Research Article

Nontrivial Solutions for a Boundary Value Problem of n th-Order Impulsive Differential Equation

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We study the existence of nontrivial solutions for n th-order boundary value problem with impulsive effects. We utilize Leray-Schauder degree theory to establish our main results. Furthermore, our nonlinear term f is allowed to grow superlinearly and sublinearly.

1. Introduction

In this paper, we consider the existence of nontrivial solutions for the following n th-order boundary value problem involving impulsive effects:

$$\begin{aligned} u^{(n)}(t) + f(t, u(t)) &= 0, \quad t \in [0, 1], \quad t \neq t_k, \\ -\Delta u^{(n-1)}|_{t=t_k} &= I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) &= \int_0^1 u(t) d\alpha(t), \quad u(1) = \int_0^1 u(t) d\beta(t), \\ u'(0) &= \dots = u^{(n-3)}(0) = u^{(n-2)}(0) = 0, \end{aligned} \quad (1)$$

where $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R}^+)$, $\int_0^1 u(t) d\alpha(t)$ and $\int_0^1 u(t) d\beta(t)$ denote the Riemann-Stieltjes integral, and α and β are right continuous on $[0, 1]$, left continuous at $t = 1$, and nondecreasing on $[0, 1]$, with $\alpha(0) = \beta(0) = 0$.

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Therefore, many research workers pay their attention to boundary value problems for impulsive differential equations; for example, see [1–10] and the references therein.

In [1], D. Guo utilized fixed point theory to establish some existence theorems of positive solutions for an infinite

boundary value problem of n th-order nonlinear impulsive singular integrodifferential equations on the half-line in Banach spaces.

In [2, 3], he dealt with multiple positive solutions for the following infinite boundary value problem of first order impulsive superlinear integrodifferential equations on the half-line:

$$\begin{aligned} u'(t) &= f(t, u(t), (Tu)(t), (Su)(t)), \quad \forall t \in J_+, \\ \Delta u|_{t=t_k} &= I_k(u(t_k)), \quad k = 1, 2, 3, \dots, \quad u(\infty) = \beta u(0). \end{aligned} \quad (2)$$

In [2], the function f is continuous, that is, no singularity, but in [3] the author discussed the singular case, that is, $f \rightarrow \infty$ as $t \rightarrow 0^+$ and $u \rightarrow 0^+$. Compare with [2], the key point of [3] is apart from the singular point $u = 0$ by some appropriate functions.

Motivated by the works cited above, in particular [1–3], in this paper, we utilize the methods in [11] to study the existence of nontrivial solutions for (1). Nevertheless, our work here improves and extends the corresponding ones in [11]. We first note the impulsive effect as a perturbation to the corresponding problem of (1) without impulse, so we can construct an integral operator for the corresponding boundary value problem without the impulsive terms and find out its first eigenvalue and eigenfunction. Then we establish a special cone associated with the Green function of

(1). Finally, by employing Leray-Schauder degree theory, two existence theorems of nontrivial solutions for (1) are obtained.

2. Preliminaries

We first offer some related preliminaries used in the ensuing section. In this work, we always assume that the following two conditions are satisfied:

(H1) $f \in C([0, 1] \times \mathbb{R}, \mathbb{R}), I_k \in C(\mathbb{R}, \mathbb{R}^+)$,

(H2) $\kappa_1 \geq 0, \kappa_4 \geq 0, \kappa := \kappa_1 \kappa_4 - \kappa_2 \kappa_3 > 0$, where

$$\begin{aligned} \kappa_1 &:= 1 - \int_0^1 (1 - t^{n-1}) d\alpha(t), & \kappa_2 &:= \int_0^1 t^{n-1} d\alpha(t), \\ \kappa_3 &:= \int_0^1 (1 - t^{n-1}) d\beta(t), & \kappa_4 &:= 1 - \int_0^1 t^{n-1} d\beta(t). \end{aligned} \quad (3)$$

For any $g \in C[0, 1]$, the boundary value problem

$$\begin{aligned} u^{(n)}(t) + g(t) &= 0, \quad t \in [0, 1], \\ u(0) = u'(0) = \dots = u^{(n-3)}(0) &= u^{(n-2)}(0) = u(1) = 0, \end{aligned} \quad (4)$$

if and only if u can be expressed by $u(t) = \int_0^1 H(t, s)g(s)ds$, where

$$\begin{aligned} H(t, s) &= \frac{1}{(n-1)!} \begin{cases} (1-s)^{n-1}t^{n-1} - (t-s)^{n-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{n-1}t^{n-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \quad (5)$$

Lemma 1 (see [12], Lemma 2.1). $H(t, s)$ has the following properties:

- (i) $0 \leq H(t, s) \leq K(s), \forall t, s \in [0, 1]$, where $K(s) := s(1-s)^{n-1}/(n-2)!$,
- (ii) $H(t, s) \geq \zeta(t)K(s), \forall t, s \in [0, 1]$, where $\zeta(t) := 1/(n-1) \min\{t^{n-1}, (1-t)t^{n-2}\}$.

Let $E := C[0, 1], \|u\| := \max_{t \in [0, 1]} |u(t)|$, and $P := \{u \in E : u(t) \geq 0, \forall t \in [0, 1]\}$. Then $(E, \|\cdot\|)$ is a real Banach space and P is a cone on E . Set $J := [0, 1], J' = [0, 1] \setminus \{t_1, t_2, \dots, t_m\}$ and introduce the following space:

$$\begin{aligned} PC^{n-1}[0, 1] &:= \{u \in C[0, 1] : u^{(n-1)}|_{(t_k, t_{k+1})} \in C(t_k, t_{k+1}), \\ &u^{(n-1)}(t_k^-) = u^{(n-1)}(t_k), \exists u^{(n-1)}(t_k^+)\}, \\ k &= 1, 2, \dots, m, \end{aligned} \quad (6)$$

with the norm $\|u\|_{PC^{n-1}} := \max\{\|u\|, \|u'\|, \dots, \|u^{(n-1)}\|\}$. Clearly, $(PC^{n-1}[0, 1], \|\cdot\|_{PC^{n-1}})$ is also a real Banach space.

Lemma 2 (see [9], Lemma 2.1). Let (H1) and (H2) hold. Then (1) is equivalent to the following integral equation:

$$u(t) := \int_0^1 G(t, s) f(s, u(s)) ds + \sum_{k=1}^m G(t, t_k) I_k(u(t_k)), \quad (7)$$

where

$$\begin{aligned} G(t, s) &= H(t, s) + \kappa^{-1} (\kappa_4 (1 - t^{n-1}) + \kappa_3 t^{n-1}) \int_0^1 H(t, s) d\alpha(t) \\ &\quad + \kappa^{-1} (\kappa_2 (1 - t^{n-1}) + \kappa_1 t^{n-1}) \int_0^1 H(t, s) d\beta(t). \end{aligned} \quad (8)$$

Let $\eta(t) := \min\{t^{n-1}, 1 - t^{n-1}\}$ and

$$\begin{aligned} \gamma(t) &:= \min \left\{ \zeta(t), \eta(t) \int_0^1 \zeta(t) d\alpha(t) (\alpha(1))^{-1}, \right. \\ &\quad \left. \eta(t) \int_0^1 \zeta(t) d\beta(t) (\beta(1))^{-1} \right\}. \end{aligned} \quad (9)$$

Here, $\zeta(t)$ is determined by Lemma 1. Clearly, $\gamma(t)$ is nonnegative and continuous on $[0, 1]$.

Lemma 3. One has $G(t, s) \geq \gamma(t)G(\tau, s), \forall t, s, \tau \in [0, 1]$.

Proof. By Lemma 1 and (8), we obtain $G(t, s) \leq K(s) + \kappa^{-1} \alpha(1)(\kappa_4 + \kappa_3)K(s) + \kappa^{-1} \beta(1)(\kappa_2 + \kappa_1)K(s) := \mathcal{F}_1(s)$, and

$$\begin{aligned} G(t, s) &\geq \zeta(t) K(s) + \kappa^{-1} (\kappa_4 (1 - t^{n-1}) + \kappa_3 t^{n-1}) \\ &\quad \times \int_0^1 \zeta(t) d\alpha(t) K(s) \\ &\quad + \kappa^{-1} (\kappa_2 (1 - t^{n-1}) + \kappa_1 t^{n-1}) \int_0^1 \zeta(t) d\beta(t) K(s) \\ &\geq \gamma(t) [K(s) + \kappa^{-1} \alpha(1) (\kappa_4 + \kappa_3) K(s) \\ &\quad + \kappa^{-1} \beta(1) (\kappa_2 + \kappa_1) K(s)] \\ &= \gamma(t) \mathcal{F}_1(s). \end{aligned} \quad (10)$$

Combining these, we can find $G(t, s) \geq \gamma(t)\mathcal{F}_1(s) \geq \gamma(t)G(\tau, s), \forall t, s, \tau \in [0, 1]$. This completes the proof. $\square$

Let

$$\begin{aligned} (Au)(t) &:= \int_0^1 G(t, s) f(s, u(s)) ds + \sum_{k=1}^m G(t, t_k) I_k(u(t_k)), \\ (Lu)(t) &:= \int_0^1 G(t, s) u(s) ds, \quad u \in E. \end{aligned} \quad (11)$$

Clearly, $A : E \rightarrow E$ is a completely continuous nonlinear operator, and the existence of the solutions of (1) is equivalent to that of fixed points of A ; $L : E \rightarrow E$ is a completely continuous linear operator, satisfying $L(P) \subset P$. Namely, L is a positive, completely continuous, linear operator. By the positivity of $G(t, s)$, $\forall t, s \in (0, 1)$, the spectral radius of L , denoted by $r(L)$, is positive. The Krein-Rutman theorem [13] then asserts that there are $p \in P \setminus \{0\}$ and $w \in L(0, 1) \setminus \{0\}$ such that

$$\begin{aligned} \int_0^1 G(t, s) p(s) ds &= r(L) p(t), \\ \int_0^1 G(t, s) w(t) dt &= r(L) w(s), \\ \int_0^1 w(t) dt &= 1. \end{aligned} \quad (12)$$

Denote $\delta := \int_0^1 \gamma(t)w(t)dt > 0$ and $P_0 := \{u \in P : \int_0^1 u(t)w(t)dt \geq \delta \|u\|\}$.

Lemma 4. One has $L(P) \subset P_0$.

Proof. Suppose that $u \in P$, then

$$\begin{aligned} &\int_0^1 (Lu)(t)w(t)dt \\ &= \int_0^1 \int_0^1 G(t, s)u(s)ds w(t)dt \\ &\geq \int_0^1 \gamma(t)w(t)dt \int_0^1 G(\tau, s)u(s)ds = \delta(Lu)(\tau), \end{aligned} \quad (13)$$

for all $\tau \in [0, 1]$ and thus $\int_0^1 (Lu)(t)w(t)dt \geq \delta \|Lu\|$. This completes the proof. $\square$

Lemma 5 (see [14]). Let E be a real Banach space and $\Omega \subset E$ a bounded open set with $0 \in \Omega$. Suppose that $A : \overline{\Omega} \rightarrow E$ is a completely continuous operator. If there is $y_0 \in E \setminus \{0\}$ such that $u \neq Au + \lambda y_0$, $\forall u \in \partial\Omega$, $\lambda \geq 0$, then $\deg(I - A, \Omega, 0) = 0$, where $\deg$ stands for the Leray-Schauder topological degree in E .

3. Main Results

Let $\lambda_1^{-1} := r(L)$. We now list our assumptions on f .

(H3) $\liminf_{u \rightarrow +\infty} (f(t, u)/u) > \lambda_1$ and $\limsup_{u \rightarrow -\infty} (f(t, u)/u) < \lambda_1$ uniformly in $t \in [0, 1]$.

(H4) There is a positive number $a_1 \in (0, \delta/(r(L)(\delta + \sum_{k=1}^m w(t_k))))$ such that

$$\begin{aligned} \limsup_{u \rightarrow 0} \frac{|f(t, u)|}{|u|} &< a_1, \quad \limsup_{u \rightarrow 0} \frac{I_k(u)}{|u|} < a_1 \\ &\text{uniformly in } t \in [0, 1]. \end{aligned} \quad (14)$$

(H5) $\liminf_{u \rightarrow 0^+} (f(t, u)/u) > \lambda_1$ and $\limsup_{u \rightarrow 0^-} (f(t, u)/u) < \lambda_1$ uniformly in $t \in [0, 1]$.

(H6) For a_1 in (H4), assume that

$$\begin{aligned} \limsup_{u \rightarrow \infty} \frac{|f(t, u)|}{|u|} &< a_1, \quad \limsup_{u \rightarrow \infty} \frac{I_k(u)}{|u|} < a_1 \\ &\text{uniformly in } t \in [0, 1]. \end{aligned} \quad (15)$$

Two Examples. Let $\alpha(t) = t$, $\beta(t) = 0$, for all $t \in [0, 1]$. Then $\kappa_1 = 1/n$, $\kappa_2 = 1/n$, $\kappa_3 = 0$, $\kappa_4 = 1$, and (H2) holds true.

(1) Let $f(t, u) = b_1 u + u^2 + u^4$, where $b_1 \in (0, a_1)$. Then f satisfies (H1). Moreover, we obtain

$$\begin{aligned} \liminf_{u \rightarrow +\infty} \frac{f(t, u)}{u} &= \liminf_{u \rightarrow +\infty} (b_1 + u + u^3) = +\infty > \lambda_1, \\ &\text{uniformly in } t \in [0, 1], \\ \limsup_{u \rightarrow -\infty} \frac{f(t, u)}{u} &= \limsup_{u \rightarrow -\infty} (b_1 + u + u^3) = -\infty < \lambda_1, \\ &\text{uniformly in } t \in [0, 1], \end{aligned} \quad (16)$$

$$\begin{aligned} \limsup_{u \rightarrow 0} \frac{|f(t, u)|}{|u|} &= \limsup_{u \rightarrow 0} |b_1 + u + u^3| = b_1 < a_1, \\ &\text{uniformly in } t \in [0, 1]. \end{aligned}$$

So, f satisfies (H3) and (H4). Also, let $I_k(u) = u^4$. Then I_k satisfies (H1) and

$$\limsup_{u \rightarrow 0} \frac{I_k(u)}{|u|} = \limsup_{u \rightarrow 0} |u|^3 = 0 < a_1. \quad (17)$$

Hence, I_k satisfies (H4). Consequently, (H1), (H3), and (H4) hold true.

(2) Let $f(t, u) = b_2 u + 2u^{2/3} + 3u^{4/5}$, $I_k(u) = \sqrt{|u|}$, where $b_2 \in (0, a_1)$. Then

$$\begin{aligned} \liminf_{u \rightarrow 0^+} \frac{f(t, u)}{u} &= \liminf_{u \rightarrow 0^+} (b_2 + 2u^{-1/3} + 3u^{-1/5}) = +\infty > \lambda_1 \\ &\text{uniformly in } t \in [0, 1], \\ \limsup_{u \rightarrow 0^-} \frac{f(t, u)}{u} &= \limsup_{u \rightarrow 0^-} (b_2 + 2u^{-1/3} + 3u^{-1/5}) = -\infty < \lambda_1 \\ &\text{uniformly in } t \in [0, 1]. \end{aligned} \quad (18)$$

Moreover,

$$\begin{aligned} \limsup_{u \rightarrow \infty} \frac{|f(t, u)|}{|u|} &= \limsup_{u \rightarrow \infty} |b_2 + 2u^{-1/3} + 3u^{-1/5}| = b_2 < a_1 \\ &\text{uniformly in } t \in [0, 1], \\ \limsup_{u \rightarrow \infty} \frac{I_k(u)}{|u|} &= \limsup_{u \rightarrow \infty} \frac{1}{\sqrt{|u|}} = 0 < a_1. \end{aligned} \quad (19)$$

So, (H1), (H5), (H6) hold.

Theorem 6. Assume that (H1)–(H4) hold, and (1) has at least one nontrivial solution.

Proof. (H3) implies that there are $\varepsilon \in (0, \lambda_1)$ and $c_1 > 0$ such that $f(t, u) \geq (\lambda_1 + \varepsilon)u - c_1, u \geq 0, t \in [0, 1]$ and $f(t, u) \geq (\lambda_1 - \varepsilon)u - c_1, u \leq 0, t \in [0, 1]$. Note that if $(t, u) \in [0, 1] \times [0, +\infty)$, $f(t, u) \geq (\lambda_1 + \varepsilon)u - c_1 \geq (\lambda_1 - \varepsilon)u - c_1$, and if $(t, u) \in [0, 1] \times (-\infty, 0]$, $f(t, u) \geq (\lambda_1 - \varepsilon)u - c_1 \geq (\lambda_1 + \varepsilon)u - c_1$. That is, we have

$$f(t, u) \geq (\lambda_1 + \varepsilon)u - c_1, \quad \forall u \in \mathbb{R}, t \in [0, 1], \quad (20)$$

$$f(t, u) \geq (\lambda_1 - \varepsilon)u - c_1, \quad \forall u \in \mathbb{R}, t \in [0, 1]. \quad (21)$$

Let $\mathcal{M}_1 := \{u \in E : u = Au + \lambda p, \lambda \geq 0\}$, where $p \in P \setminus \{0\}$ is given by (12). We claim that $\mathcal{M}_1$ is bounded in E . Indeed, if $u_0 \in \mathcal{M}_1$, there is $\lambda_0 \geq 0$ such that

$$\begin{aligned} u_0(t) &= (Au_0)(t) + \lambda_0 p(t) = \int_0^1 G(t, s) f(s, u_0(s)) ds \\ &\quad + \sum_{k=1}^m G(t, t_k) I_k(u_0(t_k)) + \lambda_0 p(t) \\ &\geq \int_0^1 G(t, s) f(s, u_0(s)) ds. \end{aligned} \quad (22)$$

Combining this with (20) yields

$$u_0(t) \geq (\lambda_1 + \varepsilon)(Lu_0)(t) - c_1(L\mathbf{1})(t), \quad (23)$$

where $\mathbf{1}$ stands for the constant function $\mathbf{1}(t) \equiv 1$. Multiply (23) by $w(t)$ on both sides and integrate over $[0, 1]$ and use (12) to obtain $\int_0^1 u_0(t)w(t)dt \geq (1 + \varepsilon r(L)) \int_0^1 u_0(t)w(t)dt - c_1 r(L)$, and thus $\int_0^1 u_0(t)w(t)dt \leq \varepsilon^{-1} c_1$. Note that (21) holds. Define the Nemytskii operator $F : E \rightarrow E$ by $(Fu)(t) := f(t, u(t))$. Now (22) is equivalent to

$$\begin{aligned} u_0(t) - (\lambda_1 - \varepsilon)(Lu_0)(t) + c_1(L\mathbf{1})(t) &= (Au_0)(t) - (\lambda_1 - \varepsilon)(Lu_0)(t) + c_1(L\mathbf{1})(t) + \lambda_0 p(t) \\ &= \int_0^1 G(t, s) [f(s, u_0(s)) - (\lambda_1 - \varepsilon)u_0(s) + c_1] ds \\ &\quad + \sum_{k=1}^m G(t, t_k) I_k(u_0(t_k)) + \lambda_0 p(t) \\ &= (L[Fu_0 - (\lambda_1 - \varepsilon)u_0 + c_1\mathbf{1}])(t) \\ &\quad + \sum_{k=1}^m G(t, t_k) I_k(u_0(t_k)) + \lambda_0 p(t). \end{aligned} \quad (24)$$

Lemma 4 implies that $u_0 - (\lambda_1 - \varepsilon)Lu_0 + c_1L\mathbf{1} \in P_0$. Therefore,

$$\begin{aligned} &\|u_0 - (\lambda_1 - \varepsilon)Lu_0 + c_1L\mathbf{1}\| \\ &\leq \delta^{-1} \int_0^1 [u_0(t) - (\lambda_1 - \varepsilon)(Lu_0)(t) + c_1(L\mathbf{1})(t)] w(t) dt \\ &\leq \varepsilon \delta^{-1} r(L) \int_0^1 u_0(t) w(t) dt + c_1 \delta^{-1} r(L) \leq 2c_1 \delta^{-1} r(L). \end{aligned} \quad (25)$$

Since $(\lambda_1 - \varepsilon)r(L) < 1$, the operator $I - (\lambda_1 - \varepsilon)L$ has the bounded inverse operator $(I - (\lambda_1 - \varepsilon)L)^{-1}$. Therefore, there is $\mathcal{N}_1 > 0$ such that $\|u\| \leq \mathcal{N}_1, \forall u \in \mathcal{M}_1$. This proves the boundedness of $\mathcal{M}_1$. For each $R > \sup_{u \in \mathcal{M}_1} \|u\|$, we have $u \neq Au + \lambda p, \forall u \in \partial B_R, \lambda \geq 0$, which leads to

$$\deg(I - A, B_R, 0) = 0. \quad (26)$$

On the other hand, (H4) implies that there is a $r > 0$ such that $|f(t, u)| \leq a_1|u|, I_k(u) \leq a_1|u|, \forall |u| \leq r, t \in [0, 1]$. Notice we may choose $r > 0$ so that $r < \sup_{u \in \mathcal{M}_1} \|u\|$. Thus we have

$$\begin{aligned} &|(Au)(t)| \\ &\leq \int_0^1 G(t, s) |f(s, u(s))| ds + \sum_{k=1}^m G(t, t_k) I_k(u(t_k)) \\ &\leq a_1 \left[\int_0^1 G(t, s) |u(s)| ds + \sum_{k=1}^m G(t, t_k) |u(t_k)| \right] \end{aligned} \quad (27)$$

for all $u \in \bar{B}_r$ and $t \in [0, 1]$. We claim that

$$u \neq \mu Au, \quad \forall u \in \partial B_r, \mu \in [0, 1]. \quad (28)$$

Suppose the contrary. Then there are $u_0 \in \partial B_r$ and $\mu_0 \in [0, 1]$ such that $u_0 = \mu_0 Au_0$. Let $v(t) = |u_0(t)|$. Then $v \in P$ and

$$v(t) \leq a_1 \left[\int_0^1 G(t, s) v(s) ds + \sum_{k=1}^m G(t, t_k) v(t_k) \right]. \quad (29)$$

Multiply (29) by $w(t)$ on both sides and integrate over $[0, 1]$ and use (12) to obtain

$$\begin{aligned} \int_0^1 v(t) w(t) dt &\leq a_1 r(L) \int_0^1 v(t) w(t) dt \\ &\quad + a_1 r(L) \sum_{k=1}^m v(t_k) w(t_k), \end{aligned} \quad (30)$$

and then by Lemma 4,

$$\begin{aligned} \delta(1 - a_1 r(L)) \|v\| &\leq (1 - a_1 r(L)) \int_0^1 v(t) w(t) dt \\ &\leq a_1 r(L) \sum_{k=1}^m v(t_k) w(t_k) \\ &\leq a_1 r(L) \|v\| \sum_{k=1}^m w(t_k), \end{aligned} \quad (31)$$

That is, $a_1 r(L)(\delta + \sum_{k=1}^m w(t_k)) \geq \delta$, which contracts to $a_1 \in (0, \delta/(r(L)(\delta + \sum_{k=1}^m w(t_k))))$. Consequently, $\|v\| = 0$, and thus $v = 0 \Rightarrow u_0 \equiv 0$. This contradicts to $u_0 \in \partial B_r$. As a result of this, (28) holds true. Hence I and $I - A$ are homotopic on ∂B_r . The homotopy invariance of topological degree implies that $\deg(I - A, B_r, 0) = 1$. This and (26) together imply that $\deg(I - A, B_R \setminus \bar{B}_r, 0) = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = -1$. Therefore the operator A has at least one point in $B_R \setminus \bar{B}_r$. Equivalently, (1) has at least one nontrivial solution. This completes the proof. $\square$

Theorem 7. Assume that (H1), (H2), (H5), and (H6) hold; (1) has at least one nontrivial solution.

Proof. (H5) implies there are $\varepsilon \in (0, \lambda_1)$ and $r > 0$ such that $f(t, u) \geq (\lambda_1 + \varepsilon)u, u \in [0, r], t \in [0, 1]$, and $f(t, u) \geq (\lambda_1 - \varepsilon)u, u \in [-r, 0], t \in [0, 1]$. Note that if $(t, u) \in [0, 1] \times [0, r]$, $f(t, u) \geq (\lambda_1 + \varepsilon)u \geq (\lambda_1 - \varepsilon)u$, and if $(t, u) \in [0, 1] \times [-r, 0]$, $f(t, u) \geq (\lambda_1 - \varepsilon)u \geq (\lambda_1 + \varepsilon)u$. That is, we have

$$f(t, u) \geq (\lambda_1 + \varepsilon)u \quad \forall u \in [-r, r], t \in [0, 1], \quad (32)$$

$$f(t, u) \geq (\lambda_1 - \varepsilon)u, \quad \forall u \in [-r, r], t \in [0, 1]. \quad (33)$$

Nowadays, we claim

$$u \neq Au + \lambda p, \quad \forall u \in \partial B_r, \lambda \geq 0, \quad (34)$$

where $p \in P \setminus \{0\}$ is given by (12). Indeed, if the claim is false, then there are $u_0 \in \partial B_r$ and $\lambda_0 \geq 0$ such that

$$\begin{aligned} u_0(t) &= (Au_0)(t) + \lambda_0 p(t) \\ &= \int_0^1 G(t, s) f(s, u_0(s)) ds \\ &\quad + \sum_{k=1}^m G(t, t_k) I_k(u_0(t_k)) + \lambda_0 p(t) \\ &\geq \int_0^1 G(t, s) f(s, u_0(s)) ds. \end{aligned} \quad (35)$$

Combining this with (32) yields

$$u_0(t) \geq (\lambda_1 + \varepsilon)(Lu_0)(t). \quad (36)$$

Multiply (36) by $w(t)$ and integrate over $[0, 1]$ and use (12) to obtain $\int_0^1 u_0(t)w(t)dt \geq (1 + \varepsilon r(L)) \int_0^1 u_0(t)w(t)dt$, and thus $\int_0^1 u_0(t)w(t)dt \leq 0$. Note that (33) holds, and then we find

$$\begin{aligned} u_0(t) - (\lambda_1 - \varepsilon)(Lu_0)(t) &= (Au_0)(t) - (\lambda_1 - \varepsilon)(Lu_0)(t) + \lambda_0 p(t) \\ &= (L[Fu_0 - (\lambda_1 - \varepsilon)u_0])(t) \\ &\quad + \sum_{k=1}^m G(t, t_k) I_k(u_0(t_k)) + \lambda_0 p(t). \end{aligned} \quad (37)$$

Lemma 4 implies that $u_0 - (\lambda_1 - \varepsilon)Lu_0 \in P_0$. Therefore, $\|u_0 - (\lambda_1 - \varepsilon)Lu_0\| \leq \delta^{-1} \int_0^1 [u_0(t) - (\lambda_1 - \varepsilon)(Lu_0)(t)]w(t)dt \leq$

$\varepsilon \delta^{-1} r(L) \int_0^1 u_0(t)w(t)dt \leq 0$. Notice that $(\lambda_1 - \varepsilon)r(L) < 1$. We have $u_0 = 0$, contradicting $u_0 \in \partial B_r$. As a result of this, (34) is true. Invoking Lemma 5 gives

$$\deg(I - A, B_r, 0) = 0. \quad (38)$$

On the other hand, (H6) implies that there is $c_2 > 0$ such that

$$\begin{aligned} |f(t, u)| &\leq a_1 |u| + c_2, I_k(u) \leq a_1 |u| + c_2, \\ \forall u \in \mathbb{R}, t \in [0, 1]. \end{aligned} \quad (39)$$

Let $\mathcal{M}_2 := \{u \in E : u = \lambda Au, \forall \lambda \in [0, 1]\}$. We shall now show that $\mathcal{M}_2$ is bounded in E . Indeed, if $u_0 \in \mathcal{M}_2$, then $u_0 = \lambda_0 Au_0$ for some $\lambda_0 \in [0, 1]$. In view of (39), let $v(t) = |u_0(t)|$, we have $v \in P$ and

$$\begin{aligned} v(t) &\leq a_1 \left[\int_0^1 G(t, s) v(s) ds + \sum_{k=1}^m G(t, t_k) v(t_k) \right] \\ &\quad + c_2 \left[\int_0^1 G(t, s) ds + \sum_{k=1}^m G(t, t_k) \right]. \end{aligned} \quad (40)$$

Multiply (40) by $w(t)$ on both sides and integrate over $[0, 1]$ and use (12) to obtain

$$\begin{aligned} \int_0^1 v(t)w(t)dt &\leq a_1 r(L) \int_0^1 v(t)w(t)dt \\ &\quad + a_1 r(L) \sum_{k=1}^m v(t_k)w(t_k) \\ &\quad + c_2 r(L) \left(1 + \sum_{k=1}^m w(t_k) \right), \end{aligned} \quad (41)$$

and then by Lemma 4,

$$\begin{aligned} \delta(1 - a_1 r(L)) \|v\| &\leq (1 - a_1 r(L)) \int_0^1 v(t)w(t)dt \\ &\leq a_1 r(L) \sum_{k=1}^m v(t_k)w(t_k) \\ &\quad + c_2 r(L) \left(1 + \sum_{k=1}^m w(t_k) \right) \\ &\leq a_1 r(L) \|v\| \sum_{k=1}^m w(t_k) \\ &\quad + c_2 r(L) \left(1 + \sum_{k=1}^m w(t_k) \right). \end{aligned} \quad (42)$$

Hence, $\|v\| \leq (c_2 r(L)(1 + \sum_{k=1}^m w(t_k)))/(\delta(1 - a_1 r(L)) - a_1 r(L) \sum_{k=1}^m w(t_k))$. This proves the boundedness of $\mathcal{M}_2$. Choosing $R > \sup_{u \in \mathcal{M}_2} \|u\|$ and $R > r$, we have $u \neq \lambda Au, \forall u \in \partial B_R, \lambda \in [0, 1]$. The homotopy invariance of Leray-Schauder topological degree implies that $\deg(I - A, B_R, 0) = 1$. Combining this with (38) we get $\deg(I - A, B_R \setminus \bar{B}_r, 0) = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = 1$. Therefore the operator A has at least one point in $B_R \setminus \bar{B}_r$. Equivalently, (1) has at least one nontrivial solution. This completes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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