Existence of Strong Coupled Fixed Points for Cyclic Coupled Ciric-Type Mappings

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In this short communication the concept of cyclic coupled Kannan-type contractions is generalized using a certain class of Ciric-type mappings.

1. Introduction and Preliminaries

The Banach contraction condition in a metric space \((X, d)\) given by
\[
d(Tx, Ty) \leq Ld(x, y), 0 \leq L < 1,\]
has so many significant generalizations which include the class of generalized contractions defined by Ciric [1] as follows. A mapping \(T : X \to X\) is called a generalized contraction if and only if there exist nonnegative numbers \(q, r, s, t\) such that
\[
d(Tx, Ty) \leq qd(x, y) + rd(x, Tx) + sd(y, Ty) + t[d(x, Ty) + d(y, Tx)];
\]
\[
\sup\{q + r + s + 2s\} < 1.
\]

(1) \(T\) is called contractive if
\[
d(Tx, Ty) < d(x, y).
\]

(2)

It is worth mentioning that the contractive condition (2) restricts applications only to the class of continuous operators while the contractive conditions (1) accommodate discontinuous operators as well. The search for contractive conditions that do not require continuity of operators culminated in 1969 with the appearance of the Kannan [2] contractive condition below:
\[
d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)],\quad 0 \leq a < \frac{1}{2}.
\]

(3)

The Chatterjea [3] contractive condition which followed is independent of both the contractive condition (2) and the

Kannan condition (3) which in turn is independent of (2). Consequently, unlike condition (2) the Kannan condition (3) does not generalize the well-known Banach condition above. In a first attempt, the three contractive conditions were combined by Zamfirescu [4] in one theorem to generalize and extend the Banach fixed point theorem. Following Zamfirescu Ciric unified contractive conditions mentioned above by introducing the larger and unifying class of operators called quasi-contractions. \(T\) is called a quasi-contraction (Cirić [5]) if there exists \(L \in (0, 1)\) such that
\[
d(Tx, Ty) \leq L \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\};
\]
\[
d(x, Ty), d(y, Tx)\).
\]

(4)

Čirić [5] observed that the class of quasi-contractions contains the class of generalized contractions as a proper subclass. Rhoades [6] noted that the Zamfirescu result is generalized by the Ciric contractive condition (4).

There have been numerous generalizations and extensions of the Banach fixed point theorem in literature and they are, basically, modifications of those mentioned above. Very recently Choudhury and Maity [7] introduced the concept of cyclic coupled Kannan-type contractions and established a strong cyclic coupled fixed point result below. We recall the following definition. Let \(A\) and \(B\) be two nonempty subsets of a given set \(X\). A mapping \(F : X \times X \to X\), such that \(F(x, y) \in B\) if \(x \in A\) and \(y \in B\) and \(F(x, y) \in A\) if \(x \in B\) and \(y \in A\), is called a cyclic mapping with respect to \(A\) and \(B\).
Definition 1 (see [8]). Let \((X, d)\) be a metric space and \(A, B \subset X\) nonempty subsets. \(T : A \cup B \rightarrow A \cup B\) is called a cyclic (or 2-cyclic) \(q\)-contraction if \(TA \subset B\) and \(TB \subset A\) and the following condition is satisfied:

\[
d(Tx, Ty) \leq q d(x, y) + (1 - q) \text{dist}(A, B)
\]

(5)

for all \(x \in A, y \in B\).

Definition 2 (see [7]). Let \(A\) and \(B\) be two nonempty subsets of a metric space \((X, d)\). A mapping \(F : X \times X \rightarrow X\) is called a cyclic coupled Kannan-type contraction with respect to \(A\) and \(B\) if \(F\) is cyclic with respect to \(A\) and \(B\) and, for some \(k \in (0, 1/2)\), \(F\) satisfies the following condition:

\[
d(F(x, y), F(u, v)) \leq k \left[ d(x, F(x, y)) + d(u, F(u, v)) \right],
\]

(6)

where \(x, v \in A\) and \(y, u \in B\).

Definition 3 (see [7]). Let \(X\) be a nonempty set. An element \((p_x, p_y) \in X \times X\) is called a coupled fixed point of a mapping \(F : X \times X \rightarrow X\) if \(p_x = F(p_x, p_y)\) and \(p_y = F(p_y, p_x)\). A point \((p, p) \in X \times X\) is called a strong coupled fixed point if \(F(p, p) = p\).

Theorem 4 (see [7]). Let \(A\) and \(B\) be two nonempty closed subsets of a complete metric space \((X, d)\). A mapping \(F : X \times X \rightarrow X\) cyclic coupled Kannan contraction with respect to \(A\) and \(B\) with \(A \cap B \neq \emptyset\). Then \(F\) has a strong coupled fixed point in \(A \cap B\).

The aim of this paper is to obtain a generalization of Theorem 4 in the context of Ciric-type contractions.

2. Main Results

Following Definition 2 we formulate the following definition.

Definition 5. Let \(A\) and \(B\) be two nonempty subsets of a metric space \((X, d)\). A mapping \(F : X \times X \rightarrow X\) is called a cyclic coupled Ciric-type mapping with respect to \(A\) and \(B\) if \(F\) is cyclic with respect to \(A\) and \(B\) and, for some constants \(q \in (0, 1)\), \(F\) satisfies the following condition:

\[
d(F(x, y), F(u, v)) \leq qM(x, u),
\]

(7)

where \(x, v \in A\) and \(y, u \in B\) and \(M(x, u)\) is given by

\[
M(x, u) = \max \left\{ d(x, u), \frac{1}{2} d(u, F(x, y)), \frac{1}{2} d(x, F(u, v)) \right\}.
\]

(8)

Examples of coupled cyclic Ciric-type mappings include coupled cyclic Kannan-type mappings studied in [7] since the later imply the former. We now prove the main theorem of this work.

Theorem 6. Let \(A\) and \(B\) be two nonempty closed subsets of a complete metric space \((X, d)\). A mapping \(F : X \times X \rightarrow X\) cyclic coupled Kannan contraction with respect to \(A\) and \(B\) with \(A \cap B \neq \emptyset\). Then \(F\) has a strong coupled fixed point in \(A \cap B\).

Proof. Let \(x_0 \in A\) and \(y_0 \in B\); we put \(x_1 = F(y_0, x_0), y_1 = F(x_0, y_0), x_2 = F(y_1, x_1), y_2 = F(x_1, x_1)\), and so forth. Then we obtain the following as estimates for respective displacements:

\[
d(x_1, y_1) = d(F(y_0, x_0), F(x_0, y_0))
\]

\[
\leq q \max \left\{ d(y_0, x_0), \frac{1}{2} d(y_0, F(x_0, y_0)), \frac{1}{2} d(x_0, F(y_0, x_0)) \right\}
\]

\[
\leq q \max \left\{ d(x_0, y_0), \frac{1}{2} d(y_0, y_1), \frac{1}{2} d(x_0, x_1) \right\}
\]

\[
\leq q \max \left\{ d(x_0, y_0), d(x_0, x_1), d(y_0, y_1) \right\},
\]

\[
d(y_0, x_1), d(x_0, y_1) = q\tau,
\]

(9)

where \(\tau = \max\{d(x_0, y_0), d(x_0, x_1), d(y_0, y_1), d(y_0, x_1), d(x_0, y_1)\}\).

Similarly, using the fact that \(d(y_1, x_2) \neq qd(y_1, x_2)\) we obtain

\[
d(y_1, x_2) = d(F(x_0, y_0), F(y_1, x_1))
\]

\[
\leq q \max \left\{ d(x_0, y_1), \frac{1}{2} d(y_1, F(x_0, y_0)), \frac{1}{2} d(x_0, F(y_1, x_1)), \frac{1}{2} d(x_0, F(x_0, y_0)) \right\}
\]

\[
\leq q \max \left\{ d(x_0, y_1), \frac{1}{2} d(x_0, x_2), \frac{1}{2} d(x_0, y_1) + d(y_1, x_2) \right\}
\]

\[
\leq q \max \left\{ d(x_0, y_1), \frac{1}{2} [d(x_0, y_1) + d(y_1, x_2)] \right\}
\]

(10)

yielding \(d(y_1, x_2) \leq qd(x_0, y_1) \leq q\tau\).

\[
d(x_1, y_2) \leq qd(y_0, x_1) \leq q\tau.
\]

(11)

\[
d(x_1, y_2) \leq qd(y_0, x_1) \leq q\tau.
\]

(12)
Next, we will use (9), (11), and (12) to obtain the following estimates for the displacements \( d(x_1, x_2) \) and \( d(y_1, y_2) \):

\[
d(x_1, x_2) = d(F(y_0, x_0), F(y_1, x_1)) \\
\leq q \max \left\{ d(y_0, y_1), \frac{1}{2} d(y_1, F(x_0, y_0)), \right. \\
\left. \frac{1}{2} d(y_0, F(y_1, x_1)), \right. \\
\left. \frac{1}{2} [d(y_0, F(x_0, y_0)) + d(y_1, F(y_1, x_1))] \right\} \\
\leq q \max \left\{ d(y_0, y_1), \frac{1}{2} d(y_0, x_2), \\
\frac{1}{2} [d(y_0, y_1) + d(y_1, x_2)] \right\} \\
\leq q \max \left\{ d(y_0, y_1), \frac{1}{2} d(y_0, x_2) + qd(x_0, y_1) \right\} \\
\leq q \max \left\{ d(y_0, y_1), \frac{1}{2} [d(y_0, y_1) + qd(x_0, y_1)] \right\} \\
\leq q \max \left\{ d(y_0, y_1), \frac{1}{2} [d(y_0, y_1) + qd(x_0, y_1)] \right\}
\]

(13)

Yielding \( d(x_1, x_2) \leq q \max \{d(y_0, y_1), qd(x_0, y_1)\} \leq qr \), \( d(y_1, y_2) \leq q \max \{d(x_0, x_1), qd(y_0, x_1)\} \leq qr \).

(14)

Similarly,

\[
d(y_2, x_3) \leq qd(F(x_1, y_1), F(y_1, x_1)) \\
\leq q \max \left\{ d(x_1, y_2), \frac{1}{2} d(y_2, F(x_1, y_1)), \right. \\
\left. \frac{1}{2} d(x_1, F(y_2, x_2)), \right. \\
\left. \frac{1}{2} [d(x_1, F(x_1, y_1)) + d(y_2, F(y_2, x_2))] \right\} \\
\leq q \max \left\{ d(x_1, y_2), \frac{1}{2} d(x_1, y_3), \\
\frac{1}{2} [d(x_1, y_2) + d(y_2, x_3)] \right\} ;
\]

that is, \( d(y_2, x_3) \leq qd(x_1, y_2) \leq qr \), by (11),

\[
d(x_2, y_3) \leq qd(y_1, x_2) \leq qr \), by (12).
\]

(16)

Now, to estimate \( d(x_m, x_m) \) and \( d(x_n, y_n) \) we proceed as follows: given that \( d(x_1, x_2) \leq qr, d(y_1, y_2) \leq qr, \)

\[
d(y_2, x_3) \leq qd(x_1, y_2) \leq qr \), and \( d(x_2, y_3) \leq qd(y_1, x_2) \leq qr \), we have the following:

\[
d(x_2, x_3) = d(F(y_1, x_1), F(y_2, x_2)) \\
\leq q \max \left\{ d(y_1, y_2), \frac{1}{2} d(y_2, F(y_1, x_1)), \right. \\
\left. \frac{1}{2} d(y_1, F(y_2, x_2)), \right. \\
\left. \frac{1}{2} [d(y_1, F(y_1, x_1)) + d(y_2, F(y_2, x_2))] \right\} \\
\leq q \max \left\{ d(y_1, y_2), \frac{1}{2} d(y_2, x_3), \\
\frac{1}{2} [d(y_1, x_2) + d(y_2, x_3)] \right\}, \text{ by (14)}
\]

yielding \( d(x_2, x_3) \leq q \max \{q\tau, q^2\tau\} = q^2\tau \),

\[
d(y_2, y_3) \leq q^2\tau \), \text{ by (15).}
\]

(17)

Inductively, we assume \( d(x_k, x_{k+1}) \leq q^k\tau \) and \( d(y_k, y_{k+1}) \leq q^k\tau \), \( k \geq 1 \). Also, we assume \( d(x_k, x_{k+1}) \leq q^k\tau \) and \( d(y_k, y_{k+1}) \leq q^k\tau \), \( k \geq 1 \). Then

\[
d(y_{k+1}, x_{k+2}) = d(F(x_k, y_k), F(y_{k+1}, x_{k+1})) \\
\leq q \max \left\{ d(x_k, y_{k+1}), \frac{1}{2} d(x_{k+1}, F(x_k, y_k)), \right. \\
\left. \frac{1}{2} d(x_k, F(y_{k+1}, x_{k+1})) \right\} \\
\leq q \max \left\{ d(x_k, y_{k+1}), \frac{1}{2} d(x_k, x_{k+2}), \\
\frac{1}{2} [d(x_k, y_{k+1}) + d(y_{k+1}, x_{k+2})] \right\} \\
\leq q \max \left\{ d(x_k, y_{k+1}), \frac{1}{2} [d(x_k, y_{k+1}) + d(y_{k+1}, x_{k+2})] \right\}
\]

(18)

yielding \( d(y_{k+1}, x_{k+2}) \leq q d(x_k, y_{k+1}) \leq q^{k+1}\tau \),

\[
d(x_{k+1}, y_{k+2}) \leq q d(y_k, x_{k+1}) \leq q^{k+1}\tau.
\]

(19)

(20)
It follows from (19) and (20) that
\[ d(x_{k+1}, x_{k+2}) = d(F(y_k, x_k), F(y_{k+1}, x_{k+1})) \]
\[ \leq q \max \left\{ d(y_k, y_{k+1}), \frac{1}{2} d(y_k, F(x_k, y_k)), \frac{1}{2} d(y_k, F(y_{k+1}, x_{k+1})), \right. \]
\[ \left. + d(y_{k+1}, F(y_{k+1}, x_{k+1})) \right\} \]
\[ \leq q \max \left\{ d(y_k, y_{k+1}), \frac{1}{2} d(y_k, x_{k+2}), \frac{1}{2} d(y_k, x_{k+2}) \right\} \]
\[ \leq q \max \{ d(y_k, y_{k+1}), d(y_k, x_{k+1}), d(y_{k+1}, x_{k+2}) \} \]  \( \text{for all } k \geq 1 \) \( \text{implying that if } p_x = \lim x_n \text{ and } p_y = \lim y_n \text{ then } p_x = p_y = p \text{ and } (p, p) \in \text{Fix}(F). \) To prove that the sequences \( \{x_n\} \) and \( \{y_n\} \) are convergent it suffices to verify that they are Cauchy sequences.

Therefore, using (19)–(23) we conclude that for all \( n \geq 1 \) we have \( d(x_n, x_{n+1}) \leq q^n \tau \) and \( d(x_n, x_{n+1}) \leq q^n \tau \). Also, using the fact that \( d(x_n, y_n) \leq d(x_n, y_{n+1}) + d(y_{n+1}, y_n) \) we conclude that \( d(x_n, y_n) \leq q^n 2 \tau \) for all \( n \geq 1 \) implying that if \( p_x = \lim x_n \) and \( p_y = \lim y_n \) then \( p_x = p_y = p \) and \( (p, p) \in \text{Fix}(F) \).

3. Conclusion

Further studies include proving our main result and Theorem 4 without the restriction \( A \cap B \neq \emptyset \). This can be studied as a best proximity point problem.