Research Article

Inequalities of Convex Functions and Self-Adjoint Operators

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Received 28 November 2013; Accepted 4 January 2014; Published 9 February 2014

Academic Editor: Palle E. Jorgensen

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The paper offers generalizations of the Jensen-Mercer inequality for self-adjoint operators and generally convex functions. The obtained results are applied to define the quasi-arithmetic operator means without using operator convexity. The version of the harmonic-geometric-arithmetic operator mean inequality is derived as an example.

1. Introduction

Throughout the paper we will use a real interval $\mathcal{J}$ with the nonempty interior and real segments $[a, b]$ and (a, b) with $a < b$.

We briefly summarize a development path of the operator form of Jensen's inequality. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let $B(\mathcal{H})$ and $B(\mathcal{K})$ be associated C^* -algebras of bounded linear operators, and let 1_H and 1_K be their identity operators.

Combining the results from [1, 2], it follows that every operator convex function $f : \mathcal{J} \rightarrow \mathbb{R}$ satisfies the Schwarz inequality

$$f(\Phi(A)) \leq \Phi(f(A)), \quad (1)$$

where $\Phi : B(\mathcal{K}) \rightarrow B(\mathcal{H})$ is a positive linear mapping such that $\Phi(1_H) = 1_K$ and $A \in B(\mathcal{K})$ is a self-adjoint operator with the spectrum $\text{Sp}(A) \subseteq \mathcal{J}$. The above inequality was extended in [3] to the inequality

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i(f(A_i)), \quad (2)$$

where $\Phi_i : B(\mathcal{K}) \rightarrow B(\mathcal{H})$ are positive linear mappings such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$ and $A_i \in B(\mathcal{K})$ are self-adjoint operators with spectra $\text{Sp}(A_i) \subseteq \mathcal{J}$. The operator inequality of (2) was formulated for convex (without operator) continuous functions in [4] assuming the spectral conditions: $\text{Sp}(A) \subseteq [a, b]$ and $\text{Sp}(A_i) \cap (a, b) = \emptyset$ for all A_i , where $A = \sum_{i=1}^n \Phi_i(A_i)$.

Including positive operators $P_i \in B(\mathcal{K})$ satisfying $\sum_{i=1}^n \Phi_i(P_i) = 1_K$, we have that every convex continuous function $f : \mathcal{J} \rightarrow \mathbb{R}$ satisfies the inequality

$$f\left(\sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})\right) \leq \sum_{i=1}^n \Phi_i(P_i^{1/2} f(A_i) P_i^{1/2}) \quad (3)$$

if provided the spectral conditions: $\text{Sp}(A) \subseteq [a, b]$ and $\text{Sp}(A_i) \cap (a, b) = \emptyset$ for all self-adjoint operators A_i , and the operator sum $A = \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})$. The inequality in (3) is possible because the operators $P_i^{1/2} A_i P_i^{1/2}$ and $P_i^{1/2} f(A_i) P_i^{1/2}$ are self-adjoint.

2. Discrete and Operator Inequalities for Convex Functions and Trinomial Affine Combinations

2.1. Discrete Variants. Every number $x \in \mathbb{R}$ can be uniquely presented as the binomial affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \quad (4)$$

which is convex if and only if the number x belongs to the interval $[a, b]$. Given the function $f : \mathbb{R} \rightarrow \mathbb{R}$, let $f_{[a,b]}^{\text{line}} : \mathbb{R} \rightarrow \mathbb{R}$ be the function of the chord line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f . Applying the affinity of $f_{[a,b]}^{\text{line}}$ to the combination in (4), we get

$$f_{[a,b]}^{\text{line}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b). \quad (5)$$

If the function f is convex, then we have the inequality

$$f(x) \leq f_{\{a,b\}}^{\text{line}}(x) \quad \text{if } x \in [a, b], \quad (6)$$

and the reverse inequality

$$f(x) \geq f_{\{a,b\}}^{\text{line}}(x) \quad \text{if } x \notin (a, b). \quad (7)$$

Let $\alpha, \beta, \gamma \in \mathbb{R}$ be coefficients such that $\alpha + \beta - \gamma = 1$. Let $a, b, c \in \mathbb{R}$ be points where $a < b$. We consider the affine combination $\alpha a + \beta b - \gamma c$. Inserting the affine combination $c = \lambda a + \mu b$ assuming that $\lambda + \mu = 1$, we get the binomial form

$$\alpha a + \beta b - \gamma c = (\alpha - \gamma\lambda)a + (\beta - \gamma\mu)b. \quad (8)$$

Lemma 1. *Let $\alpha, \beta, \gamma \in [0, 1]$ be coefficients such that $\alpha + \beta - \gamma = 1$. Let $a, b, c \in \mathbb{R}$ be points such that $a < b$ and $c \in [a, b]$.*

Then the affine combination

$$\alpha a + \beta b - \gamma c \in [a, b], \quad (9)$$

and every convex function $f : [a, b] \rightarrow \mathbb{R}$ satisfies the inequality

$$f(\alpha a + \beta b - \gamma c) \leq \alpha f(a) + \beta f(b) - \gamma f(c). \quad (10)$$

Proof. The condition $c = \lambda a + \mu b \in [a, b]$ involves $\lambda, \mu \in [0, 1]$. Then the binomial combination of the right-hand side in (8) is convex since its coefficients $\alpha - \gamma\lambda \geq \alpha - \gamma = 1 - \beta \geq 0$ and also $\beta - \gamma\mu \geq 0$. So, the combination $\alpha a + \beta b - \gamma c$ belongs to $[a, b]$. Applying the inequality in (6) and the affinity of $f_{\{a,b\}}^{\text{line}}$, we get

$$\begin{aligned} f(\alpha a + \beta b - \gamma c) &\leq f_{\{a,b\}}^{\text{line}}(\alpha a + \beta b - \gamma c) \\ &= \alpha f(a) + \beta f(b) - \gamma f_{\{a,b\}}^{\text{line}}(c) \\ &\leq \alpha f(a) + \beta f(b) - \gamma f(c) \end{aligned} \quad (11)$$

because $f_{\{a,b\}}^{\text{line}}(c) \geq f(c)$. $\square$

Lemma 1 is trivially true if $a = b$. It is also valid for $\gamma \in [-1, 1]$ because then the observed affine combinations with $\gamma \leq 0$ become convex, and associated inequalities follow from Jensen's inequality. The similar combinations including $\gamma \in [-1, 1]$ were observed in [5, Corollary 11 and Theorem 12] additionally using a monotone function g . If $\alpha = \beta = \gamma = 1$, then the inequality in (10) is reduced to simple Mercer's variant of Jensen's inequality obtained in [6].

Lemma 2. *Let $\alpha, \beta, \gamma \in [1, \infty)$ be coefficients such that $\alpha + \beta - \gamma = 1$. Let $a, b, c \in \mathbb{R}$ be points such that $a < b$ and $c \notin (a, b)$.*

Then the affine combination

$$\alpha a + \beta b - \gamma c \notin (a, b), \quad (12)$$

and every convex function $f : \mathcal{F} \rightarrow \mathbb{R}$, where $\mathcal{F} = \text{conv}\{a, b, c\}$ satisfies the inequality

$$f(\alpha a + \beta b - \gamma c) \geq \alpha f(a) + \beta f(b) - \gamma f(c). \quad (13)$$

Proof. The condition $c = \lambda a + \mu b \notin (a, b)$ entails $\lambda \leq 0$ or $\lambda \geq 1$, and the coefficients of the binomial form of (8) satisfy $\alpha - \gamma\lambda \geq \alpha \geq 1$ if $\lambda \leq 0$, or $\alpha - \gamma\lambda \leq \alpha - \gamma = 1 - \beta \leq 0$ if $\lambda \geq 1$. So, the combination $\alpha a + \beta b - \gamma c$ does not belong to (a, b) . Applying the inequality in (7), we get the series of inequalities as in (11) but with the reverse inequality signs. $\square$

It is not necessary to require $\gamma \in [1, \infty)$ in Lemma 2, because it follows from the other coefficient conditions.

2.2. Operator Variants. We write $A \leq B$ for self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ if the inner product inequality $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ holds for every vector $x \in \mathcal{H}$. A self-adjoint operator A is positive (nonnegative) if it is greater than or equal to null operator ($A \geq 0$). If $\text{Sp}(A) \subseteq \mathcal{F}$ and $f, g : \mathcal{F} \rightarrow \mathbb{R}$ are continuous functions such that $f(x) \leq g(x)$ for every $x \in \text{Sp}(A)$, then the operator inequality $f(A) \leq g(A)$ is valid. The bounds of a self-adjoint operator A are defined with

$$a_A = \inf_{\|x\|=1} \langle Ax, x \rangle, \quad b_A = \sup_{\|x\|=1} \langle Ax, x \rangle, \quad (14)$$

and its spectrum $\text{Sp}(A)$ is contained in $[a_A, b_A]$ wherein we have the operator inequality

$$a_A 1_H \leq A \leq b_A 1_H. \quad (15)$$

More details on the theory of bounded operators and their inequalities can be found in [7]. The operator versions of Lemmas 1 and 2 follow.

Corollary 3. *Let $\alpha, \beta, \gamma \in [0, 1]$ be coefficients such that $\alpha + \beta - \gamma = 1$. Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator such that $\text{Sp}(A) \subseteq [a, b]$.*

Then

$$\text{Sp}(\alpha a 1_H + \beta b 1_H - \gamma A) \subseteq [a, b], \quad (16)$$

and every convex continuous function $f : [a, b] \rightarrow \mathbb{R}$ satisfies the inequality

$$f(\alpha a 1_H + \beta b 1_H - \gamma A) \leq \alpha f(a) 1_H + \beta f(b) 1_H - \gamma f(A). \quad (17)$$

Proof. The spectral inclusion in (16) follows from the inclusion in (9). Using the affinity of the function $f_{\{a,b\}}^{\text{line}}$ and the operator inequalities $f_{\{a,b\}}^{\text{line}}(\cdot) \geq f(\cdot)$, we can replace the discrete inequalities in (11) with the operator inequalities. $\square$

Corollary 4. *Let $\alpha, \beta, \gamma \in [1, \infty)$ be coefficients such that $\alpha + \beta - \gamma = 1$. Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator such that $\text{Sp}(A) \cap (a, b) = \emptyset$.*

Then

$$\text{Sp}(\alpha a 1_H + \beta b 1_H - \gamma A) \cap (a, b) = \emptyset, \quad (18)$$

and every convex continuous function $f : \mathcal{F} \rightarrow \mathbb{R}$, where $\mathcal{F}$ contains $\text{Sp}(A)$ and $[a, b]$, satisfies the inequality

$$f(\alpha a 1_H + \beta b 1_H - \gamma A) \geq \alpha f(a) 1_H + \beta f(b) 1_H - \gamma f(A). \quad (19)$$

3. Main Results

We want to extend and generalize the inequalities in (17) and (19) including positive operators and positive linear mappings. The main results are Theorems 8 and 9.

Lemma 5. Let $\Phi_i : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ be linear mappings and let $P_i \in B(\mathcal{H})$ be positive linear operators so that $\sum_{i=1}^n \Phi_i(P_i) = 1_K$. Let $A_i \in B(\mathcal{H})$ be self-adjoint operators.

Then every affine function $g(x) = ux + v$, where u and v are real constants, satisfies the operator equality

$$g\left(\sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})\right) = \sum_{i=1}^n \Phi_i(P_i^{1/2} g(A_i) P_i^{1/2}). \quad (20)$$

Proof. Applying the affinity of the function g and the assumption $\sum_{i=1}^n \Phi_i(P_i) = 1_K$, it follows that

$$\begin{aligned} g\left(\sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})\right) &= u \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2}) + v 1_K \\ &= u \sum_{i=1}^n \Phi_i(P_i A_i) + v \sum_{i=1}^n \Phi_i(P_i) \\ &= \sum_{i=1}^n \Phi_i(P_i^{1/2} (u A_i + v 1_H) P_i^{1/2}) \\ &= \sum_{i=1}^n \Phi_i(P_i^{1/2} g(A_i) P_i^{1/2}) \end{aligned} \quad (21)$$

achieving the equality in (20). $\square$

Lemma 6. Let $\Phi_i : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ be positive linear mappings and let $P_i \in B(\mathcal{H})$ be positive linear operators so that $\sum_{i=1}^n \Phi_i(P_i) = 1_K$. Let $A_i \in B(\mathcal{H})$ be self-adjoint operators such that $\text{Sp}(A_i) \subseteq [a, b]$.

Then the spectrum of the operator sum $A = \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})$ is contained in $[a, b]$.

Proof. Applying the positive operators P_i and the positive mappings Φ_i to the assumed spectral inequalities

$$a 1_H \leq A_i \leq b 1_H, \quad (22)$$

we get

$$a \Phi_i(P_i) \leq \Phi_i(P_i^{1/2} A_i P_i^{1/2}) \leq b \Phi_i(P_i). \quad (23)$$

Summing the above inequalities and using the assumption $\sum_{i=1}^n \Phi_i(P_i) = 1_K$, we have

$$a 1_K \leq \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2}) \leq b 1_K \quad (24)$$

which provides that $\text{Sp}(A) \subseteq [a, b]$. $\square$

Corollary 7. Let $\Phi_i : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ be positive linear mappings and let $P_i \in B(\mathcal{H})$ be positive linear operators so

that $\sum_{i=1}^n \Phi_i(P_i) = 1_K$. Let $A_i \in B(\mathcal{H})$ be self-adjoint operators such that $\text{Sp}(A_i) \subseteq [a, b]$.

Then every convex continuous function $f : [a, b] \rightarrow \mathbb{R}$ satisfies the inequality

$$\begin{aligned} &\max \left\{ f\left(\sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})\right), \sum_{i=1}^n \Phi_i(P_i^{1/2} f(A_i) P_i^{1/2}) \right\} \\ &\leq f_{[a,b]}^{\text{line}}\left(\sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})\right) \\ &\leq \max \{f(a) 1_K, f(b) 1_K\}. \end{aligned} \quad (25)$$

Proof. The inequality in (25) is the consequence of Lemmas 5 and 6, and the discrete inequality

$$\begin{aligned} &\max \left\{ f\left(\sum_{i=1}^n p_i x_i\right), \sum_{i=1}^n p_i f(x_i) \right\} \\ &\leq f_{[a,b]}^{\text{line}}\left(\sum_{i=1}^n p_i x_i\right) \\ &\leq \max \{f(a), f(b)\}, \end{aligned} \quad (26)$$

where $x_i \in [a, b]$ are points and $p_i \in [0, 1]$ are coefficients of the sum equal to 1. $\square$

Theorem 8. Let $\alpha, \beta, \gamma \in [0, 1]$ be coefficients such that $\alpha + \beta - \gamma = 1$. Let $\Phi_i : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ be positive linear mappings and let $P_i \in B(\mathcal{H})$ be positive linear operators so that $\sum_{i=1}^n \Phi_i(P_i) = 1_K$. Let $A_i \in B(\mathcal{H})$ be self-adjoint operators such that $\text{Sp}(A_i) \subseteq [a, b]$.

Then the spectrum of the operator

$$\bar{A} = \alpha a 1_K + \beta b 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2}) \quad (27)$$

is contained in $[a, b]$, and every convex continuous function $f : [a, b] \rightarrow \mathbb{R}$ satisfies the inequality

$$f(\bar{A}) \leq \alpha f(a) 1_K + \beta f(b) 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} f(A_i) P_i^{1/2}). \quad (28)$$

If the function f is concave, then the reverse inequality is valid in (28).

Proof. Taking the operator sum $A = \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})$, the spectral inclusion $\text{Sp}(\bar{A}) \subseteq [a, b]$ follows from Lemma 6 and

the inclusion in (16). Assuming and applying the convexity of f and the affinity of $f_{\{a,b\}}^{\text{line}}$ according to Lemma 5, we get

$$\begin{aligned}
 f(\bar{A}) &\leq f_{\{a,b\}}^{\text{line}} \left(\alpha a 1_K + \beta b 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2}) \right) \\
 &= \alpha f(a) 1_K + \beta f(b) 1_K \\
 &\quad - \gamma f_{\{a,b\}}^{\text{line}} \left(\sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2}) \right) \\
 &= \alpha f(a) 1_K + \beta f(b) 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} f_{\{a,b\}}^{\text{line}}(A_i) P_i^{1/2}) \\
 &\leq \alpha f(a) 1_K + \beta f(b) 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} f(A_i) P_i^{1/2})
 \end{aligned} \tag{29}$$

because $f_{\{a,b\}}^{\text{line}}(A_i) \geq f(A_i)$. $\square$

The version of Theorem 8 for $\alpha = \beta = \gamma = 1$ and all $P_i = 1_H$ was obtained in [8] as the main result.

Theorem 9. Let $\alpha, \beta, \gamma \in [1, \infty)$ be coefficients such that $\alpha + \beta - \gamma = 1$. Let $\Phi_i : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ be positive linear mappings and let $P_i \in B(\mathcal{H})$ be positive linear operators so that $\sum_{i=1}^n \Phi_i(P_i) = 1_K$. Let $A_i \in B(\mathcal{H})$ be self-adjoint operators such that $\text{Sp}(A_i) \cap (a, b) = \emptyset$, and let $A = \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2})$ be the operator sum such that $\text{Sp}(A) \cap (a, b) = \emptyset$.

Then the spectrum of the operator

$$\bar{A} = \alpha a 1_K + \beta b 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2}) \tag{30}$$

satisfies the relation $\text{Sp}(\bar{A}) \cap (a, b) = \emptyset$, and every convex continuous function $f : \mathcal{J} \rightarrow \mathbb{R}$, where $\mathcal{J}$ contains all spectra and $[a, b]$, satisfies the inequality

$$\begin{aligned}
 f(\bar{A}) &\geq \alpha f(a) 1_K + \beta f(b) 1_K \\
 &\quad - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} f(A_i) P_i^{1/2}).
 \end{aligned} \tag{31}$$

If the function f is concave, then the reverse inequality is valid in (31).

Proof. The relation $\text{Sp}(\bar{A}) \cap (a, b) = \emptyset$ is the consequence of the relation in (18). Assuming and using the convexity of f and the affinity of $f_{\{a,b\}}^{\text{line}}$, as well as the inequalities $f_{\{a,b\}}^{\text{line}}(A_i) \leq f(A_i)$, we get the series of inequalities as in (29) but with the reverse inequality signs. $\square$

4. Application to Quasi-Arithmetic Means

In applications of convexity to quasi-arithmetic means, we use strictly monotone continuous functions $\varphi, \psi : \mathcal{J} \rightarrow \mathbb{R}$ such that the function $\psi \circ \varphi^{-1}$ is convex, in which case we say

that ψ is φ -convex. A similar notation is used for concavity. This terminology is taken from [9, Definition 1.19].

A continuous function $f : \mathcal{J} \rightarrow \mathbb{R}$ is said to be operator increasing on $\mathcal{J}$ if $A \leq B$ implies $f(A) \leq f(B)$ for every pair of self-adjoint operators $A, B \in B(\mathcal{H})$ with spectra in $\mathcal{J}$. A function f is said to be operator decreasing if the function $-f$ is operator increasing.

Take an operator affine combination

$$\bar{A} = \alpha a 1_K + \beta b 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} A_i P_i^{1/2}) \tag{32}$$

as in Theorem 8. If $\varphi : [a, b] \rightarrow \mathbb{R}$ is a strictly monotone continuous function, we define the φ -quasi-arithmetic mean of the combination $\bar{A}$ as the operator

$$\begin{aligned}
 M_\varphi(\bar{A}) &= \varphi^{-1} \left(\alpha \varphi(a) 1_K + \beta \varphi(b) 1_K \right. \\
 &\quad \left. - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} \varphi(A_i) P_i^{1/2}) \right).
 \end{aligned} \tag{33}$$

The spectrum of the operator $M_\varphi(\bar{A})$ is contained in $[a, b]$ because the spectrum of the operator

$$\bar{A}_\varphi = \alpha \varphi(a) 1_K + \beta \varphi(b) 1_K - \gamma \sum_{i=1}^n \Phi_i(P_i^{1/2} \varphi(A_i) P_i^{1/2}) \tag{34}$$

is contained in $\varphi([a, b])$. The quasi-arithmetic means defined in (33) are invariant with respect to the affinity; that is, the equality

$$M_{u\varphi+v}(\bar{A}) = M_\varphi(\bar{A}) \tag{35}$$

holds for all pairs of real numbers $u \neq 0$ and v . Indeed, if $\psi(x) = u\varphi(x) + v$, then

$$\begin{aligned}
 \bar{A}_\psi &= u \bar{A}_\varphi + v 1_K, \\
 \psi^{-1}(x) &= \varphi^{-1} \left(\frac{1}{u} (x - v) \right),
 \end{aligned} \tag{36}$$

and therefore, it follows that

$$\begin{aligned}
 M_\psi(\bar{A}) &= \psi^{-1}(\bar{A}_\psi) = \varphi^{-1} \left(\frac{1}{u} (\bar{A}_\varphi - v 1_K) \right) \\
 &= \varphi^{-1}(\bar{A}_\varphi) = M_\varphi(\bar{A}).
 \end{aligned} \tag{37}$$

The order of the pair of quasi-arithmetic means M_φ and M_ψ depends on convexity of the function $\psi \circ \varphi^{-1}$ and monotonicity of the function ψ . Theorem 8 can be applied to operator means as follows.

Corollary 10. Let $\bar{A}$ be an affine combination as in (32) satisfying the assumptions of Theorem 8. Let $\varphi, \psi : [a, b] \rightarrow \mathbb{R}$ be strictly monotone continuous functions.

If ψ is either φ -convex with operator increasing ψ^{-1} or φ -concave with operator decreasing ψ^{-1} , then one has the inequality

$$M_{\varphi}(\bar{A}) \leq M_{\psi}(\bar{A}). \quad (38)$$

If ψ is either φ -convex with operator decreasing ψ^{-1} or φ -concave with operator increasing ψ^{-1} , then one has the reverse inequality in (38).

Proof. Let us prove the case in which ψ is φ -convex with operator increasing ψ^{-1} . Put $[c, d] = \varphi([a, b])$. Applying the inequality in (28) of Theorem 8 to the affine combination $\bar{A}_{\varphi}$ of (34) with $\text{Sp}(\bar{A}_{\varphi}) \subseteq [c, d]$ and the convex function $f = \psi \circ \varphi^{-1} : [c, d] \rightarrow \mathbb{R}$, we get

$$\psi \circ \varphi^{-1}(\bar{A}_{\varphi}) \leq \bar{A}_{\psi}. \quad (39)$$

Assigning the increasing function ψ^{-1} to the above inequality, we attain

$$M_{\varphi}(\bar{A}) = \varphi^{-1}(\bar{A}_{\varphi}) \leq \psi^{-1}(\bar{A}_{\psi}) = M_{\psi}(\bar{A}) \quad (40)$$

which finishes the proof. $\square$

Using Corollary 10 we get the following version of the harmonic-geometric-arithmetic mean inequality for operators.

Corollary 11. *If $\bar{A}$ is an affine operator combination as in (32) satisfying the assumptions of Theorem 8 with the addition that $[a, b] \subset (0, \infty)$, then one has the harmonic-geometric-arithmetic operator inequality*

$$\begin{aligned} & \left(\left(\frac{\alpha}{a} + \frac{\beta}{b} \right) 1_K - \gamma \sum_{i=1}^n \Phi_i \left(P_i^{1/2} A_i^{-1} P_i^{1/2} \right) \right)^{-1} \\ & \leq \ln a^{\alpha} b^{\beta} 1_K - \gamma \exp \left(\sum_{i=1}^n \Phi_i \left(P_i^{1/2} (\ln A_i) P_i^{1/2} \right) \right) \\ & \leq (\alpha a + \beta b) 1_K - \gamma \sum_{i=1}^n \Phi_i \left(P_i^{1/2} A_i P_i^{1/2} \right). \end{aligned} \quad (41)$$

Proof. To prove the left-hand side of the inequality in (41) we use the functions $\varphi(x) = \ln x$ and $\psi(x) = x^{-1}$. Then $\psi \circ \varphi^{-1}(x) = \exp(-x)$ and $\psi^{-1}(x) = x^{-1}$, so ψ is φ -convex and $\psi^{-1}(x) = x^{-1}$ is operator decreasing. Applying Corollary 10 to this case, we have

$$M_{\ln x}(\bar{A}) \geq M_{x^{-1}}(\bar{A}). \quad (42)$$

To prove the right-hand side we use the functions $\varphi(x) = \ln x$ and $\psi(x) = x$. Then $\psi \circ \varphi^{-1}(x) = \exp x$ and $\psi^{-1}(x) = x$, so ψ is φ -convex and $\psi^{-1}(x) = x$ is operator increasing. Applying the inequality in (38), we get

$$M_{\ln x}(\bar{A}) \leq M_x(\bar{A}). \quad (43)$$

The double inequality in (41) follows by connecting the inequalities in (42) and (43). $\square$

Quasi-arithmetic operator means without applying operator convexity were also investigated in [4, 10].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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