Research Article

Coefficient Estimates for New Subclasses of Meromorphic Bi-Univalent Functions

Serap Bulut

Civil Aviation College, Kocaeli University, Arslanbey Campus, İzmit, 41285 Kocaeli, Turkey

Correspondence should be addressed to Serap Bulut; bulutserap@yahoo.com

Received 18 March 2014; Accepted 7 July 2014; Published 29 October 2014

Academic Editor: Cédric Join

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We introduce and investigate two new subclasses \( M_{\sigma}(\alpha, \lambda) \) and \( M_{\sigma}(\beta, \lambda) \) of meromorphic bi-univalent functions defined on \( \Delta = \{ z : z \in \mathbb{C}, 1 < |z| < \infty \} \). For functions belonging to these classes, estimates on the initial coefficients are obtained.

1. Introduction

Let \( A \) denote the class of all functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disk

\[
\mathbb{U} = \{ z : z \in \mathbb{C}, |z| < 1 \}.
\]

We also denote by \( \mathcal{S} \) the class of all functions in the normalized analytic function class \( A \) which are univalent in \( \mathbb{U} \).

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk \( \mathbb{U} \). In fact, the Koebe one-quarter theorem [1] ensures that the image of \( \mathbb{U} \) under every univalent function \( f \in \mathcal{S} \) contains a disk of radius \( 1/4 \). Thus, every function \( f \in A \) has an inverse \( f^{-1} \), which is defined by

\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),
\]

\[
f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4).
\]

In fact, the inverse function \( f^{-1} \) is given by

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + \cdots.
\]

A function \( f \in A \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{U} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1). For a brief history and interesting examples of functions in the class \( \Sigma \), see [2] (see also [3, 4]). In fact, the aforesaid work of Srivastava et al. [2] essentially revived the investigation of various subclasses of the bi-univalent function class \( \Sigma \) in recent years; it was followed by such works as those by Murugusundaramoorthy et al. [5], Frasin and Aouf [6], Çağlar et al. [7], and others (see, e.g., [8–15]).

In this paper, the concept of bi-univalency is extended to the class of meromorphic functions defined on

\[
\Delta = \{ z : z \in \mathbb{C}, 1 < |z| < \infty \}.
\]

For this purpose, let \( \sigma \) denote the class of all meromorphic univalent functions \( g \) of the form

\[
g(z) = z + \sum_{n=0}^{\infty} b_n z^n,
\]

defined on the domain \( \Delta \). Since \( g \in \sigma \) is univalent, it has an inverse \( g^{-1} \) that satisfies

\[
g^{-1}(g(z)) = z \quad (z \in \Delta),
\]

\[
g(g^{-1}(w)) = w \quad (M < |w| < \infty; M > 0).
\]
Furthermore, the inverse function \( g^{-1} \) has a series expansion of the form

\[
g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n},
\]

where \( M < |w| < \infty \). Analogous to the bi-univalent analytic functions, a function \( g \in \sigma \) is said to be meromorphically bi-univalent if both \( g \) and \( g^{-1} \) are meromorphically univalent in \( \Delta \). We denote by \( \sigma_{\Delta} \) the class of all meromorphic bi-univalent functions in \( \Delta \) given by (6). A simple calculation shows that

\[
g^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_1 b_0}{w^2} - \cdots .
\]

The coefficient problem was investigated for various interesting subclasses of the meromorphic univalent functions (see, e.g., [16–18]).

In the present investigation, certain subclasses of meromorphic bi-univalent functions are introduced and estimates for the coefficients \( b_0 \) and \( b_1 \) of functions in the newly introduced subclasses are obtained.

**Definition 1.** A function \( g \) given by (6) is said to be in the class \( \mathcal{M}_\sigma(\alpha, \lambda) \) if the following conditions are satisfied:

\[
g \in \sigma_{\Delta}, \quad \left| \frac{z g'(z)}{(1 - \lambda) g(z) + \lambda z g'(z)} \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, 0 \leq \lambda < 1, z \in \Delta),
\]

\[
\left| \frac{w h'(w)}{(1 - \lambda) h(w) + \lambda w h'(w)} \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, 0 \leq \lambda < 1, w \in \Delta),
\]

where the function \( h = g^{-1} \) is given by (9).

**Definition 2.** A function \( g \) given by (6) is said to be in the class \( \mathcal{M}_\sigma(\beta, \lambda) \) if the following conditions are satisfied:

\[
g \in \sigma_{\Delta}, \quad \Re \left\{ \frac{z g'(z)}{(1 - \lambda) g(z) + \lambda z g'(z)} \right\} > \beta, \quad (0 \leq \beta < 1, 0 \leq \lambda < 1, z \in \Delta),
\]

\[
\Re \left\{ \frac{w h'(w)}{(1 - \lambda) h(w) + \lambda w h'(w)} \right\} > \beta, \quad (0 \leq \beta < 1, 0 \leq \lambda < 1, w \in \Delta),
\]

where the function \( h = g^{-1} \) is given by (9).

**Remark 3.** We note that, for \( \lambda = 0 \), the classes \( \mathcal{M}_\sigma(\alpha, \lambda) \) and \( \mathcal{M}_\sigma(\beta, \lambda) \) reduce to the classes

\[
\mathcal{M}_\sigma(\alpha, 0) = \mathcal{M}_\sigma^*(\alpha),
\]

\[
\mathcal{M}_\sigma(\beta, 0) = \Sigma_{\Delta}(\beta),
\]

respectively, introduced and studied by Halim et al. [17].

The object of the present paper is to extend the concept of bi-univalent to the class of meromorphic functions defined on \( \Delta \) and find estimates on the coefficients \( |b_0| \) and \( |b_1| \) for functions in the above-defined classes \( \mathcal{M}_\sigma(\alpha, \lambda) \) and \( \mathcal{M}_\sigma(\beta, \lambda) \) of the function class \( \sigma_{\Delta} \).

Firstly, in order to derive our main results, we need the following lemma.

**Lemma 4** (see [19]). If \( p \in \mathcal{P} \), then \( |c_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p \) analytic in \( \mathbb{U} \) for which

\[
\Re \left\{ \frac{p(z)}{(1 - \lambda) p(z) + \lambda p'(z)} \right\} > \beta, \quad (0 < \beta < 1, p(z) \in \mathbb{U}),
\]

for \( z \in \mathbb{U} \).

**2. Coefficient Estimates**

We begin this section by finding the estimates on the coefficients \( |b_0| \) and \( |b_1| \) for functions in the class \( \mathcal{M}_\sigma(\alpha, \lambda) \).

**Theorem 5.** Let the function \( g(z) \) given by the series expansion (6) be in the function class

\[
\mathcal{M}_\sigma(\alpha, \lambda), \quad 0 < \alpha \leq 1, 0 \leq \lambda < 1.
\]

Then

\[
|b_0| \leq \frac{\sqrt{2\alpha}}{1 - \lambda}, \quad (0 < \alpha \leq 1),
\]

\[
|b_1| \leq \frac{\sqrt{2\alpha}^2 - \alpha}{1 - \lambda}, \quad 0 < \alpha \leq \frac{1}{\sqrt{2}},
\]

**Proof.** It follows from (10) that

\[
\frac{z g'(z)}{(1 - \lambda) g(z) + \lambda z g'(z)} = \left[ \frac{p(z)}{1 - \lambda} \right]^\alpha, \quad (z \in \Delta),
\]

\[
\frac{w h'(w)}{(1 - \lambda) h(w) + \lambda w h'(w)} = \left[ \frac{q(w)}{1 - \lambda} \right]^\alpha, \quad (w \in \Delta),
\]

respectively, where \( p(z) \) and \( q(w) \) are functions with positive real part in \( \Delta \) and have the forms

\[
p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots,
\]

\[
q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \cdots,
\]

respectively. Now, upon equating the coefficients in (17), we get

\[
-(1 - \lambda) b_0 = \alpha p_1, \quad (20)
\]

\[
(1 - \lambda) \left[ (1 - \lambda) b_0^2 - 2b_1 \right] = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \quad (21)
\]

\[
(1 - \lambda) b_0 = \alpha q_1, \quad (22)
\]

\[
(1 - \lambda) \left[ (1 - \lambda) b_0^2 + 2b_1 \right] = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2, \quad (23)
\]
From (20) and (22), we find that
\[ p_1 = -q_1, \]  \( (24) \)
\[ 2(1-\lambda)^2 b_0^2 = \alpha^2 \left( p_1^2 + q_1^2 \right). \]  \( (25) \)

Also, from (21) and (23) we obtain
\[ 2(1-\lambda)^2 b_0^2 = \alpha \left( p_2 + q_2 \right) + \frac{\alpha(\alpha-1)}{2} \left( p_1^2 + q_1^2 \right). \]  \( (26) \)

Since \( \Re(p(z)) > 0 \) and \( \Re(q(z)) > 0 \) in \( \Delta \), the functions \( p(1/z), q(1/z) \in \mathcal{P} \) and hence the coefficients \( p_k \) and \( q_k \) for each \( k \) satisfy the inequality in Lemma 4. Applications of triangle inequality followed by Lemma 4 in (25) and (26) give us the required estimates on \( |b_0| \) as asserted in (15).

Next, in order to find the bound on the coefficient \( |b_1| \), we subtract (23) from (21). We then get
\[ -4(1-\lambda) b_1 = \alpha \left( p_2 - q_2 \right). \]  \( (27) \)

Hence
\[ |b_1| \leq \frac{\alpha}{1-\lambda}. \]  \( (28) \)

On the other hand, using (21) and (23) yields
\[ 2(1-\lambda)^2 \left[ (1-\lambda)^2 b_0^4 + 4b_1^2 \right] = \frac{\alpha^2(\alpha-1)^2}{4} \left( p_1^4 + q_1^4 \right) + \alpha^2 \left( p_2^2 + q_2^2 \right) + \alpha^2(\alpha-1) \left( p_1^2p_2 + q_1^2q_2 \right). \]  \( (29) \)

By using (25) we have from the above equality
\[ b_1^2 = \frac{\alpha^2(\alpha-1)^2}{32(1-\lambda)^2} \left( p_1^4 + q_1^4 \right) + \frac{\alpha^2}{8(1-\lambda)^2} \left( p_2^2 + q_2^2 \right) \]
\[ + \frac{\alpha^2(\alpha-1)}{8(1-\lambda)^2} \left( p_1^2p_2 + q_1^2q_2 \right) \]
\[ - \frac{\alpha^4}{16(1-\lambda)^2} \left( p_1^4 + q_1^4 \right) - \frac{\alpha^4}{8(1-\lambda)^2} p_1^2q_1^2. \]  \( (30) \)

From Lemma 4, we obtain
\[ |b_1| \leq \frac{\sqrt{\alpha^2}}{1-\lambda}. \]  \( (31) \)

Also, by using (26) we have, from equality (29),
\[ |b_1| \leq \frac{\sqrt{2\alpha^2}}{1-\lambda}. \]  \( (32) \)

Comparing (28), (31), and (32) we get the desired estimate on the coefficient \( |b_1| \) as asserted in (16).

\\[ \square \]

For \( \lambda = 0 \), we have the following corollary of Theorem 5.

**Corollary 6.** Let the function \( g(z) \) given by the series expansion (6) be in the function class \( \widetilde{\Sigma}_{\beta}^* (\alpha) \) \( (0 < \alpha \leq 1) \).

Then
\[ |b_0| \leq \sqrt{2\alpha}, \]  \( (33) \)
\[ |b_1| \leq \begin{cases} \sqrt{2\alpha^2}, \quad 0 < \alpha \leq \frac{1}{\sqrt{2}} \\ \alpha, \quad \frac{1}{\sqrt{2}} < \alpha \leq 1. \end{cases} \]  \( (34) \)

**Remark 7.** Corollary 6 is an improvement of the following estimates which were given by Halim et al. [17].

**Corollary 8** (see [17]). Let the function \( g(z) \) given by the series expansion (6) be in the function class \( \widetilde{\Sigma}_{\beta}^* (\alpha) \) \( (0 < \alpha \leq 1) \).

Then
\[ |b_0| \leq 2\alpha, \]  \( (35) \)
\[ |b_1| \leq \sqrt{5\alpha^2}. \]  \( (36) \)

**Remark 9.** Corollary 6 is also an improvement of the estimates which were given by Panigrahi [20, Corollary 2.3].

Next we estimate the coefficients \( |b_0| \) and \( |b_1| \) for functions in the class \( \mathcal{M}_\sigma(\beta, \lambda) \).

**Theorem 10.** Let the function \( g(z) \) given by the series expansion (6) be in the function class \( \mathcal{M}_\sigma(\beta, \lambda) \) \( (0 \leq \beta < 1, 0 \leq \lambda < 1) \).

Then
\[ |b_0| \leq \frac{\sqrt{2(1-\beta)}}{1-\lambda}, \]  \( (37) \)
\[ |b_1| \leq \frac{1-\beta}{1-\lambda}. \]  \( (38) \)

**Proof.** It follows from (11) that
\[ \frac{zg'(z)}{(1-\lambda)g(z)+\lambda zg'(z)} = \beta + (1-\beta)p(z) \quad (z \in \Delta), \]
\[ \frac{wh'(w)}{(1-\lambda)h(w)+\lambda wh'(w)} = \beta + (1-\beta)q(w) \quad (w \in \Delta), \]  \( (39) \)
respectively, where \( p(z) \) and \( q(w) \) are functions with positive real part in \( \Delta \) and have the forms (18) and (19), respectively. Now, upon equating the coefficients in (39), we get
\[ -(1-\lambda)b_0 = (1-\beta)p_1, \]  \( (34) \)
\[ (1-\lambda)\left[(1-\lambda)b_0^2 - 2b_1\right] = (1-\beta)p_2, \]  \( (35) \)
\[ (1-\lambda)b_0 = (1-\beta)q_1, \]  \( (36) \)
\[ (1-\lambda)\left[(1-\lambda)b_0^2 + 2b_1\right] = (1-\beta)q_2. \]  \( (37) \)
From (41) and (43), we obtain
\[ p_1 = -q_1, \quad (45) \]
\[ 2(1 - \lambda)^2 b_0^2 = (1 - \beta)^2 \left( p_1^2 + q_1^2 \right). \quad (46) \]

Also, from (42) and (44), we obtain
\[ 2(1 - \lambda)^2 b_0^2 = (1 - \beta) \left( p_2 + q_2 \right). \quad (47) \]

Since \( \Re(p(z)) > 0 \) and \( \Re(q(z)) > 0 \) in \( \Delta \), the functions \( p(1/z), q(1/z) \in \mathcal{P} \) and hence the coefficients \( p_k \) and \( q_k \) for each \( k \) satisfy the inequality in Lemma 4. Therefore, we find from (46) and (47) that
\[ |b_0| \leq \frac{2(1 - \beta)}{1 - \lambda}, \quad (48) \]
\[ |b_0| \leq \frac{\sqrt{2}(1 - \beta)}{1 - \lambda}, \]
respectively. So we get the desired estimate on the coefficient \( |b_0| \) as asserted in (38).

Next, in order to find the bound on the coefficient \( |b_1| \), we subtract (44) from (42). We thus get
\[ -4(1 - \lambda) b_1 = (1 - \beta) \left( p_2 - q_2 \right). \quad (49) \]

Hence
\[ |b_1| \leq \frac{1 - \beta}{1 - \lambda}. \quad (50) \]

On the other hand, using (42) and (44) yields
\[ (1 - \lambda)^2 \left[ (1 - \lambda)^2 b_0^2 - 4b_1^2 \right] = (1 - \beta)^2 p_2 q_2 \quad (51) \]
or equivalently
\[ 4b_1^2 = (1 - \lambda)^2 b_0^4 - (1 - \beta)^2 p_2 q_2. \quad (52) \]

Upon substituting the value of \( b_0^2 \) from (46) and (47) into (52), respectively, it follows that
\[ |b_1| \leq \frac{1 - \beta}{1 - \lambda} \sqrt{4\beta^2 - 8\beta + 5}, \quad (53) \]
\[ |b_1| \leq \frac{\sqrt{2}(1 - \beta)}{1 - \lambda}. \]

Comparing (50) and (53), we get the desired estimate on the coefficient \( |b_1| \) as asserted in (39).

For \( \lambda = 0 \), we have the following corollary of Theorem 10.

**Corollary 11.** Let the function \( g(z) \) given by the series expansion (6) be in the function class
\[ \Sigma^\ast_{\beta} \left( \begin{array}{c} 0 \leq \beta < 1 \end{array} \right). \quad (54) \]

Then
\[ |b_0| \leq \sqrt{2(1 - \beta)}, \quad (55) \]
\[ |b_1| \leq 1 - \beta. \]

**Remark 12.** Corollary 11 is an improvement of the following estimates which were given by Halim et al. [17].

**Corollary 13** (see [17]). Let the function \( g(z) \) given by the series expansion (6) be in the function class
\[ \Sigma^\ast_{\beta} \left( \begin{array}{c} 0 \leq \beta < 1 \end{array} \right). \quad (56) \]

Then
\[ |b_0| \leq 2 \left( 1 - \beta \right), \quad (57) \]
\[ |b_1| \leq (1 - \beta) \sqrt{4\beta^2 - 8\beta + 5}. \]

**Remark 14.** Corollary 11 is also an improvement of the estimates which were given by Panigrahi [20, Corollary 3.3].

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


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