Research Article

Coefficient Estimates for New Subclasses of Meromorphic Bi-Univalent Functions

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We introduce and investigate two new subclasses $\mathcal{M}_\sigma(\alpha, \lambda)$ and $\mathcal{M}_\sigma(\beta, \lambda)$ of meromorphic bi-univalent functions defined on $\Delta = \{z : z \in \mathbb{C}, 1 < |z| < \infty\}$. For functions belonging to these classes, estimates on the initial coefficients are obtained.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}. \quad (2)$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [1] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus, every function $f \in \mathcal{A}$ has an inverse f^{-1} , which is defined by

$$\begin{aligned} f^{-1}(f(z)) &= z \quad (z \in \mathbb{U}), \\ f(f^{-1}(w)) &= w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right). \end{aligned} \quad (3)$$

In fact, the inverse function f^{-1} is given by

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \end{aligned} \quad (4)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class Σ , see [2] (see also [3, 4]). In fact, the aforementioned work of Srivastava et al. [2] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Murugusundaramoorthy et al. [5], Frasin and Aouf [6], Çağlar et al. [7], and others (see, e.g., [8–15]).

In this paper, the concept of bi-univalence is extended to the class of meromorphic functions defined on

$$\Delta = \{z : z \in \mathbb{C}, 1 < |z| < \infty\}. \quad (5)$$

For this purpose, let σ denote the class of all meromorphic univalent functions g of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \quad (6)$$

defined on the domain Δ . Since $g \in \sigma$ is univalent, it has an inverse g^{-1} that satisfies

$$\begin{aligned} g^{-1}(g(z)) &= z \quad (z \in \Delta), \\ g(g^{-1}(w)) &= w \quad (M < |w| < \infty; M > 0). \end{aligned} \quad (7)$$

Furthermore, the inverse function g^{-1} has a series expansion of the form

$$g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \quad (8)$$

where $M < |w| < \infty$. Analogous to the bi-univalent analytic functions, a function $g \in \sigma$ is said to be meromorphically bi-univalent if both g and g^{-1} are meromorphically univalent in Δ . We denote by $\sigma_{\mathcal{M}}$ the class of all meromorphic bi-univalent functions in Δ given by (6). A simple calculation shows that

$$g^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \dots \quad (9)$$

The coefficient problem was investigated for various interesting subclasses of the meromorphic univalent functions (see, e.g., [16–18]).

In the present investigation, certain subclasses of meromorphic bi-univalent functions are introduced and estimates for the coefficients b_0 and b_1 of functions in the newly introduced subclasses are obtained.

Definition 1. A function g given by (6) is said to be in the class $\mathcal{M}_{\sigma}(\alpha, \lambda)$ if the following conditions are satisfied:

$$\begin{aligned} g \in \sigma_{\mathcal{M}}, \quad & \left| \arg \left(\frac{zg'(z)}{(1-\lambda)g(z) + \lambda zg'(z)} \right) \right| < \frac{\alpha\pi}{2} \\ & (0 < \alpha \leq 1, 0 \leq \lambda < 1, z \in \Delta), \\ & \left| \arg \left(\frac{wh'(w)}{(1-\lambda)h(w) + \lambda wh'(w)} \right) \right| < \frac{\alpha\pi}{2} \\ & (0 < \alpha \leq 1, 0 \leq \lambda < 1, w \in \Delta), \end{aligned} \quad (10)$$

where the function $h = g^{-1}$ is given by (9).

Definition 2. A function g given by (6) is said to be in the class $\mathcal{M}_{\sigma}(\beta, \lambda)$ if the following conditions are satisfied:

$$\begin{aligned} g \in \sigma_{\mathcal{M}}, \quad & \Re \left\{ \frac{zg'(z)}{(1-\lambda)g(z) + \lambda zg'(z)} \right\} > \beta \\ & (0 \leq \beta < 1, 0 \leq \lambda < 1, z \in \Delta), \\ & \Re \left\{ \frac{wh'(w)}{(1-\lambda)h(w) + \lambda wh'(w)} \right\} > \beta \\ & (0 \leq \beta < 1, 0 \leq \lambda < 1, w \in \Delta), \end{aligned} \quad (11)$$

where the function $h = g^{-1}$ is given by (9).

Remark 3. We note that, for $\lambda = 0$, the classes $\mathcal{M}_{\sigma}(\alpha, \lambda)$ and $\mathcal{M}_{\sigma}(\beta, \lambda)$ reduce to the classes

$$\begin{aligned} \mathcal{M}_{\sigma}(\alpha, 0) &= \tilde{\Sigma}_{\mathcal{B}}^*(\alpha), \\ \mathcal{M}_{\sigma}(\beta, 0) &= \Sigma_{\mathcal{B}}^*(\beta), \end{aligned} \quad (12)$$

respectively, introduced and studied by Halim et al. [17].

The object of the present paper is to extend the concept of bi-univalent to the class of meromorphic functions defined on Δ and find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the above-defined classes $\mathcal{M}_{\sigma}(\alpha, \lambda)$ and $\mathcal{M}_{\sigma}(\beta, \lambda)$ of the function class $\sigma_{\mathcal{M}}$.

Firstly, in order to derive our main results, we need the following lemma.

Lemma 4 (see [19]). If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which

$$\Re(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \dots, \quad (13)$$

for $z \in \mathbb{U}$.

2. Coefficient Estimates

We begin this section by finding the estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\mathcal{M}_{\sigma}(\alpha, \lambda)$.

Theorem 5. Let the function $g(z)$ given by the series expansion (6) be in the function class

$$\mathcal{M}_{\sigma}(\alpha, \lambda) \quad (0 < \alpha \leq 1, 0 \leq \lambda < 1). \quad (14)$$

Then

$$|b_0| \leq \frac{\sqrt{2}\alpha}{1-\lambda}, \quad (15)$$

$$|b_1| \leq \begin{cases} \frac{\sqrt{2}\alpha^2}{1-\lambda}, & 0 < \alpha \leq \frac{1}{\sqrt{2}}, \\ \frac{\alpha}{1-\lambda}, & \frac{1}{\sqrt{2}} \leq \alpha \leq 1. \end{cases} \quad (16)$$

Proof. It follows from (10) that

$$\frac{zg'(z)}{(1-\lambda)g(z) + \lambda zg'(z)} = [p(z)]^{\alpha} \quad (z \in \Delta), \quad (17)$$

$$\frac{wh'(w)}{(1-\lambda)h(w) + \lambda wh'(w)} = [q(w)]^{\alpha} \quad (w \in \Delta),$$

respectively, where $p(z)$ and $q(w)$ are functions with positive real part in Δ and have the forms

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \dots, \quad (18)$$

$$q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \dots, \quad (19)$$

respectively. Now, upon equating the coefficients in (17), we get

$$-(1-\lambda)b_0 = \alpha p_1, \quad (20)$$

$$(1-\lambda)[(1-\lambda)b_0^2 - 2b_1] = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \quad (21)$$

$$(1-\lambda)b_0 = \alpha q_1, \quad (22)$$

$$(1-\lambda)[(1-\lambda)b_0^2 + 2b_1] = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \quad (23)$$

From (20) and (22), we find that

$$p_1 = -q_1, \quad (24)$$

$$2(1-\lambda)^2 b_0^2 = \alpha^2 (p_1^2 + q_1^2). \quad (25)$$

Also, from (21) and (23) we obtain

$$2(1-\lambda)^2 b_0^2 = \alpha (p_2 + q_2) + \frac{\alpha(\alpha-1)}{2} (p_1^2 + q_1^2). \quad (26)$$

Since $\Re(p(z)) > 0$ and $\Re(q(z)) > 0$ in Δ , the functions $p(1/z), q(1/z) \in \mathcal{P}$ and hence the coefficients p_k and q_k for each k satisfy the inequality in Lemma 4. Applications of triangle inequality followed by Lemma 4 in (25) and (26) give us the required estimates on $|b_0|$ as asserted in (15).

Next, in order to find the bound on the coefficient $|b_1|$, we subtract (23) from (21). We thus get

$$-4(1-\lambda)b_1 = \alpha(p_2 - q_2). \quad (27)$$

Hence

$$|b_1| \leq \frac{\alpha}{1-\lambda}. \quad (28)$$

On the other hand, using (21) and (23) yields

$$\begin{aligned} & 2(1-\lambda)^2 [(1-\lambda)^2 b_0^4 + 4b_1^2] \\ &= \frac{\alpha^2(\alpha-1)^2}{4} (p_1^4 + q_1^4) + \alpha^2 (p_2^2 + q_2^2) \\ &+ \alpha^2 (\alpha-1) (p_1^2 p_2 + q_1^2 q_2). \end{aligned} \quad (29)$$

By using (25) we have from the above equality

$$\begin{aligned} b_1^2 &= \frac{\alpha^2(\alpha-1)^2}{32(1-\lambda)^2} (p_1^4 + q_1^4) + \frac{\alpha^2}{8(1-\lambda)^2} (p_2^2 + q_2^2) \\ &+ \frac{\alpha^2(\alpha-1)}{8(1-\lambda)^2} (p_1^2 p_2 + q_1^2 q_2) \\ &- \frac{\alpha^4}{16(1-\lambda)^2} (p_1^4 + q_1^4) - \frac{\alpha^4}{8(1-\lambda)^2} p_1^2 q_1^2. \end{aligned} \quad (30)$$

From Lemma 4, we obtain

$$|b_1| \leq \frac{\sqrt{5}\alpha^2}{1-\lambda}. \quad (31)$$

Also, by using (26) we have, from equality (29),

$$|b_1| \leq \frac{\sqrt{2}\alpha^2}{1-\lambda}. \quad (32)$$

Comparing (28), (31), and (32) we get the desired estimate on the coefficient $|b_1|$ as asserted in (16). $\square$

For $\lambda = 0$, we have the following corollary of Theorem 5.

Corollary 6. Let the function $g(z)$ given by the series expansion (6) be in the function class

$$\tilde{\Sigma}_{\mathcal{B}}^*(\alpha) \quad (0 < \alpha \leq 1). \quad (33)$$

Then

$$\begin{aligned} |b_0| &\leq \sqrt{2}\alpha, \\ |b_1| &\leq \begin{cases} \sqrt{2}\alpha^2, & 0 < \alpha \leq \frac{1}{\sqrt{2}}, \\ \alpha, & \frac{1}{\sqrt{2}} \leq \alpha \leq 1. \end{cases} \end{aligned} \quad (34)$$

Remark 7. Corollary 6 is an improvement of the following estimates which were given by Halim et al. [17].

Corollary 8 (see [17]). Let the function $g(z)$ given by the series expansion (6) be in the function class

$$\tilde{\Sigma}_{\mathcal{B}}^*(\alpha) \quad (0 < \alpha \leq 1). \quad (35)$$

Then

$$\begin{aligned} |b_0| &\leq 2\alpha, \\ |b_1| &\leq \sqrt{5}\alpha^2. \end{aligned} \quad (36)$$

Remark 9. Corollary 6 is also an improvement of the estimates which were given by Panigrahi [20, Corollary 2.3].

Next we estimate the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\mathcal{M}_{\sigma}(\beta, \lambda)$.

Theorem 10. Let the function $g(z)$ given by the series expansion (6) be in the function class

$$\mathcal{M}_{\sigma}(\beta, \lambda) \quad (0 \leq \beta < 1, 0 \leq \lambda < 1). \quad (37)$$

Then

$$|b_0| \leq \frac{\sqrt{2(1-\beta)}}{1-\lambda}, \quad (38)$$

$$|b_1| \leq \frac{1-\beta}{1-\lambda}. \quad (39)$$

Proof. It follows from (11) that

$$\begin{aligned} \frac{zg'(z)}{(1-\lambda)g(z) + \lambda zg'(z)} &= \beta + (1-\beta)p(z) \quad (z \in \Delta), \\ \frac{wh'(w)}{(1-\lambda)h(w) + \lambda wh'(w)} &= \beta + (1-\beta)q(w) \quad (w \in \Delta), \end{aligned} \quad (40)$$

respectively, where $p(z)$ and $q(w)$ are functions with positive real part in Δ and have the forms (18) and (19), respectively. Now, upon equating the coefficients in (40), we get

$$-(1-\lambda)b_0 = (1-\beta)p_1, \quad (41)$$

$$(1-\lambda)[(1-\lambda)b_0^2 - 2b_1] = (1-\beta)p_2, \quad (42)$$

$$(1-\lambda)b_0 = (1-\beta)q_1, \quad (43)$$

$$(1-\lambda)[(1-\lambda)b_0^2 + 2b_1] = (1-\beta)q_2. \quad (44)$$

From (41) and (43), we obtain

$$p_1 = -q_1, \quad (45)$$

$$2(1-\lambda)^2 b_0^2 = (1-\beta)^2 (p_1^2 + q_1^2). \quad (46)$$

Also, from (42) and (44), we obtain

$$2(1-\lambda)^2 b_0^2 = (1-\beta) (p_2 + q_2). \quad (47)$$

Since $\Re(p(z)) > 0$ and $\Re(q(z)) > 0$ in Δ , the functions $p(1/z), q(1/z) \in \mathcal{P}$ and hence the coefficients p_k and q_k for each k satisfy the inequality in Lemma 4. Therefore, we find from (46) and (47) that

$$\begin{aligned} |b_0| &\leq \frac{2(1-\beta)}{1-\lambda}, \\ |b_0| &\leq \frac{\sqrt{2(1-\beta)}}{1-\lambda}, \end{aligned} \quad (48)$$

respectively. So we get the desired estimate on the coefficient $|b_0|$ as asserted in (38).

Next, in order to find the bound on the coefficient $|b_1|$, we subtract (44) from (42). We thus get

$$-4(1-\lambda)b_1 = (1-\beta)(p_2 - q_2). \quad (49)$$

Hence

$$|b_1| \leq \frac{1-\beta}{1-\lambda}. \quad (50)$$

On the other hand, using (42) and (44) yields

$$(1-\lambda)^2 [(1-\lambda)^2 b_0^4 - 4b_1^2] = (1-\beta)^2 p_2 q_2 \quad (51)$$

or equivalently

$$4b_1^2 = (1-\lambda)^2 b_0^4 - \frac{(1-\beta)^2}{(1-\lambda)^2} p_2 q_2. \quad (52)$$

Upon substituting the value of b_0^2 from (46) and (47) into (52), respectively, it follows that

$$\begin{aligned} |b_1| &\leq \frac{1-\beta}{1-\lambda} \sqrt{4\beta^2 - 8\beta + 5}, \\ |b_1| &\leq \frac{\sqrt{2}(1-\beta)}{1-\lambda}. \end{aligned} \quad (53)$$

Comparing (50) and (53), we get the desired estimate on the coefficient $|b_1|$ as asserted in (39). $\square$

For $\lambda = 0$, we have the following corollary of Theorem 10.

Corollary 11. *Let the function $g(z)$ given by the series expansion (6) be in the function class*

$$\Sigma_{\mathcal{B}}^*(\beta) \quad (0 \leq \beta < 1). \quad (54)$$

Then

$$\begin{aligned} |b_0| &\leq \sqrt{2(1-\beta)}, \\ |b_1| &\leq 1-\beta. \end{aligned} \quad (55)$$

Remark 12. Corollary 11 is an improvement of the following estimates which were given by Halim et al. [17].

Corollary 13 (see [17]). *Let the function $g(z)$ given by the series expansion (6) be in the function class*

$$\Sigma_{\mathcal{B}}^*(\beta) \quad (0 \leq \beta < 1). \quad (56)$$

Then

$$\begin{aligned} |b_0| &\leq 2(1-\beta), \\ |b_1| &\leq (1-\beta) \sqrt{4\beta^2 - 8\beta + 5}. \end{aligned} \quad (57)$$

Remark 14. Corollary 11 is also an improvement of the estimates which were given by Panigrahi [20, Corollary 3.3].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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