begin{aligned}
&\Phi(u) - \lambda\Psi(u) \\
&= \frac{1}{2} \|u\|^2 + \int_{\Omega} G(u(x)) dx - \lambda \int_{\Omega} F(x, u(x)) dx \\
&\geq \frac{1 - K^2 Lm(\Omega)}{2} \|u\|^2 - \lambda \frac{1 - K^2 Lm(\Omega)}{2K^2} \\
&\quad \times \left(\eta \int_{\Omega} |u(x)|^2 dx + \vartheta \right) \\
&\geq \frac{1 - K^2 Lm(\Omega)}{2} \\
&\quad \times \left(1 - \eta \frac{r}{\int_{\Omega} \sup_{|t| \leq K\sqrt{2r/(1-LK^2m(\Omega))}} F(x, t) dx} \right) \\
&\quad \times \|u\|^2 - \frac{1 - K^2 Lm(\Omega)}{2} \vartheta,
\end{aligned} \tag{63}$$

and thus

$$\lim_{\|u\| \rightarrow +\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty. \tag{64}$$

On the other hand, if $\eta \leq 0$, clearly we obtain $\lim_{\|u\| \rightarrow +\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty$. Both cases lead to the coercivity of functional $\Phi - \lambda\Psi$.

So, assumptions (A1) and (A2) in Theorem 3 are satisfied. Hence, by using Theorem 3, the problem (1) admits at least three distinct weak solutions in X . $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities (no. XDJK2013D007), the Scientific Research Fund of SUSE (no. 2011KY03), and the Scientific Research Fund of Sichuan Provincial Education Department (no. 12ZB081).

References

- [1] G. A. Afrouzi, S. Heidarkhani, and D. O'Regan, "Existence of three solutions for a doubly eigenvalue fourth-order boundary value problem," *Taiwanese Journal of Mathematics*, vol. 15, no. 1, pp. 201–210, 2011.
- [2] G. Bonanno and B. Di Bella, "A boundary value problem for fourth-order elastic beam equations," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 2, pp. 1166–1176, 2008.
- [3] G. Bonanno and B. Di Bella, "A fourth-order boundary value problem for a Sturm-Liouville type equation," *Applied Mathematics and Computation*, vol. 217, no. 8, pp. 3635–3640, 2010.
- [4] G. Bonanno and B. Di Bella, "Infinitely many solutions for a fourth-order elastic beam equation," *Nonlinear Differential Equations and Applications*, vol. 18, no. 3, pp. 357–368, 2011.
- [5] G. Bonanno, B. Di Bella, and D. O'Regan, "Non-trivial solutions for nonlinear fourth-order elastic beam equations," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1862–1869, 2011.
- [6] P. Candito, L. Li, and R. Livrea, "Infinitely many solutions for a perturbed nonlinear Navier boundary value problem involving the p -biharmonic," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 17, pp. 6360–6369, 2012.
- [7] P. Candito and R. Livrea, "Infinitely many solution for a nonlinear Navier boundary value problem involving the p -biharmonic," *Studia Universitatis Babes-Bolyai Mathematica*, vol. 55, no. 4, pp. 41–51, 2010.
- [8] P. Candito and G. M. Bisci, "Multiple solutions for a Navier boundary value problem involving the p -biharmonic operator," *Discrete and Continuous Dynamical Systems S*, vol. 5, no. 4, pp. 741–751, 2012.
- [9] J. R. Graef, S. Heidarkhani, and L. Kong, "Multiple solutions for a class of $(p_1 \cdots p_n)$ -biharmonic systems," *Communications on Pure and Applied Analysis*, vol. 12, no. 3, pp. 1393–1406, 2013.
- [10] S. Heidarkhani, "Existence of solutions for a two-point boundary-value problem of a fourth-order Sturm-Liouville type," *Electronic Journal of Differential Equations*, No. 84, 15 pages, 2012.
- [11] S. Heidarkhani, "Non-trivial solutions for a class of $(p_1, \dots, p_n)$ -biharmonic systems with Navier boundary conditions," *Annales Polonici Mathematici*, vol. 105, no. 1, pp. 65–76, 2012.
- [12] S. Heidarkhani, "Non-trivial solutions for two-point boundary-value problems of fourth-order Sturm-Liouville type equations," *Electronic Journal of Differential Equations*, No. 27, 9 pages, 2012.
- [13] S. Heidarkhani, "Three solutions for a class of $(p_1, \dots, p_n)$ -biharmonic systems via variational methods," *Thai Journal of Mathematics*, vol. 10, no. 3, pp. 497–515, 2012.

- [14] S. Heidarkhani, Y. Tian, and C. Tang, "Existence of three solutions for a class of $(p_1, \dots, p_n)$ -biharmonic systems with Navier boundary conditions," *Annales Polonici Mathematici*, vol. 104, no. 3, pp. 261–277, 2012.
- [15] S. M. Khalkhali, S. Heidarkhani, and A. Razani, "Infinitely many solutions for a fourth-order boundary-value problem," *Electronic Journal of Differential Equations*, no. 164, p. 14, 2012.
- [16] C. Li and C.-L. Tang, "Three solutions for a Navier boundary value problem involving the p -biharmonic," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1339–1347, 2010.
- [17] L. Li, "Existence of three solutions for a class of Navier quasilinear elliptic systems involving the $(p_1, \dots, p_n)$ -biharmonic," *Bulletin of the Korean Mathematical Society*, vol. 50, no. 1, pp. 57–71, 2013.
- [18] L. Li and S. Heidarkhani, "Existence of three solutions to a double eigenvalue problem for the p -biharmonic equation," *Annales Polonici Mathematici*, vol. 104, no. 1, pp. 71–80, 2012.
- [19] L. Li and S. Heidarkhani, "Multiplicity solutions to a doubly eigenvalue fourth-order equation," *The Journal of the Indian Mathematical Society*, vol. 80, no. 1-2, pp. 99–110, 2013.
- [20] L. Li and C.-L. Tang, "Existence of three solutions for $(p-q)$ -biharmonic systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 3, pp. 796–805, 2010.
- [21] H. Liu and N. Su, "Existence of three solutions for a p -biharmonic problem," *Dynamics of Continuous, Discrete & Impulsive Systems A: Mathematical Analysis*, vol. 15, no. 3, pp. 445–452, 2008.
- [22] X.-L. Liu and W.-T. Li, "Existence and multiplicity of solutions for fourth-order boundary value problems with parameters," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 1, pp. 362–375, 2007.
- [23] M. Massar, E. M. Hssini, and N. Tsouli, "Infinitely many solutions for class of Navier boundary $(p; q)$ -biharmonic systems," *Electronic Journal of Differential Equations*, vol. 2012, no. 163, pp. 1–9, 2012.
- [24] B. Ricceri, "On a three critical points theorem," *Archiv der Mathematik*, vol. 75, no. 3, pp. 220–226, 2000.
- [25] G. Bonanno, "A critical points theorem and nonlinear differential problems," *Journal of Global Optimization*, vol. 28, no. 3-4, pp. 249–258, 2004.
- [26] G. Bonanno and S. A. Marano, "On the structure of the critical set of non-differentiable functions with a weak compactness condition," *Applicable Analysis*, vol. 89, no. 1, pp. 1–10, 2010.
- [27] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. III: Variational Methods and Optimization*, Springer, New York, NY, USA, 1985, Translated from the German by L. F. Boron.
- [28] E. Zeidler, *Nonlinear Functional Analysis and Its Applications: II/A: Linear Monotone Operators*, Springer, New York, NY, USA, 1990, translated by E. Zeidler and L. F. Boron.
- [29] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B, Nonlinear Monotone Operators*, Springer, New York, NY, USA, 1990, Translated by E. Zeidler and L. F. Boron.