Research Article

The Relatively Free Groups $F(N_c \wedge A^2)$ Satisfy Noncentral Commutative Transitivity

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Received 22 May 2014; Accepted 25 July 2014; Published 1 October 2014

Academic Editor: Antonio M. Cegarra

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We prove that a free group, $F(N_c \wedge A^2)$, relative to the variety, $N_c \wedge A^2$, of all groups simultaneously nilpotent of class at most c and metabelian is such that the centralizer of every noncentral element is abelian. We relate that result to the model theory of such groups as well as a quest to find a relative analog in $N_c \wedge A^2$ of a classical theorem of Benjamin Baumslag. We also touch briefly on similar considerations in the varieties N_c of nilpotent groups.

1. Introduction

1.1. Separation and Discrimination

Definition 1. Let Γ be a group and $\mathcal{X}$ a nonempty class of groups closed under isomorphism. $\mathcal{X}$ *separates* Γ provided to every nontrivial element $\gamma \in \Gamma \setminus \{1\}$; there is a group $G \in \mathcal{X}$ and a homomorphism $\varphi : \Gamma \rightarrow G$ such that $\varphi(\gamma) \neq 1$. If n is a positive integer, then $\mathcal{X}$ *n-separates* Γ provided to every $(\gamma_1, \dots, \gamma_n) \in (\Gamma \setminus \{1\})^n$; there is a group $G \in \mathcal{X}$ and a homomorphism $\varphi : \Gamma \rightarrow G$ such that $\varphi(\gamma_i) \neq 1$ for all $i = 1, \dots, n$. $\mathcal{X}$ *discriminates* Γ provided $\mathcal{X}$ *n-separates* Γ for every positive integer n . Note that separation is the same as 1-separation. In the event that $\mathcal{X}$ is the isomorphism class of a single group G we may say that G separates (*n-separates*, discriminates) Γ for $\mathcal{X}$ separates (*n-separates*, discriminates) Γ .

Observe that $\mathcal{X}$ separating Γ is equivalent to Γ being embeddable in a direct product of groups in $\mathcal{X}$. In particular, this is so if Γ is isomorphic to a subdirect product of groups in $\mathcal{X}$ (see [1]). Here $G \leq \prod_{i \in I} G_i$ is a *subdirect product of the family* $(G_i)_{i \in I}$ provided that, for each fixed $i_0 \in I$, the restriction $\rho_{i_0} \mid G$ of the projection onto the i_0 th coordinate remains surjective.

Definition 2. Let $\mathcal{P}$ be a nonempty class of groups closed under isomorphism. $\mathcal{P}$ is a *prevariety* if it is closed under taking subgroups and direct products.

B. Fine observed that $\mathcal{P}$ is a prevariety if and only if it is a *separating family* in the sense that every group Γ separated by $\mathcal{P}$ must already lie in $\mathcal{P}$.

Definition 3. Let $\mathcal{X}$ be a nonempty class of groups closed under isomorphism. For a given group Γ let $\mathcal{X}(\Gamma)$ be the set $\{N \trianglelefteq \Gamma : \Gamma/N \in \mathcal{X}\}$. $\mathcal{X}$ is a *radical class* provided, for every group Γ ,

$$\bigcap_{N \in \mathcal{X}(\Gamma)} N \in \mathcal{X}(\Gamma). \quad (1)$$

If $\mathcal{X}$ is a radical class, then $\text{Rad}_{\mathcal{X}}(\Gamma) = \bigcap_{N \in \mathcal{X}(\Gamma)} N$ is the *radical* of Γ relative to $\mathcal{X}$.

We observe that every prevariety $\mathcal{P}$ is a radical class with

$$\text{Rad}_{\mathcal{P}}(\Gamma) = \text{Ker} \left(\Gamma \rightarrow \prod_{N \in \mathcal{P}(\Gamma)} \left(\frac{\Gamma}{N} \right) \right). \quad (2)$$

Here $\gamma \mapsto (N\gamma)_{N \in \mathcal{P}(\Gamma)}$.

Let us call the isomorphism class of the one element group *trivial*. Every nontrivial prevariety admits free objects of every rank. Indeed, if F is a free group of rank r , then $F_r(\mathcal{P}) = F/\text{Rad}_{\mathcal{P}}(F)$ is free of rank r relative to $\mathcal{P}$.

Definition 4. One will say a group is *freely separated* (*freely discriminated*) provided it is separated (discriminated) by the class of free groups.

Definition 5. A group Γ is *commutative transitive* or *CT* provided the binary relation $C \subseteq (\Gamma \setminus \{1\})^2$, given by xCy if and only if $xy = yx$, is transitive.

Remark 6. Clearly Γ is CT if and only if the centralizer in Γ of any nontrivial element $\gamma \in \Gamma \setminus \{1\}$ is abelian. This result was originally stated by Harrison [2].

Theorem 7 (Baumslag [3]). *Let Γ be a freely separated group. Then Γ is freely discriminated if and only if it is CT.*

Corollary 8 (Remeslennikov). *Γ is freely discriminated if and only if it is 2-freely separated.*

Proof. It will suffice to show that every 2-freely separated group is CT. Suppose not. Let Γ be 2-freely separated. If Γ is not CT, then there are $x, y, z \in \Gamma$ such that $y \neq 1$, $[x, y] = 1$, and $[y, z] = 1$, but $[x, z] \neq 1$. (Here, of course, $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$.) Then there is a free group F and a homomorphism $\varphi : \Gamma \rightarrow F$ such that simultaneously $\varphi(y) \neq 1$ and $[\varphi(x), \varphi(z)] = \varphi([x, z]) \neq 1$. But this is impossible since $[\varphi(x), \varphi(y)] = \varphi([x, y]) = [\varphi(y), \varphi(z)] = \varphi([y, z]) = 1$ and free groups are CT. The contradiction shows that Γ is CT and hence freely discriminated. $\square$

Our terminology differs from that of Benjamin Baumslag who used residually- $\mathcal{X}$ which we use separated by $\mathcal{X}$. He also used n -residually- $\mathcal{X}$ where we use n separated by $\mathcal{X}$ and fully residually- $\mathcal{X}$ which we use discriminated by $\mathcal{X}$. We prefer to reserve residually- $\mathcal{X}$ for separation by a family of surjective homomorphisms into $\mathcal{X}$ (equivalently being isomorphic to a subdirect product of groups in $\mathcal{X}$) as well as using fully residually- $\mathcal{X}$ for discrimination by a family of surjective homomorphisms into $\mathcal{X}$. We believe there are good historical grounds for this choice of terminology. Compare this to the discussion on page 101 of Cohn [4]. B. Fine posed the question of analogues of Benjamin Baumslag's theorem in varieties of metabelian and nilpotent groups. The authors in [5] tried to treat that question in the case of the metabelian variety. In retrospect it would have been more productive to not insist our homomorphisms be surjective. Although, because of the Nielsen-Schreier subgroup theorem, the concepts of residually free and freely separated coincide as do the concepts of fully residually free and freely discriminated, the analogue of the Nielsen-Schreier subgroup theorem can fail to hold in varieties of groups and the relativized concepts need not coincide.

1.2. Metabelian and Nilpotent Groups. Basically in this section, we fix notation. The left-normed or simple commutator

$[x_1, \dots, x_n]$ is defined recursively by $[x_1] = x_1$ and if $[x_1, \dots, x_{n-1}]$ has already been defined, $[x_1, \dots, x_{n-1}, x_n] = [x_1, \dots, x_{n-1}]^{-1}x_n^{-1}[x_1, \dots, x_{n-1}]x_n$. If A and B are (not necessarily distinct) subgroups of a group Γ , then $[A, B]$ is the subgroup of Γ generated by all commutators $[a, b]$ as (a, b) varies over the Cartesian product $A \times B$. The derived group or commutator subgroup of Γ is the subgroup $\Gamma' = [\Gamma, \Gamma]$. The second derived group Γ'' is $(\Gamma')'$. The group Γ is *metabelian* provided it is an extension of an abelian group by an abelian group. Equivalently, Γ is metabelian provided $\Gamma'' = \{1\}$.

A nonincreasing sequence of subgroups normal in Γ

$$\Gamma = \Gamma_1 \geq \Gamma_2 \geq \dots \geq \Gamma_n \geq \dots \quad (3)$$

is a *descending central series* for Γ provided that, for each positive integer n ,

$$\frac{\Gamma_n}{\Gamma_{n+1}} \text{ is central in } \frac{\Gamma}{\Gamma_{n+1}}. \quad (4)$$

An example of such is the *lower central series* of Γ defined recursively by $\gamma_1(\Gamma) = \Gamma$ and if $\gamma_{n-1}(\Gamma)$ has already been defined, $\gamma_n(\Gamma) = [\gamma_{n-1}(\Gamma), \Gamma]$. By Lemma 33.35 on page 86 of [1], $\gamma_n(\Gamma)$ may also be characterized as the subgroup of Γ generated by the simple commutators $[g_1, \dots, g_n]$ as $(g_1, \dots, g_n)$ varies over the Cartesian power Γ^n .

A nondescending sequence of subgroups normal in Γ

$$\{1\} = \Gamma_0 \leq \Gamma_1 \leq \dots \leq \Gamma_n \leq \dots \quad (5)$$

is an *ascending central series* for Γ provided, for each positive integer n ,

$$\frac{\Gamma_n}{\Gamma_{n-1}} \text{ is central in } \frac{\Gamma}{\Gamma_{n-1}}. \quad (6)$$

An example of such is the *upper central series* of Γ defined recursively by $Z_0(\Gamma) = \{1\}$ and, if $Z_{n-1}(\Gamma)$ has already been defined,

$$Z\left(\frac{\Gamma}{Z_{n-1}(\Gamma)}\right) = \frac{Z_n(\Gamma)}{Z_{n-1}(\Gamma)}. \quad (7)$$

That is, $Z_n(\Gamma)$ is the complete preimage in Γ of the center of $\Gamma/Z_{n-1}(\Gamma)$. One can prove that, for an integer $c \geq 0$, $\gamma_{c+1}(\Gamma) = \{1\}$ if and only if $Z_c(\Gamma) = \Gamma$. In that event, Γ is said to be *nilpotent* of class at most c . (The nilpotency class of a nilpotent group Γ is the least $c \geq 0$ such that $\gamma_{c+1}(\Gamma) = \{1\}$.) More is true. If Γ is nilpotent of class at most c , then, for all integers n , $1 \leq n \leq c + 1$, $\gamma_n(\Gamma) \leq Z_{c+1-n}(\Gamma)$.

1.3. Concluding Introductory Remarks. This paper is in good measure expository. However there is a genuinely new result expounded in the title. The exposition is intended to explain the result's significance in a wider context. Another *raison d'être* of this paper is to ultimately list some unsolved problems as questions.

2. Preliminaries from Logic and Varieties of Groups

2.1. Quasivarieties and Varieties. We identify the set $\omega = \{0, 1, 2, \dots\}$ of nonnegative integers with the first limit ordinal. Adopting the von Neumann definition of a cardinal as an ordinal not in bijective correspondence with any prior ordinal, we also view ω as the first infinite cardinal $\aleph_0$.

Let L_0 be the first order language with equality containing a binary operation symbol $\bullet$, a unary operation symbol $^{-1}$, and a constant symbol 1 . Thus, an L_0 -structure G is a set provided with a binary operation $G^2 \rightarrow G, (g_1, g_2) \mapsto g_1 g_2$, a unary operation $G \rightarrow G, g \mapsto g^{-1}$, and a distinguished element $1 \in G$. We may view every group as an L_0 -structure.

Suppose that $X = \{x_1, x_2, \dots\} = \{x_{n+1} : n \in \omega\}$ is the set of variables of L_0 . We remark that being first order means that, in the interpretation of the variables in any L_0 -structure G , they are allowed to roam over individual elements of G only but never, for example, over subsets of G nor functions defined on G . The phrase “with equality” means that L_0 contains the logical symbol $=$ which is always to be interpreted as the identity relation in any L_0 -structure.

A *universal sentence* of L_0 is one of the form $\forall \bar{x} \varphi(\bar{x})$ where $\bar{x}$ is a tuple of distinct variables and $\varphi(\bar{x})$ is a formula of L_0 containing no quantifiers and containing occurrences of (at most) the variables in $\bar{x}$. Since vacuous quantifications are permitted, quantifier free sentences of L_0 are considered special cases of universal sentences. For example, $1 = 1$ is viewed as a universal sentence of L_0 . Less trivial examples are the group axioms; namely,

$$\begin{aligned} \forall x_1, x_2, x_3 ((x_1 \bullet x_2) \bullet x_3 &= x_1 \bullet (x_2 \bullet x_3)) \\ \forall x_1 (x_1 \bullet 1 &= x_1) \\ \forall x_1 (x_1 \bullet x_1^{-1} &= 1). \end{aligned} \quad (8)$$

Henceforth, we will suppress the symbol $\bullet$ and indicate the binary operation by juxtaposition. Furthermore, in all that follows, we will tacitly assume that all sets of sentences under consideration are consistent and contain the group axioms above so that we are restricting our L_0 -structures to groups. That allows us to write every term of L_0 as a word $w \in (X \cup X^{-1})^*$ on the free monoid on $X \cup X^{-1}$ (here $X^{-1} = \{x^{-1} : x \in X\}$), where the empty word is identified with the constant symbol 1 . Modulo the group axioms we make some definitions.

Definition 9. A *quasi-identity* or *quasilaw* of L_0 is a universal sentence of the form

$$\forall \bar{x} \left(\bigwedge_i (w_i(\bar{x}) = 1) \longrightarrow (w(\bar{x}) = 1) \right). \quad (9)$$

A *quasivariety* is the model class of a set of quasi-identities.

Remark 10. Quasi-identities are easily seen to be preserved under taking subgroups and direct products. Thus, every quasivariety is a prevariety. The converse is false. An example

of a prevariety which is not a quasivariety is the class of abelian groups containing no nonzero divisible subgroup.

Definition 11. A universal sentence of L_0 of the form $\forall \bar{x} (w(\bar{x}) = 1)$ is a *law* or *identity*. A *variety* is the model class of a set of laws.

Remark 12. Generally we omit the universal quantifiers and write $w(\bar{x}) = 1$ for the law $\forall \bar{x} (w(\bar{x}) = 1)$.

Remark 13. The law $\forall \bar{x} (w(\bar{x}) = 1)$ is equivalent to the quasi-identity:

$$\forall \bar{x} ((1 = 1) \longrightarrow (w(\bar{x}) = 1)). \quad (10)$$

Hence, every variety is a quasivariety. The converse is false. An example of a quasivariety that is not a variety is the class of torsion-free groups.

Remark 14. A classic theorem of Garrett Birkhoff asserts that a nonempty class of groups is a variety if and only if it is closed under taking subgroups, homomorphic images, and direct products.

Definition 15. Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of groups. Then the *product class* $\mathcal{UV}$ is the class of all groups G which are extensions of a group $U \in \mathcal{U}$ by a group $V \in \mathcal{V}$.

One can show that $\mathcal{UV}$ is a variety called the *product variety* of $\mathcal{U}$ by $\mathcal{V}$ (see [1]). We now give some examples of varieties of groups.

- (1) The isomorphism class $\mathcal{E}$ of the trivial group $\{1\}$ is the *trivial variety*. All other varieties of groups are *nontrivial*. $\mathcal{E}$ is determined by the law $x_1 = 1$.
- (2) The class $\mathcal{O}$ of all groups is a variety. It is determined by the law $1 = 1$.

Remark 16. Multiplication of varieties is associative. $\mathcal{E}$ is the identity element and $\mathcal{O}$ is a zero. That is, for any variety $\mathcal{V}$, $\mathcal{E}\mathcal{V} = \mathcal{V}\mathcal{E} = \mathcal{V}$ and $\mathcal{O}\mathcal{V} = \mathcal{V}\mathcal{O} = \mathcal{O}$.

- (3) The class $\mathcal{A}$ of all abelian groups is a variety. It is determined by the law $[x_1, x_2] = 1$.
- (4) The product variety $\mathcal{A}^2$ is the class of all metabelian groups. Since a group is metabelian if and only if commutators commute, $\mathcal{A}^2$ is determined by the law $[[x_1, x_2], [x_3, x_4]] = 1$.
- (5) For each integer $c \geq 0$, the class $\mathcal{N}_c$ of all groups nilpotent of class at most c is a variety. It is determined by the law $[x_1, \dots, x_{c+1}] = 1$.
- (6) Since the intersection of any family of varieties is again a variety, we have, for each fixed integer $c \geq 0$, that the class $\mathcal{N}_c \wedge \mathcal{A}^2$ of groups simultaneously nilpotent of class at most c and metabelian is a variety. $\mathcal{N}_c \wedge \mathcal{A}^2$ is determined by the laws $[x_1, \dots, x_{c+1}] = 1$ and $[[x_1, x_2], [x_3, x_4]] = 1$.

Since every variety is a prevariety, we have that every nontrivial variety $\mathcal{V} \neq \mathcal{E}$ admits free objects (a.k.a. *relatively free groups*)

$$F_r(\mathcal{V}) = \frac{F}{\text{Rad}_{\mathcal{V}}(F)} \quad (11)$$

of all ranks r . Here F is an absolutely free group of rank r .

Definition 17. If $\mathcal{V}$ is a variety of groups and Γ is a group, then $\text{Rad}_{\mathcal{V}}(\Gamma)$ is the *verbal subgroup* of Γ corresponding to $\mathcal{V}$.

If one is given a set $W \subseteq (X \cup X^{-1})^*$ such that $\{w(\bar{x}) = 1 : w \in W\}$ is a set of laws determining the variety $\mathcal{V}$, then one has a good description of $\text{Rad}_{\mathcal{V}}(\Gamma)$ as the subgroup of Γ generated by all elements of the form $w(\bar{g})$ as w varies over W and $\bar{g}$ varies over tuples from Γ .

Remark 18. A variety $\mathcal{V}$ can be determined by more than one set of laws. If we wish a unique such set we can take W to be the set of all $w \in (X \cup X^{-1})^*$ such that every group in $\mathcal{V}$ satisfies the law $w(\bar{x}) = 1$, but this set of laws is unnecessarily large.

We now give some examples of verbal subgroups.

- (1) $\text{Rad}_{\mathcal{E}}(\Gamma) = \Gamma$.
- (2) $\text{Rad}_{\mathcal{O}}(\Gamma) = \{1\}$.
- (3) $\text{Rad}_{\mathcal{A}}(\Gamma) = \Gamma'$.
- (4) $\text{Rad}_{\mathcal{A}^2}(\Gamma) = \Gamma''$.
- (5) $\text{Rad}_{\mathcal{N}_c}(\Gamma) = \gamma_{c+1}(\Gamma)$.
- (6) $\text{Rad}_{\mathcal{N}_c \wedge \mathcal{A}^2}(\Gamma) = \gamma_{c+1}(\Gamma) \cdot \Gamma''$.

Definition 19. Let $\mathcal{V}$ be a nontrivial variety of groups and r a cardinal. Let the r -generator group Γ lie in $\mathcal{V}$. Let $\langle a_1, a_2, \dots; \rangle_{\mathcal{V}}$ be the group $F_r(\mathcal{V})$ free in $\mathcal{V}$ on the set $A = \{a_{\xi+1} : \xi < r\}$. Let $\varphi : F_r(\mathcal{V}) \rightarrow \Gamma$ be a surjective homomorphism. If $\text{Ker}(\varphi)$ is the normal closure of $\{R_{\xi+1} : \xi < s\}$ in $F_r(\mathcal{V})$, then

$$\langle a_1, a_2, \dots; R_1 = R_2 = \dots = 1 \rangle_{\mathcal{V}} \quad (12)$$

is a *presentation* of Γ relative to $\mathcal{V}$. If r and s are finite, then Γ is said to be *finitely presented* relative to $\mathcal{V}$.

Remark 20. Of course, if $\Gamma \in \mathcal{V}$ is finitely presented relative to the variety $\mathcal{O}$ of all groups, then Γ is finitely presented relative to $\mathcal{V}$.

We next paraphrase a definition due to G. Baumslag, B. H. Neumann, H. Neumann, and P. M. Neumann [N].

Definition 21. Let $\mathcal{V}$ be a nontrivial variety of groups. A group $\Gamma \in \mathcal{V}$ *discriminates* $\mathcal{V}$ provided that Γ discriminates its free group $F_{\omega}(\mathcal{V})$ of countably infinite rank.

Definition 22. Let $\mathcal{V}$ be a nontrivial variety of groups and r a positive integer. $\mathcal{V}$ is *r -discriminated* provided that $\mathcal{V}$ is discriminated by its free group $F_r(\mathcal{V})$ of rank r .

Remark 23. It follows from Benjamin Baumslag's original paper [3] that the variety $\mathcal{O}$ of all groups is 2-discriminated.

Definition 24. Let $\mathcal{V}$ be a nontrivial variety of groups and Γ a group in $\mathcal{V}$. Γ is *$\mathcal{V}$ -freely separated* provided to each $\gamma \in \Gamma \setminus \{1\}$; there is a group $F(\mathcal{V})$ free in $\mathcal{V}$ and a homomorphism $\varphi : \Gamma \rightarrow F(\mathcal{V})$ such that $\varphi(\gamma) \neq 1$. If n is a positive integer, Γ is *$\mathcal{V}$ - n -freely separated* provided for every $(\gamma_1, \dots, \gamma_n) \in (\Gamma \setminus \{1\})^n$ there is a group $F(\mathcal{V})$ free in $\mathcal{V}$ and a homomorphism $\varphi : \Gamma \rightarrow F(\mathcal{V})$ such that $\varphi(\gamma_i) \neq 1$ for all $i = 1, \dots, n$. Γ is *$\mathcal{V}$ -freely discriminated* provided it is $\mathcal{V}$ - n -freely separated for every positive integer n .

2.2. Universal Theories. Observe that commutative transitivity is rendered by the universal sentence

$$\begin{aligned} \forall x, y, z ((y \neq 1) \wedge ([x, y] = 1) \wedge ([y, z] = 1)) \\ \longrightarrow ([x, z] = 1). \end{aligned} \quad (13)$$

Every nonabelian group Γ with a nontrivial center must violate this property for if $ab \neq ba$ and $z \in Z(\Gamma) \setminus \{1\}$, then a commutes with $z \neq 1$ which commutes with b , but a and b do not commute. We introduce *noncentral commutative transitivity*, abbreviated NZCT, as the universal property:

$$\begin{aligned} \forall x, y, z, w_1 ((([w_1, y] \neq 1) \wedge ([x, y] = 1) \wedge ([y, z] = 1)) \\ \longrightarrow ([x, z] = 1)) \end{aligned} \quad (14)$$

asserting that the centralizer of a noncentral element $g \in \Gamma \setminus Z(\Gamma)$ be abelian. Our main result, proven in the next section, is that the relatively free groups $F(\mathcal{N}_c \wedge \mathcal{A}^2)$ satisfy NZCT.

Remark 25. NZCT was first introduced by Levin and Rosenberger in [6] where it was called commutation transitivity. However, Remeslennikov and Stöhr in [7] call CT commutation transitivity. Thus, we have chosen to avoid the terminology “commutation transitivity” altogether.

CT and NZCT are the 0th and 1st terms of the following hierarchy of universal sentences $\text{CT}(n)$ introduced in [8]:

$$\begin{aligned} \forall x, y, z, w_1, \dots, w_n ((([w_1, \dots, w_n, y] \neq 1) \wedge ([x, y] = 1) \\ \wedge ([y, z] = 1)) \longrightarrow ([x, z] = 1)). \end{aligned} \quad (15)$$

Remark 26. If $m < n$, then $\text{CT}(m)$ implies $\text{CT}(n)$.

Remark 27. The interpretation of $\text{CT}(n)$ in a group Γ is that the centralizers of an element $g \in \Gamma \setminus Z_n(\Gamma)$ outside of the n th term of the upper central series be abelian.

It was observed in [8] that, for an integer $c \geq 1$, the free nilpotent groups $F(\mathcal{N}_c)$ satisfy $\text{CT}(c - 1)$.

Definition 28. Let G be a group. The set $\text{Th}_{\forall}(G)$ of all universal sentences of L_0 true in G is the *universal theory* of G . One says G and H are *universally equivalent* and writes $G \equiv_{\forall} H$ provided that $\text{Th}_{\forall}(G) = \text{Th}_{\forall}(H)$.

Analogous to the definition of a universal sentence, we define an *existential sentence* of L_0 to be one of the form $\exists \bar{x} \varphi(\bar{x})$, where $\bar{x}$ is a tuple of distinct variables and $\varphi(\bar{x})$ is a formula of L_0 containing no quantifiers and involving (at most) the variables in $\bar{x}$. The meanings of $\text{Th}_{\exists}(G)$ and $G \equiv_{\exists} H$ are apparent. Since an existential sentence is logically equivalent to the negation of a universal sentence and vice versa one concludes that $G \equiv_{\forall} H$ if and only if $G \equiv_{\exists} H$. The following, for which we provide a proof, is standard fare.

Lemma 29. *Let G be a group and H a subgroup of G . Then $G \equiv_{\forall} H$ if and only if, to every finite system (call it $(\otimes)$)*

$$\begin{aligned} u_i(\bar{x}) &= 1, & 1 \leq i \leq I, \\ w_j(\bar{x}) &\neq 1, & 1 \leq j \leq J \end{aligned} \quad (\otimes)$$

of equations and inequations in a finite tuple $\bar{x} = (x_1, \dots, x_k)$ of unknowns, it is the case that $(\otimes)$ has a solution in H whenever it has a solution in G .

Proof. Suppose first that $G \equiv_{\forall} H$. Then $G \equiv_{\exists} H$. So,

$$\exists \bar{x} \left(\bigwedge_i (u_i(\bar{x}) = 1) \wedge \bigwedge_j (w_j(\bar{x}) \neq 1) \right) \quad (16)$$

is true in G ; it must also be true in H .

Now suppose that $(\otimes)$ has a solution in H whenever it has a solution in G . It will suffice to show $G \equiv_{\exists} H$. Since $H \subseteq G$, every existential sentence of L_0 true in H must also be true in G . Suppose that $\exists \bar{x} \varphi(\bar{x})$ is true in G . We may assume that $\varphi(\bar{x})$ is written in *disjunctive normal form*; namely,

$$\bigvee_n \left(\bigwedge_{n,i} (u_{n,i}(\bar{x}) = 1) \wedge \bigwedge_{n,j} (w_{n,j}(\bar{x}) \neq 1) \right). \quad (17)$$

Then $\exists \bar{x} \varphi(\bar{x})$ is logically equivalent to

$$\bigvee_n \exists \bar{x} \left(\bigwedge_{n,i} (u_{n,i}(\bar{x}) = 1) \wedge \bigwedge_{n,j} (w_{n,j}(\bar{x}) \neq 1) \right). \quad (18)$$

Since a disjunction is true if and only if at least one of the disjuncts is true, for at least one fixed n , $\exists \bar{x} (\bigwedge_{n,i} (u_{n,i}(\bar{x}) = 1) \wedge \bigwedge_{n,j} (w_{n,j}(\bar{x}) \neq 1))$ is true in G , then the system

$$\begin{aligned} u_{n,i}(\bar{x}) &= 1, \\ w_{n,j}(\bar{x}) &\neq 1 \end{aligned} \quad (19)$$

has a solution in G and hence also a solution in H . Working backwards, so to speak, $\exists \bar{x} \varphi(\bar{x})$ is true in H . $\square$

Corollary 30. *Let G be a group and H a subgroup of G . If H discriminates G , then $G \equiv_{\forall} H$.*

Proof. Suppose the system

$$\begin{aligned} u_i(\bar{x}) &= 1, & 1 \leq i \leq I, \\ w_j(\bar{x}) &\neq 1, & 1 \leq j \leq J \end{aligned} \quad (20)$$

in the tuple $\bar{x} = (x_1, \dots, x_k)$ has a solution $\bar{x} = \bar{g} = (g_1, \dots, g_k) \in G^k$. Since H discriminates G , there is a homomorphism $\varphi : G \rightarrow H$ which does not annihilate any $w_j(g_1, \dots, g_k)$ for $1 \leq j \leq J$. But then $\bar{x} = (\varphi(g_1), \dots, \varphi(g_k))$ is a solution to the system in H . $\square$

Proposition 31. *Let $\mathcal{V}$ be a nontrivial variety of groups and let r be an infinite cardinal. Then the free group $F_{\omega}(\mathcal{V})$ of countably infinite rank in $\mathcal{V}$ discriminates $F_r(\mathcal{V})$. In particular, $F_r(\mathcal{V}) \equiv_{\forall} F_{\omega}(\mathcal{V})$ for all infinite cardinals r .*

Remark 32. Before giving a proof of Proposition 31, we remark that the universal equivalence of free algebras of infinite rank is also a very special consequence of a general theorem of Vaught, which is recorded as Theorem 4 of Section 3.8 of Grätzer [9] and proven therein.

Proof of Proposition 31. Let $A = \{a_{\xi+1} : 0 \leq \xi < r\}$ freely generate $F_r(\mathcal{V})$ and view $F_{\omega}(\mathcal{V})$ as the subgroup generated (necessarily freely) by the initial segment $B = \{a_{n+1} : 0 \leq n < \omega\}$. Let $C = A \setminus B$. Let $g_1, \dots, g_k$ be finitely many nontrivial elements of $F_r(\mathcal{V})$. For each $i = 1, \dots, k$, fix a word w_i on A representing g_i . Each w_i can involve only finitely many generators from B and only finitely many generators from C . Let B_0 be the set of generators from B involved in $w_1, \dots, w_k$ and C_0 the set of generators from C involved in $w_1, \dots, w_k$ so that B_0 and C_0 are finite. Say $B_0 = \{b_1, \dots, b_m\}$ and $C_0 = \{c_1, \dots, c_p\}$. We may assume that $b_j = a_{n(j)}$, $j = 1, \dots, m$, where $1 \leq n(1) < \dots < n(m) < \omega$ and $c_j = a_{\xi(j)}$, $j = 1, \dots, p$, where $\omega \leq \xi(1) < \dots < \xi(p)$. We may view each w_i as a word on $a_1, \dots, a_{n(\omega)}, a_{\xi(1)}, \dots, a_{\xi(p)}$. Since every map on A into any group in $\mathcal{V}$ extends to a homomorphism, we may map into $F_{\omega}(\mathcal{V})$ (and onto $\langle a_1, \dots, a_{n(m)+p} \rangle$ which we may identify with $F_{n(m)+p}(\mathcal{V})$) by fixing $\{a_1, \dots, a_{n(m)+p}\}$, annihilating $(B \setminus \{a_1, \dots, a_{n(m)+p}\}) \cup (C \setminus C_0)$, and mapping $a_{\xi(j)}$ to $a_{n(m)+j}$ for all $j = 1, \dots, p$. The image of $g_i = w_i(a_1, \dots, a_{n(\omega)}, a_{\xi(1)}, \dots, a_{\xi(p)})$ is $w_i(a_1, \dots, a_{n(m)+p}) \in F_{n(m)+p}(\mathcal{V}) \leq F_{\omega}(\mathcal{V})$.

Suppose that, for some fixed i_0 , $1 \leq i_0 \leq k$, we would have $w_{i_0}(a_1, \dots, a_{n(m)+p}) = 1$. Since $F_{n(m)+p}$ is free on $\{a_1, \dots, a_{n(m)+p}\}$ in $\mathcal{V}$ we would have, for any $\Gamma \in \mathcal{V}$ and any $(\gamma_1, \dots, \gamma_{n(m)+p}) \in \Gamma^{n(m)+p}$, that $w_{i_0}(\gamma_1, \dots, \gamma_{n(m)+p}) = 1$. That is, $w_{i_0}(x_1, \dots, x_{n(m)+p}) = 1$ is a law in $\mathcal{V}$. But that contradicts the fact that $g_{i_0} = w_{i_0}(a_1, \dots, a_{n(\omega)}, a_{\xi(1)}, \dots, a_{\xi(p)}) \neq 1$ in $F_r(\mathcal{V})$. The contradiction shows that all of the images $w_i(a_1, \dots, a_{n(m)+p})$ are nontrivial in $F_{\omega}(\mathcal{V})$, $i = 1, \dots, k$. Hence, $F_{\omega}(\mathcal{V})$ discriminates $F_r(\mathcal{V})$. $\square$

Given a nontrivial variety $\mathcal{V}$, every universal sentence true in $F_{\omega}(\mathcal{V})$ must also be true in $F_n(\mathcal{V}) \leq F_{\omega}(\mathcal{V})$ for all $0 \leq n < \omega$ (where we view $F_0(\mathcal{V}) = \{1\}$ to be free on the empty set). The $F_n(\mathcal{V})$ can conceivably satisfy more universal sentences of L_0 than $F_{\omega}(\mathcal{V})$, but they must satisfy at least those true in $F_{\omega}(\mathcal{V})$ and hence true in $F_r(\mathcal{V})$ for all cardinals $r \geq \omega$. It therefore makes sense to call $\text{Th}_{\forall}(F_{\omega}(\mathcal{V}))$ the universal theory of the free groups in $\mathcal{V}$ for that is the set of universal sentences of L_0 that must be true in every group $F(\mathcal{V})$ free in $\mathcal{V}$. In some cases, we can do better than rank ω . Let us say that $\mathcal{V}$ is *finitely discriminated* if there is

a positive integer r such that $\mathcal{V}$ is r -discriminated. In that event, $F_r(\mathcal{V})$ discriminates $F_\omega(\mathcal{V})$ and since, for any finite $r \leq s < \omega$, we may view a map $F_\omega(\mathcal{V}) \rightarrow F_r(\mathcal{V})$ as a map $F_\omega(\mathcal{V}) \rightarrow F_s(\mathcal{V})$, such $F_s(\mathcal{V})$ also discriminates $F_\omega(\mathcal{V})$. It follows that $F_s(\mathcal{V}) \equiv_{\mathcal{V}} F_\omega(\mathcal{V})$ for all $r \leq s < \omega$. Therefore, if we let $N(\mathcal{V}) = \omega$ when $\mathcal{V}$ is not finitely discriminated and $N(\mathcal{V}) = \min\{r : \mathcal{V} \text{ is } r\text{-discriminated}\}$ otherwise, then $\text{Th}_{\mathcal{V}}(F_{N(\mathcal{V})}(\mathcal{V}))$ is precisely the set of universal sentences of L_0 true in every group $F(\mathcal{V})$ free in $\mathcal{V}$. In particular, since quasi-identities are instances of universal sentences, if $\mathcal{Q}(\mathcal{V})$ is the set of all quasi-identities of L_0 true in every $F(\mathcal{V})$ free in $\mathcal{V}$, then $\mathcal{Q}(\mathcal{V})$ is precisely the set of all quasi-identities of L_0 true in $F_{N(\mathcal{V})}(\mathcal{V})$.

Definition 33. Let $\mathcal{V}$ be a nontrivial variety of groups with the above notation $\text{Th}_{\mathcal{V}}(F_{N(\mathcal{V})}(\mathcal{V}))$ being the *universal theory of the free groups in $\mathcal{V}$* .

Proposition 34. Let $\mathcal{V}$ be a nontrivial variety of groups. Suppose that for some integer $m \geq 0$ the free groups in $\mathcal{V}$ satisfy the universal sentence $\text{CT}(m)$. Let $\Gamma \in \mathcal{V}$ be $\mathcal{V}$ -2-freely separated. Then Γ satisfies $\text{CT}(m)$.

Proof. Suppose not. Then there are $a, b, c, d_1, \dots, d_m \in \Gamma$ such that

$$[d_1, \dots, d_m, b] \neq 1, \quad (21)$$

and $[a, b] = 1$ and $[b, c] = 1$, but $[a, c] \neq 1$. Since Γ is $\mathcal{V}$ -2-freely separated, there is a group $F(\mathcal{V})$ free in $\mathcal{V}$ and a homomorphism $\varphi : \Gamma \rightarrow F(\mathcal{V})$ such that simultaneously $[\varphi(d_1), \dots, \varphi(d_m), \varphi(b)] = \varphi([d_1, \dots, d_m, b]) \neq 1$ and $[\varphi(a), \varphi(c)] = \varphi([a, c]) \neq 1$, but that contradicts the fact that $F(\mathcal{V})$ satisfies $\text{CT}(m)$ since $[\varphi(a), \varphi(b)] = \varphi([a, b]) = \varphi([b, c]) = [\varphi(b), \varphi(c)] = 1$ in $F(\mathcal{V})$. The contradiction shows that Γ satisfies $\text{CT}(m)$. $\square$

Remark 35. We may as well take m in Proposition 34 to be the least integer n such that the free groups in $\mathcal{V}$ satisfy $\text{CT}(n)$ as $\text{CT}(m)$ implies $\text{CT}(n)$ whenever $n \geq m$.

It was shown in [10] that the universal theory of the groups free in the variety $\mathcal{O}$ of all groups is axiomatizable by the quasi-identities $\mathcal{Q}(\mathcal{O})$ true in the free groups together with CT. It was shown in [7, 11] that the universal theory of the free metabelian groups is axiomatizable by the quasi-identities $\mathcal{Q}(\mathcal{A}^2)$ true in the free metabelian groups together with CT. With those facts in mind, we assert the next proposition.

Proposition 36. Let $\mathcal{V}$ be a nontrivial variety of groups. Suppose that there is an integer $m \geq 0$ (which we may assume minimal) such that the universal theory of the free groups in $\mathcal{V}$ is axiomatizable by a set $\mathcal{Q}(\mathcal{V})$ of quasi-identities together with $\text{CT}(m)$. If $\Gamma \in \mathcal{V}$ has a finite presentation relative to $\mathcal{V}$, is $\mathcal{V}$ -freely separated, and satisfies $\text{CT}(m)$, then Γ is $\mathcal{V}$ -freely discriminated.

Proof. Let

$$\begin{aligned} \langle a_1, a_2, \dots, a_m; R_1(a_1, a_2, \dots, a_m) = \dots \\ = R_n(a_1, a_2, \dots, a_m) = 1 \rangle_{\mathcal{V}} \end{aligned} \quad (22)$$

be a finite presentation of Γ relative to $\mathcal{V}$. Suppose that Γ is not $\mathcal{V}$ -freely discriminated. Then there exist finitely many nontrivial elements $g_1, \dots, g_k$ (with $k > 1$) of Γ such that every homomorphism $\varphi : \Gamma \rightarrow F(\mathcal{V})$ must annihilate at least one of $g_1, \dots, g_k$. Let $g_i = w_i(a_1, a_2, \dots, a_m)$, $i = 1, \dots, k$. It follows that the following universal sentence $(\otimes)$ of L_0 is true in every $F(\mathcal{V})$:

$$\begin{aligned} \forall x_1, \dots, x_m \left(\bigwedge_{i=1}^n (R_i(x_1, \dots, x_m) = 1) \right. \\ \left. \rightarrow \bigvee_{j=1}^k (w_j(x_1, \dots, x_m) = 1) \right). \end{aligned} \quad (\otimes)$$

Since Γ is $\mathcal{V}$ -freely separated, it is isomorphic to a subgroup of a direct product of groups free in $\mathcal{V}$. Quasi-identities are preserved in direct products and subgroups. Hence, Γ is a model of $\mathcal{Q}(\mathcal{V})$. But Γ also satisfies $\text{CT}(m)$ and so, by hypothesis, is a model of the universal theory of the free groups in $\mathcal{V}$. In particular, Γ must satisfy $(\otimes)$. But then at least one of $w_1(a_1, \dots, a_m)$, or $w_k(a_1, \dots, a_m)$ must be trivial in Γ -contradicting the choice of $g_1, \dots, g_k$. The contradiction shows that Γ is $\mathcal{V}$ -freely discriminated. $\square$

Philip Hall showed (recorded as Observation (2) on page 22 of [12]) that every finitely generated metabelian group has a finite presentation relative to the variety $\mathcal{A}^2$ of all metabelian groups. Putting that observation together with the fact that the universal theory of the free metabelian groups is axiomatizable by $\mathcal{Q}(\mathcal{A}^2) \cup \{\text{CT}\}$ and Propositions 34 and 36, we immediately deduce the following metabelian analogue of Benjamin Baumslag's classic theorem.

Theorem 37. Let Γ be a finitely generated metabelian group which is $\mathcal{A}^2$ -freely separated. Then Γ is $\mathcal{A}^2$ -freely discriminated if and only if it is CT.

With the same proof as Corollary 8, we immediately deduce the following.

Corollary 38. Let Γ be a finitely generated metabelian group. Then Γ is $\mathcal{A}^2$ -freely discriminated if and only if it is $\mathcal{A}^2$ -2-freely separated.

Lyndon showed in [13] that every finitely generated nilpotent group is finitely presented. Anticipating our main result, proven in the next section, that the relatively free groups $F(\mathcal{N}_c \wedge \mathcal{A}^2)$ satisfy $\text{NZCT} = \text{CT}(1)$, we may ask whether the universal theory of the free groups in $\mathcal{N}_c \wedge \mathcal{A}^2$ is axiomatizable by $\mathcal{Q}(\mathcal{N}_c \wedge \mathcal{A}^2) \cup \{\text{NZCT}\}$. If it is, then we have the analogue of Theorem 37 for the varieties $\mathcal{N}_c \wedge \mathcal{A}^2$. Furthermore, following a suggestion of A. G. Myasnikov that the universal theory of the free groups in $\mathcal{N}_c$ may be

axiomatizable by $\mathcal{Q}(\mathcal{N}_c) \cup \{\text{CT}(m(c))\}$ for a suitable $m(c)$ (which we may assume minimal), we may ask whether or not this is so. If true, then we also have the analogue of Theorem 37 for the varieties $\mathcal{N}_c$.

3. The Main Result

3.1. Philip Hall's Basic Commutators. The general reference for this section as well as the justification for all subsequent commutator calculations will be in Chapter 5 of [14]. We begin by fixing an integer $r \geq 2$. Let F be a rank r free group and suppose that $(c_1, \dots, c_r) \in F^r$ such that F is freely generated by the set $\{c_1, \dots, c_r\}$. F is residually nilpotent. That is,

$$\bigcap_{n \in \omega} \gamma_{n+1}(F) = \{1\}. \quad (23)$$

It follows that, for every element $x \in F \setminus \{1\}$, there is a positive integer n such that $x \notin \gamma_{n+1}(F)$. The least such positive integer n is called the *weight* of x and is denoted by $\text{wt}(x)$.

For each positive integer n , the quotient group $\gamma_n(F)/\gamma_{n+1}(F)$ is a finitely generated free abelian group. Suppose that we order a basis for each such quotient. Say $b_{q(n-1)+1}, \dots, b_{q(n)}$ (here $q(n)$ is the number of basis elements of weight $\leq n$; see Theorem 40 below) is a basis for $\gamma_n(F)/\gamma_{n+1}(F)$ ordered according to their subscripts. Then every element of $\gamma_n(F)$ is uniquely represented modulo $\gamma_{n+1}(F)$ in the form

$$\widehat{b_{q(n-1)+1}}^{e_1} \cdots \widehat{b_{q(n)}}^{e_{q(n)-q(n-1)}}, \quad (24)$$

where $\widehat{b}$ are fixed preimages in $\gamma_n(F)$ of the b modulo $\gamma_{n+1}(F)$. Furthermore, if we order the union of these preimages over all n by asserting that those of smaller weight precede those of larger weight, we have for each fixed integer $c \geq 1$ that every element of F is uniquely represented modulo $\gamma_{c+1}(F)$ in the form

$$\widehat{b_1}^{e_1} \cdots \widehat{b_r}^{e_r} \widehat{b_{r+1}}^{e_{r+1}} \cdots \widehat{b_{q(c-1)+1}}^{e_{q(c-1)+1}} \cdots \widehat{b_{q(c)}}^{e_{q(c)}}. \quad (25)$$

Equivalently, every element of the nilpotent free group $F/\gamma_{c+1}(F) = F_r(\mathcal{N}_c)$ is uniquely represented in the form $b_1^{e_1} \cdots b_r^{e_r} b_{r+1}^{e_{r+1}} \cdots b_{q(c-1)+1}^{e_{q(c-1)+1}} \cdots b_{q(c)}^{e_{q(c)}}$.

Starting with $(c_1, \dots, c_r)$ as before, we extend to an infinite sequence of distinct terms:

$$(c_1, \dots, c_r, c_{r+1}, \dots, c_{n+1}, \dots) = (c_{n+1})_{n \in \omega} \in F^\omega \quad (26)$$

according to a construction which guarantees that the terms of any fixed weight $n \geq 1$ in the sequence not only are pairwise incongruent modulo $\gamma_{n+1}(F)$ but also project to a basis for $\gamma_n(F)$ modulo $\gamma_{n+1}(F)$.

Although it is now generally agreed that Philip Hall never himself actually constructed such a sequence (Marshall Hall, Jr. did), the sequence constructed according to the recipe to follow is named in his honor. The terms of such a sequence are called (*Hall*) *basic commutators*.

Remark 39. The terminology “basic commutators” is standard. However, technically, it is actually the sequence rather than the individual commutators which is basic.

Let us observe, for example, that

$$\begin{aligned} [x, xy] &= x^{-1}(xy)^{-1}x(xy) \\ &= x^{-1}y^{-1}x^{-1}xxy \\ &= x^{-1}y^{-1}xy \\ &= [x, y]. \end{aligned} \quad (27)$$

So the representation of a commutator need not be unique. In order to finesse questions about the uniqueness of representations of basic commutators in terms of prior ones, we construct formal basic commutators in an abstract setting. With apologies to our topologist friends who wish to reserve the word “groupoid” for a concept in category theory, we follow the usage of Baumslag in [15] and let *groupoid* be a set provided with a binary operation. Let $\mathcal{G}$ be the class of all groupoids and let $F_r(\mathcal{G})$ be free of rank r relative to $\mathcal{G}$. Suppose that

$$(b_1, \dots, b_r) \in (F_r(\mathcal{G}))^r \quad (28)$$

such that $\{b_1, \dots, b_r\}$ freely generates $F_r(\mathcal{G})$. Thus, every element of $F_r(\mathcal{G})$ is uniquely a nonassociative word on $b_1, \dots, b_r$. If the length of such a nonassociative word b is n , then we say b has *weight* n and write $\text{wt}(b) = n$.

Starting with $(b_1, \dots, b_r) \in (F_r(\mathcal{G}))^r$, we extend to a *basic sequence* of distinct terms

$$(b_1, \dots, b_r, b_{r+1}, \dots, b_{n+1}, \dots) = (b_{n+1})_{n \in \omega} \in (F_r(\mathcal{G}))^\omega \quad (29)$$

according to the recipe below.

- (1) $b_1, \dots, b_r$ are the formal basic commutators of weight 1 and are ordered according to their subscripts.
- (2) If $n > 1$, then the formal basic commutators of weight less than n precede those of weight n while the formal basic commutators of weight n are ordered arbitrarily among themselves.
- (3) If $n > 1$, then $b = (b_i)(b_j)$ is a formal basic commutator of weight n provided that the following conditions are satisfied:

- (i) $\text{wt}(b_i) + \text{wt}(b_j) = n$;
- (ii) $b_j < b_i$ in the ordering of formal basic commutators;
- (iii) if $b_j = (b_k)(b_l)$, then $b_l \leq b_j$ in the ordering of formal basic commutators.

Notice that different orderings can lead to different basic sequences. If we wish to pin down a unique basic sequence given $(b_1, \dots, b_r)$, we can, for example, insist on the lexicographical ordering of terms of each fixed weight $n > 1$ in the sequence. That is, $(b_{i_1})(b_{j_1})$ and $(b_{i_2})(b_{j_2})$, each of weight n , will be ordered according to

$$(b_{i_1})(b_{j_1}) < (b_{i_2})(b_{j_2}) \quad (30)$$

provided either $b_{i_1} < b_{i_2}$ or $b_{i_1} = b_{i_2}$ and $b_{j_1} < b_{j_2}$.

Now F may be viewed as a groupoid under the derived operation of commutation. That is, we define a new binary operation $\circ$ on F by $x \circ y = [x, y] = x^{-1}y^{-1}xy$. The assignment $b_i \mapsto c_i$, $i = 1, \dots, r$, extends to a groupoid homomorphism from $F_r(\mathcal{F})$ into F . The images of the formal basic commutators (given a fixed basic sequence) are the *basic commutators* in F (with respect to a fixed basic sequence).

For example, if $F = \langle c_1, c_2, c_3 \rangle$ has rank 3, then (if we insist on the lexicographical ordering) the basic commutators of weights 1 through 4 are the following.

Weight 1:

$$c_1 < c_2 < c_3 \quad (31)$$

Weight 2:

$$[c_2, c_1] < [c_3, c_1] < [c_3, c_2] \quad (32)$$

Weight 3:

$$\begin{aligned} [c_2, c_1, c_1] &< [c_2, c_1, c_2] < [c_2, c_1, c_3] \\ &< [c_3, c_1, c_1] < [c_3, c_1, c_2] < [c_3, c_1, c_3] \\ &< [c_3, c_2, c_2] < [c_3, c_2, c_3] \end{aligned} \quad (33)$$

Weight 4:

$$\begin{aligned} [[c_3, c_1], [c_2, c_1]] &< [[c_3, c_2], [c_2, c_1]] < [[c_3, c_2], [c_3, c_1]] \\ &< [c_2, c_1, c_1, c_1] < [c_2, c_1, c_1, c_2] \\ &< [c_2, c_1, c_1, c_3] < [c_2, c_1, c_2, c_2] \\ &< [c_2, c_1, c_2, c_3] < [c_2, c_1, c_3, c_3] \\ &< [c_3, c_1, c_1, c_1] < [c_3, c_1, c_1, c_2] \\ &< [c_3, c_1, c_1, c_3] < [c_3, c_1, c_2, c_2] \\ &< [c_3, c_1, c_2, c_3] < [c_3, c_1, c_3, c_3] \\ &< [c_3, c_2, c_2, c_2] < [c_3, c_2, c_2, c_3] \\ &< [c_3, c_2, c_3, c_3]. \end{aligned} \quad (34)$$

These project modulo $\gamma_5(F)$ to a basic sequence in the free nilpotent group $F/\gamma_5(F) = F_3(\mathcal{N}_4)$. A result of Magnus recorded as Theorem 36.32 of [1] asserts that if F is a non-abelian free group, then the left-normed basic commutators of weight $\leq c$ project modulo $\gamma_{c+1}(F) \cdot F''$ to a basic sequence in the relatively free group $F/\gamma_{c+1}(F) \cdot F'' = F(\mathcal{N}_c \wedge \mathcal{A}^2)$. Thus, in our concrete example, if we omit the non-left-normed terms, namely, $[[c_3, c_1], [c_2, c_1]]$, $[[c_3, c_2], [c_2, c_1]]$, and $[[c_3, c_2], [c_3, c_1]]$, then the remaining terms are basic modulo $\gamma_5(F) \cdot F''$ and so project to a basic sequence in the relatively free group $F/\gamma_5(F) \cdot F'' = F_3(\mathcal{N}_4 \wedge \mathcal{A}^2)$.

3.2. Deducing the Main Result. If the $\max\{c, r\} < 2$, then $F_r(\mathcal{N}_c \wedge \mathcal{A}^2)$ is abelian, so it certainly satisfies NZCT. So suppose that each of c and r is at least 2. Then $\mathcal{N}_c \wedge \mathcal{A}^2$ is 2-discriminated (see [16]). Thus, every universal sentence of L_0 true in $F_2(\mathcal{N}_c \wedge \mathcal{A}^2)$ must also be true in every group free in $\mathcal{N}_c \wedge \mathcal{A}^2$. NZCT is a universal sentence of L_0 ; hence, it suffices to show that $F_2 \in (\mathcal{N}_c \wedge \mathcal{A}^2)$ satisfies NZCT. Let F be an absolutely free group of rank 2. Let $G = F/\gamma_{c+1}(F) = F_2(\mathcal{N}_c)$ be free of rank 2 in $\mathcal{N}_c$. Then $G'' = \gamma_{c+1}(F) \cdot F''/\gamma_{c+1}(F)$ and G/G'' is isomorphic to $F/(\gamma_{c+1}(F) \cdot F'') = F_2(\mathcal{N}_c \wedge \mathcal{A}^2)$.

Our strategy will be to work in G modulo G'' . We must show that, modulo G'' , the centralizer of an element which projects to a noncentral element modulo G'' is abelian. It will be useful to know what subgroups of free nilpotent groups look like. Our principal tool will be the following theorem of Moran whose proof may be found in [17].

Theorem 40 (Moran's subgroup theorem). *Let $G = F_r(\mathcal{N}_c)$ and let B be a subgroup of G . Then $B = \langle B(1), \dots, B(c) \rangle$, where $B(v) = F_{n_v}(\mathcal{N}_{\lfloor c/v \rfloor})$,*

$$\begin{aligned} [B(v_1), B(v_2)] \\ \leq \langle B(v_1 + v_2), B(v_1 + v_2 + 1), \dots, B(c) \rangle \end{aligned} \quad (35)$$

if $v_1 + v_2 \leq c$ and $[B(v_1), B(v_2)] = \{1\}$ if $v_1 + v_2 > c$. The $B(v)$ are constructed by taking as generators individual preimages in $B \cap \gamma_v(G)$ of basis elements of the free abelian group $\gamma_{v+1}(G) \cdot (B \cap \gamma_v(G)) / \gamma_{v+1}(G) \leq \gamma_v(G) / \gamma_{v+1}(G)$. The group $\gamma_v(G) / \gamma_{v+1}(G)$ is free abelian of rank given by the Witt formula, $(1/v) \sum_{d|v} \mu(d) r^{v/d}$, where μ is the Möbius function defined for positive integers v by the rules $\mu(1) = 1$ and if $v = p_1^{\epsilon_1} \cdots p_s^{\epsilon_s}$ with the p 's being distinct primes, $\mu(v) = 0$ if any $\epsilon > 1$, and $\mu(p_1 \cdots p_s) = (-1)^s$.

Remark 41. Of course, if F is an absolutely free group of finite rank $r \geq 2$, and $G = F/\gamma_{c+1}(F) = F_r(\mathcal{N}_c)$, then, for all v , $1 \leq v \leq c+1$, $\gamma_v(G) = \gamma_v(F)/\gamma_{c+1}(F)$ and so, for all v , $1 \leq v \leq c$, $\gamma_v(G)/\gamma_{v+1}(G)$ is isomorphic to $\gamma_v(F)/\gamma_{v+1}(F)$. In the application at hand r will be 2.

Lemma 42. *The free groups in $\mathcal{N}_c \wedge \mathcal{A}^2$ are torsion free.*

Proof. Let $n > 1$ be a positive integer and suppose that $x^n = 1$ in $\Gamma = F(\mathcal{N}_c \wedge \mathcal{A}^2)$ and assume that $x \neq 1$. Let k be the least integer such that $x \in \gamma_k(\Gamma) \setminus \gamma_{k+1}(\Gamma)$. Let x map to $\bar{x}$ in $\bar{\Gamma} = \Gamma/\gamma_{k+1}(\Gamma) = F(\mathcal{N}_k \wedge \mathcal{A}^2)$. Then $\bar{x} \in \gamma_k(\bar{\Gamma})$ and so, by Theorem 36.32 of [1], $\bar{x}$ is uniquely the product of powers of weight k left-normed basic commutators $\bar{c}_1^{e_1} \cdots \bar{c}_m^{e_m}$ in $\bar{\Gamma}$. But $\bar{c}_1, \dots, \bar{c}_m$ are central in $\bar{\Gamma}$. Thus,

$$(\bar{c}_1^{e_1} \cdots \bar{c}_m^{e_m})^n = \bar{c}_1^{ne_1} \cdots \bar{c}_m^{ne_m} \neq 1 \quad (36)$$

in $\bar{\Gamma}$ again by the uniqueness of the representation. That contradicts $x^n = 1$. $\square$

Definition 43. A group G is *power commutative* provided for ordered pairs of positive integers (m, n) and all $(a, b) \in G^2$ such that $a^m \neq 1$, $b^n \neq 1$, and $[a^m, b^n] = 1$ it is the case that $[a, b] = 1$.

Definition 44 (Baumslag [15]). A group G is a U -group provided for every integer $n > 1$ it is the case that $a = b$ whenever $a^n = b^n$ in G .

Lemma 45. *Every U -group is power commutative.*

Proof. Suppose that $a^m \neq 1$, $b^n \neq 1$, and $[a^m, b^n] = 1$ in the U -group G . Then $(a^{-m}ba^m)^n = a^{-m}b^n a^m = b^n$, so $a^{-m}ba^m = b$ and a^m commutes with b . Thus, $(b^{-1}ab)^m = b^{-1}a^m b = a^m$, so $b^{-1}ab = a$ and consequently a and b commute. $\square$

By a result of A. I. Mal'cev recorded as Lemma 2.1 in [15], it follows that torsion-free nilpotent groups are U -groups. We therefore conclude.

Corollary 46. *The free groups in $\mathcal{N}_c \wedge \mathcal{A}^2$ are power commutative.*

The next result is Lemma 34.51 of [1].

Lemma 47. *If G is a metabelian group and $x \in G'$, then $[x, y_1, \dots, y_n] = [x, y_{\pi(1)}, \dots, y_{\pi(n)}]$ for every permutation π of $\{1, 2, \dots, n\}$.*

Theorem 48 (the main theorem). *The free groups in $\mathcal{N}_c \wedge \mathcal{A}^2$ satisfy NZCT.*

Proof. If $\max\{c, r\} < 2$, then $F_r(\mathcal{N}_c \wedge \mathcal{A}^2)$ is abelian, so it certainly satisfies NZCT. We assume that each of r and c is at least 2. Since $\mathcal{N}_c \wedge \mathcal{A}^2$ is 2-discriminated, every universal sentence of L_0 true in $F_2(\mathcal{N}_c \wedge \mathcal{A}^2)$ is true in every group free in $\mathcal{N}_c \wedge \mathcal{A}^2$. Thus it suffices to show that the universal sentence NZCT is true in $F_2(\mathcal{N}_c \wedge \mathcal{A}^2)$.

Let $G = \langle a_1, a_2; \rangle_{\mathcal{N}_c} = F_2(\mathcal{N}_c)$ and let

$$\bar{G} = F_2(\mathcal{N}_c \wedge \mathcal{A}^2) = \frac{G}{G''} = \langle \bar{a}_1, \bar{a}_2; \rangle_{\mathcal{N}_c \wedge \mathcal{A}^2}, \quad (37)$$

where $\bar{a}_i = a_i G''$, $i = 1, 2$.

Since $\bar{G}$ is metabelian commutators which commute in $\bar{G}$, it suffices to determine what can commute with an element $\bar{y} \in \bar{G} \setminus \bar{G}'$. Since $\bar{G}$ is power commutative, we may take $\bar{y}$ not to be a proper power in $\bar{G}$. Now for integers $1 \leq p < q \leq c$ let $\gamma(p, q)$ be the left-normed basic commutator

$$\left[\underbrace{a_2, a_1, \dots, a_1}_{(q-p)\text{-times}}, \underbrace{a_2, \dots, a_2}_{(p-1)\text{-times}} \right] \quad (38)$$

and let $\bar{\gamma}(p, q)$ be its image in $\bar{G}$. Note that, when $q = c$, the weight c commutators $\gamma(p, c)$ and $\bar{\gamma}(p, c)$ ($1 \leq p < c$) are central in G and $\bar{G}$, respectively. In determining commutativity, we may ignore central factors, so we can assume that $\bar{y}$ contains no factors $\bar{\gamma}(p, c)$.

Thus

$$\bar{y} = \bar{a}_1^{e_1} \bar{a}_2^{e_2} \prod_{q=2}^{c-1} \prod_{p=1}^{q-1} \bar{\gamma}(p, q)^{\gamma(p, q)}, \quad (39)$$

where $(e_1, e_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ and $\bar{y}$ is not a proper power in $\bar{G}$. Let y be the preimage $a_1^{e_1} a_2^{e_2} \prod_{q=2}^{c-1} \prod_{p=1}^{q-1} \gamma(p, q)^{\gamma(p, q)}$ of $\bar{y}$ in G . Note that y cannot be a proper power in G ; otherwise, $\bar{y}$ would be a proper power in $\bar{G}$. Let $\bar{B}$ be the centralizer in $\bar{G}$ of $\bar{y}$. We wish to show that $\bar{B}$ is abelian.

Let $B = \{x \in G : [y, x] \in G''\}$ be the preimage of $\bar{B}$ in G and note that $\gamma_c(G) \cdot G'' \leq \bar{B}$. Our strategy will be to use Moran's subgroup theorem to show that B is commutative modulo G'' . We can take for generators of $B(1)$ preimages in B of a basis for

$$\gamma_2(G) \cdot \frac{(B \cap G)}{\gamma_2(G)} = \gamma_2(G) \cdot \frac{B}{\gamma_2(G)}. \quad (40)$$

So suppose that $x \in B$ and $x \equiv a_1^{x_1} a_2^{x_2} [a_2, a_1]^{x(1,2)} \pmod{\gamma_3(G)}$. (Note: $[a_2, a_1] = \gamma(1, 2)$.) Now $G'' \leq \gamma_3(G)$, so $[y, x] \in G''$ implies $[y, x] \in \gamma_3(G)$. So by collecting a_1 and a_2 to the front in $[y, x]$, we see that $\det \begin{bmatrix} e_1 & e_2 \\ x_1 & x_2 \end{bmatrix} = 0$ and (e_1, e_2) and (x_1, x_2) are scalar multiples of a common nonzero ordered pair $(t_1, t_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ with $\text{GCD}(t_1, t_2) = 1$. There is a least positive integer n such that $z \equiv a_1^{nt_1} a_2^{nt_2} \pmod{\gamma_2(G)}$ lies in B . Then $\gamma_2(G) \cdot B / \gamma_2(G)$ is infinite cyclic generated by $a_1^{nt_1} a_2^{nt_2} \gamma_2(G)$ and we may take $B(1) = \langle z \rangle$.

Now let $2 \leq v \leq c-1$. Suppose that x lies in $B \cap \gamma_v(G)$ and

$$x = m \prod_{q=v}^c \prod_{p=1}^{q-1} \gamma(p, q)^{x(p, q)}, \quad (41)$$

where $m \in G''$. By Theorem 36.32 of [1], we may write x in this form. Now x will commute with y modulo G'' if and only if $m^{-1}x$ does. Since $y \equiv a_1^{e_1} a_2^{e_2} \pmod{G'}$ and $m^{-1}x$ has weight at least $v \geq 2$, $[y, m^{-1}x]$ will lie in G'' if and only if $[a_1^{e_1} a_2^{e_2}, m^{-1}x]$ does. But

$$m^{-1}x \equiv \prod_{p=1}^{v-1} \gamma(p, v)^{x(p, v)} \pmod{\gamma_{v+1}(G)} \quad (42)$$

and hence,

$$[a_1^{e_1} a_2^{e_2}, m^{-1}x] \equiv \left[a_1^{e_1} a_2^{e_2}, \prod_{p=1}^{v-1} \gamma(p)^{x(p, v)} \right] \pmod{\gamma_{v+2}(G)}. \quad (43)$$

Thus, if x is to commute with y modulo G'' , we had better have

$$\left[a_1^{e_1} a_2^{e_2}, \prod_{p=1}^{v-1} \gamma(p)^{x(p, v)} \right] \equiv 1 \pmod{G''}. \quad (44)$$

We seek conditions on the exponents $x(p, v)$ for this to occur:

$$\begin{aligned} & \left[a_1^{e_1} a_2^{e_2}, \prod_{p=1}^{v-1} \gamma(p)^{x(p, v)} \right] \\ & \equiv \prod_{p=1}^{v-1} [\gamma(p, v), a_1]^{-e_1 x(p, v)} \\ & \quad \cdot \prod_{p=1}^{v-1} [\gamma(p, v), a_2]^{-e_2 x(p, v)} \pmod{\gamma_{v+2}(G)}. \end{aligned} \quad (45)$$

Now

$$[\gamma(p, v), a_2] = \gamma(p + 1, v + 1) \quad (46)$$

while, by Lemma 47,

$$[\gamma(p, v), a_1] \equiv \gamma(p, v + 1) \pmod{G''}. \quad (47)$$

Thus, if x is to commute with y modulo G'' , we had better have

$$\begin{aligned} & \prod_{p=1}^{v-1} \gamma(p, v + 1)^{-e_1 x(p, v)} \cdot \prod_{p=1}^{v-1} \gamma(p + 1, v + 1)^{-e_2 x(p, v)} \\ & \equiv 1 \pmod{(\gamma_{v+2}(G))}. \end{aligned} \quad (48)$$

Now, modulo $\gamma_{v+2}(G)$, the above expression is congruent to

$$\begin{aligned} & \gamma(1, v + 1)^{-e_1 x(1, v)} \\ & \times \left(\prod_{p=2}^{v-1} \gamma(p, v + 1)^{-e_1 x(p, v) - e_2 x(p-1, v)} \right) \\ & \cdot \gamma(v, v + 1)^{-e_2 x(v-1, v)}. \end{aligned} \quad (49)$$

This will be trivial modulo $\gamma_{v+2}(G)$ if and only if $-e_1 x(1, v) = -e_1 x(2, v) - e_2 x(1, v) = \dots = -e_1 x(v-1, v) - e_2 x(v-2, v) = e_2 x(v-1, v) = 0$. Now $(e_1, e_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. If $e_1 = 0$, then $e_2 \neq 0$ and from $-e_2 x(1, v) = -e_2 x(2, v) = \dots = -e_2 x(v-1, v) = 0$, we conclude $x(1, v) = \dots x(v-1, v) = 0$. Similarly, if $e_2 = 0$ we draw the same conclusion. If neither e_1 nor e_2 is 0, then “telescoping” we still have the same conclusion. So, up to multiples from G'' , x is trivial modulo $\gamma_{v+1}(G)$ and thus $B(v) \leq G''$ if $2 \leq v \leq c-1$.

Now $\gamma_{c+1}(G) = \{1\}$, so we get generators for $B(c)$ by taking generators for $B \cap \gamma_c(G)$. But $B \cap \gamma_c(G)$ is central in G , so $B(c) \leq \gamma_c(G)$ is central in G . It follows that B is contained in the product of its center and a cyclic group modulo G'' and hence $\bar{B}$ is abelian.

Finally consider the centralizer in $\bar{G}$ of an element $\bar{x} \in \bar{G}'$. If $\bar{x}$ commutes with an element $\bar{y}$ of weight 1, then $\bar{x}$ must lie in the centralizer of $\bar{y}$, so be a product of a power of a weight 1 element and a central element. $\bar{x} \in \bar{G}'$ forces $\bar{x}$ to have weight at least 2 and that in turn forces $\bar{x}$ to be central. Thus only elements of $\bar{G}'$ can commute with a noncentral element of $\bar{G}$ and therefore the centralizer of every noncentral element $\bar{x} \in \bar{G}'$ is contained in $\bar{G}'$ (and thus equal to $\bar{G}'$) since $\bar{G}$ is metabelian then $\bar{G}'$ is abelian. $\square$

4. Questions

- (1) Is the universal theory of the free groups in $\mathcal{N}_c \wedge \mathcal{A}^2$ axiomatizable by a set of quasi-identities together with NZCT?
- (2) If Γ lies in $\mathcal{N}_c \wedge \mathcal{A}^2$, is finitely generated, and is $\mathcal{N}_c \wedge \mathcal{A}^2$ -2-freely separated must it be $\mathcal{N}_c \wedge \mathcal{A}^2$ -freely discriminated?

(3) (A. G. Myasnikov) Is the universal theory of the groups free in $\mathcal{N}_c$ axiomatizable by a set of quasi-identities together with CT(m) for a suitable integer m , $0 \leq m \leq c-1$?

(4) If Γ lies in $\mathcal{N}_c$, is finitely generated, and is $\mathcal{N}_c$ -2-freely separated must Γ be $\mathcal{N}_c$ -freely discriminated?

(5) (A. G. Myasnikov) Fix an integer $c \geq 2$, let $\mathcal{Q}(c)$ be the quasivariety generated by the free groups in $\mathcal{N}_c$, and suppose that $\Gamma \in \mathcal{N}_c$ is a model of the universal theory of the groups free in $\mathcal{N}_c$. Fix an element $\gamma \in \Gamma \setminus \{1\}$; form the relative centralizer extension

$$\Gamma_1 = \langle \Gamma, t; \text{rel}(\Gamma), t^{-1}zt = z \ \forall z \in C_\Gamma(\gamma) \rangle_{\mathcal{N}_c}. \quad (50)$$

Let $\Gamma_2 = \Gamma_1 / \text{Rad}_{\mathcal{Q}(c)}(\Gamma_1)$. Must Γ_2 also be a model of the universal theory of the free groups in $\mathcal{N}_c$?

(6) Given that $G = F_r(\mathcal{A}^2)$ for some fixed r , $2 \leq r < \omega$. Let $L_0[G]$ be the extension of L_0 formed by adjoining the elements of $G \setminus \{1\}$ as new constant symbols. Is the universal theory of G in the language $L_0[G]$ axiomatizable by the set of quasi-identities in the language $L_0[G]$ true in G together with CT?

(7) Let $\mathcal{V}$ be any nontrivial variety of groups such that the groups free in $\mathcal{V}$ are U -groups. Is the universal theory of the groups free in the product variety $\mathcal{A}\mathcal{V}$ axiomatizable by the quasi-identities satisfied by the groups free in that variety together with CT?

(8) Let $\mathcal{V}$ be any nontrivial variety of groups such that the groups free in $\mathcal{V}$ are U -groups. Suppose that Γ lies in $\mathcal{A}\mathcal{V}$, is finitely presented relative to $\mathcal{A}\mathcal{V}$, and is $\mathcal{A}\mathcal{V}$ -2-freely separated. Must Γ be $\mathcal{A}\mathcal{V}$ -freely discriminated?

Appendix

The Opposite Direction

Let $c \geq 2$ be a fixed but arbitrary integer. Let $d \in \{0, 1\}$. Let $\mathcal{N}_{(c,d)}$ be $\mathcal{N}_c \wedge \mathcal{A}^2$ if $d = 0$ and $\mathcal{N}_c$ if $d = 1$. Let $f(c, d)$ be $(c-1) - (1-d)(c-2)$ so that $f(c, 0) = 1$ while $f(c, 1) = c-1$. Fix $m(c, d)$ where $1 \leq m(c, d) \leq f(c, d)$ so that $m(c, 0) = 1$ while $1 \leq m(c, 1) \leq c-1$. Let $Q(c, d)$ be the set of all quasi-identities of L_0 true in $F_\omega(\mathcal{N}_{(c,d)})$. Since, up to logical equivalence, the group axioms and the laws holding in $F_\omega(\mathcal{N}_{(c,d)})$ are quasi-identities of L_0 , it is harmless to commit the abuse of taking those sentences to lie in $Q(c, d)$. Now fix $c \geq 2, d \in \{0, 1\}$ and $m(c, d)$, where $1 \leq m(c, d) \leq f(c, d) = (c-1) - (1-d)(c-2)$.

Let the *axiomatizable property* $A(c, d, m(c, d))$ be the assertion that

$$A(c, d, m(c, d)): \text{Th}_\forall(F_\omega(\mathcal{N}_{(c,d)})) \text{ is axiomatizable by } Q(c, d) \cup \text{CT}(m(c, d))$$

and the *Baumslag property* $B(c, d, m(c, d))$ be the assertion that

$$B(c, d, m(c, d)): \text{if the finitely generated group } \Gamma \text{ is } \mathcal{N}_{(c,d)}\text{-freely separated, then it is } \mathcal{N}_{(c,d)}\text{-freely discriminated if and only if it satisfies CT}(m(c, d)).$$

We have already observed that $A(c, d, m(c, d))$ implies $B(c, d, m(c, d))$. This can be found in the paragraph following Corollary 38. In this appendix, we prove the converse. That is, we show that $B(c, d, m(c, d))$ implies $A(c, d, m(c, d))$. We will need the following well-known consequence of the classical Gödel completeness theorem.

Theorem A.1. *If T is a set of sentences of L_0 and σ is a sentence of L_0 , then σ is a formal consequence of T if and only if every model of T is also a model of σ .*

Our approach will be to prove the contrapositive of

$$B(c, d, m(c, d)) \implies A(c, d, m(c, d)). \quad (\text{A.1})$$

So assume that $\text{Th}_\forall(F_\omega(\mathcal{N}_{(c,d)}))$ is not axiomatizable by $Q(c, d) \cup \{\text{CT}(m(c, d))\}$. Then there is a universal sentence $u \in \text{Th}_\forall(F_\omega(\mathcal{N}_{(c,d)}))$ which is not a formal consequence of $Q(c, d) \cup \{\text{CT}(m(c, d))\}$. Therefore, there is a model Γ of $Q(c, d) \cup \{\text{CT}(m(c, d))\}$ in which u does not hold. Therefore, Γ is a group in $\mathcal{N}_{(c,d)}$ in which the negation $\sim u$ of u holds. $\sim u$ is logically equivalent to an existential sentence of L_0 . Exactly as in the proof of Lemma 29, we may assume that an existential sentence of the form

$$\exists \bar{x} \left(\bigwedge_{i=1}^m (u_i(\bar{x}) = 1) \wedge \bigwedge_{j=1}^n (w_j(\bar{x}) \neq 1) \right) \quad (\text{A.2})$$

is true in Γ but false in $F_\omega(\mathcal{N}_{(c,d)})$. Here $\bar{x} = (x_1, \dots, x_k)$ is a tuple of distinct variables.

Suppose that $(g_1, \dots, g_k) \in \Gamma^k$ instantiates the above sentence in Γ . That is, assume that $u_i(g_1, \dots, g_k) = 1$, $1 \leq i \leq m$, and $w_j(g_1, \dots, g_k) \neq 1$, $1 \leq j \leq n$. Let $\Gamma_0 = \langle g_1, \dots, g_k \rangle$ be the subgroup of Γ generated by $\{g_1, \dots, g_k\}$. Since universal sentences are preserved in subgroups and the elements of $Q(c, d) \cup \{\text{CT}(m(c, d))\}$ are universal, Γ_0 is also a model of $Q(c, d) \cup \{\text{CT}(m(c, d))\}$. Now the finitely generated nilpotent group Γ_0 has a finite presentation:

$$\langle g_1, \dots, g_k; R_1(g_1, \dots, g_k) = \dots = R_p(g_1, \dots, g_k) = 1 \rangle_{\mathcal{N}_{(c,d)}} \quad (\text{A.3})$$

relative to the variety $\mathcal{N}_{(c,d)}$. Let $g \in \Gamma_0 \setminus \{1\}$. Suppose that g is annihilated by every homomorphism from Γ_0 into any group G free in $\mathcal{N}_{(c,d)}$. Write g as a word $v(g_1, \dots, g_k)$ in the generators $g_1, \dots, g_k$ for Γ_0 . Then the following quasi-identity q must hold in every $\mathcal{N}_{(c,d)}$ -free group:

$$\forall x_1, \dots, x_k \left(\bigwedge_{i=1}^p (R(x_1, \dots, x_k) = 1) \implies (v(x_1, \dots, x_k) = 1) \right). \quad (\text{A.4})$$

But then q must hold in Γ_0 . That is impossible. The contradiction shows that the finitely generated model Γ_0 of $Q(c, d) \cup \{\text{CT}(m(c, d))\}$ must be $\mathcal{N}_{(c,d)}$ -freely separated. Could Γ_0 be $\mathcal{N}_{(c,d)}$ -freely discriminated? Then, since $w_j(g_1, \dots, g_k) \neq 1$

for $1 \leq j \leq n$, there would be a $\mathcal{N}_{(c,d)}$ -free group G and a homomorphism $\varphi : \Gamma_0 \rightarrow G$ such that $w_j(\varphi(g_1), \dots, \varphi(g_k)) = \varphi(w_j(g_1, \dots, g_k)) \neq 1$, $1 \leq j \leq n$. Moreover, $u_i(g_1, \dots, g_k) = 1$, $1 \leq i \leq m$, entails that $u_i(\varphi(g_1), \dots, \varphi(g_k)) = \varphi(u_i(g_1, \dots, g_k)) = 1$, $1 \leq i \leq m$. Thus, the existential sentence e

$$\exists x_1, \dots, x_k \left(\bigwedge_{i=1}^m (u_i(x_1, \dots, x_k) = 1) \wedge \bigwedge_{j=1}^n (w_j(x_1, \dots, x_k) \neq 1) \right) \quad (\text{A.5})$$

must hold in G . Since existential sentences are preserved in overgroups, e must hold in $F_r(\mathcal{N}_{(c,d)})$ for all $r \geq \text{rank}(G)$, but e implies $\sim u$, so $\sim u$ must hold in $F_r(\mathcal{N}_{(c,d)})$ for all cardinals $r \geq \text{rank}(G)$. But that contradicts the fact that u holds in $F_r(\mathcal{N}_{(c,d)})$ for all cardinals $r \geq \omega$. The contradiction shows that Γ_0 is not $\mathcal{N}_{(c,d)}$ -freely discriminated. In other words, we have succeeded in showing that the falsity of $A(c, d, m(c, d))$ implies the falsity of $B(c, d, m(c, d))$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank Ben Fine and Alexei Myasnikov for posing some of these questions to them. Furthermore, they are especially grateful to Vladimir Remeslennikov for directing them to the reference [7].

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