Research Article

Dynamic Cournot Duopoly Game with Delay

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The delay Cournot duopoly game is studied. Dynamical behaviors of the game are studied. Equilibrium points and their stability are studied. The results show that the delayed system has the same Nash equilibrium point and the delay can increase the local stability region.

Dedicated to Professor H. N. Agiza on the occasion of his 60th birthday

1. Introduction

It is well known that the duopoly game is one of the fundamental oligopoly games [1, 2]. Even the duopoly situation is an oligopoly of two producers that can be more complex than one might imagine since the duopolists have to take into account their actions and reactions when decisions are made [3]. Oligopoly theory is one of the oldest branches of mathematical economics dating back to 1838 when its basic model was proposed by Bischi et al. [4]. In repeated duopoly game all players maximize their profits. Recently, the dynamics of duopoly game has been studied [5–11]. Bischi and Naimzada [8] gave the general formula of duopoly game with a form of bounded rationality. Agiza et al. [10] examined the dynamical behavior of Bowley's model with bounded rationality. They also have studied the complex dynamics of bounded rationality duopoly game with nonlinear demand function [11]. In general, a player, in order to adjust his output, can choose his strategy rule among many available techniques. Naïve, adaptive, and boundedly rational strategies are only a few examples. When literature deals with duopoly games, most papers focus on games with homogeneous players. Another branch of literature is interested in games with heterogeneous players. In this type of literatures, the

assumption of players adopting heterogeneous rules to decide their production is, in our opinion, more realistic than the opposite case; see [12]. Agiza and Elsadany [13, 14] were of the first authors who studied games with heterogeneous players, and in particular they analyze the dynamic behaviors emerging in this kind of games. Recently, Zhang et al. [15] and Dubiel-Teleszynski [16] used the same technique to analyze a duopoly game with heterogeneous players and nonlinear cost function. In addition, Angelini et al. [17] and Tramontana [18] studied a duopoly game with heterogeneous players with isoelastic demand function. Other studies on the dynamics of oligopoly models with more firms and other modifications have been studied [19–21]. Also in the past decade, there has been a great deal of interest in chaos control of duopoly games because of its complexity [22–25]. Recently Askar [26] has shown complex dynamics such as bifurcation and chaos, in a Cournot duopoly game with log-concave demand function. Zhao and Zhang [27] studied a duopoly game with heterogeneous players participating in carbon emission trading and analyzed the asymptotic stability of the equilibrium points of the game. In these literatures, adjustment speed or other parameters are taken as bifurcation parameters, and complex results such as period doubling bifurcation, unstable period orbits, and chaos are found.

The delayed discrete monopoly game is considered by Ahmed et al. [28]. Yassen and Agiza [29] and Hassan [30] showed that the stability region of the Nash equilibrium can become larger in a delayed duopoly game. Even so, a more realistic case of bounded rationality game with delay was introduced and discussed in [31]. Yali [32] reconsidered the duopoly game in Elsadany [31] for the case of increasing marginal costs instead of constant marginal costs. Recently, continuous dynamic monopoly with a single or two fixed delays is investigated in [33]. The cases of continuously distributed delays are studied in [34]. Time delay on discrete monopoly models is also considered in [35, 36].

In this work, the study is based on the duopoly market with delay, to investigate situations that one player considers delayed decisions and another player makes without delayed decisions. According to the analysis of numerical simulations, the parameters effects on the stability of this system are obtained. We formulate a duopoly game with heterogeneous bounded rationality: one is boundedly rational player without delay and the other is delayed boundedly rational player and studying its dynamical behaviors. This idea combines two realistic ideas; in our opinion, the first is heterogeneous players and the second is bounded rationality method with delay.

This paper is organized as follows. In Section 2, heterogeneous bounded rationality is briefly described. In Section 3, we analyze the dynamics for simple case of duopoly game with heterogeneous bounded rationality. Explicit parametric conditions of the existence and local stability of equilibrium points will be given. In Section 4, we present the numerical simulations, to verify our results which take place by the theoretical analysis. Finally, some remarks are confined in Section 5.

2. Heterogeneous Bounded Rationality

We consider a Cournot duopoly game where q_i denotes the quantity supplied by firm i , $i = 1, 2$. In addition let $P(q_i + q_j)$, $i \neq j$; denote by a twice differentiable and nonincreasing inverse demand function and let $C_i(q_i)$ denote by the twice differentiable increasing cost function. For firm i the profit resulting from the above Cournot game is given by

$$\Pi_i(t) = P(q_i(t) + q_j^e(t+1))q_i(t) - C_i(q_i(t)). \quad (1)$$

Maximizing profit function of player i , taking quantity supplied the opponent j , $i \neq j$, as given, results in the well-known reaction function for firm i ,

$$\begin{aligned} q_i(t+1) &= r_i(q_j^e(t+1)) \\ &= \arg \max_{q_i} [P(q_i(t) + q_j^e(t+1))q_i(t) - C_i(q_i(t))]. \end{aligned} \quad (2)$$

Cournot assumed that the expected quantity in the next time step $q_j^e(t+1)$ is given by

$$q_j^e(t+1) = q_j(t). \quad (3)$$

Then the Cournot duopoly game defined as a discrete dynamical system has the form

$$q_i(t+1) = R_i(q_j(t)), \quad i, j = 1, 2, i \neq j. \quad (4)$$

However, as pointed out by Bischi and Naimzada [8] there is an unrealistic assumption in this approach. It is implicitly assumed that the duopolist knows the market demand function. A more realistic approach is to assume bounded rationality, that is, each firm (say i th one) modifies its production according to its marginal profit $\partial \Pi_i(q_i(t), q_j^e(t+1))/\partial q_i$, $i = 1, 2, i \neq j$. This will help each player to increase or decrease the quantity produced at time $t+1$ depending on whether its own marginal profit at time t is positive or negative. Hence the dynamical system of duopoly game with bounded rationality is

$$\begin{aligned} q_i(t+1) &= q_i(t) + \alpha_i(q_i) \frac{\partial \Pi_i(q_i(t), q_j^e(t+1))}{\partial q_i}, \\ i &= 1, 2, i \neq j, \end{aligned} \quad (5)$$

where $\alpha_i(q_i)$ is the adjustment of the i th firm $i = 1, 2$. They assumed that $q_j^e(t+1) = q_j(t)$ in the term of bounded rationality and also assumed $\alpha_i(q_i) = \alpha_i q_i$. Then Bischi-Naimzada bounded rationality duopoly game has the form

$$\begin{aligned} q_i(t+1) &= q_i(t) + \alpha_i q_i \frac{\partial \Pi_i(q_i(t), q_j(t))}{\partial q_i}, \\ i &= 1, 2, i \neq j. \end{aligned} \quad (6)$$

This means that if the marginal profit is positive/negative the i th player increases/decreases its production q_i in the next period output.

Also they assumed that the expected product of a firm $q^e(t+1)$ is equal to its previous quantity $q(t)$ in bounded rationality term. However it may make more sense to use previous productions, that is, $q(t-1), q(t-2), \dots, q(t-T)$ with different weights. Ahmed et al. [28] and Agiza et al. [10] have examined this point in monopoly case only. In [29, 30], they assumed that delay was put in the full term of bounded rationality for all players in the game. In the duopoly game discussed in [31], each producer considers time delay only for opponents output and averages every opponent's previous outputs. We suggested putting delay in the term of bounded rationality for all players excepting the i th player. Here both realistic ideas of bounded rationality and delay are combined. The dynamical system will be

$$q_i(t+1) = q_i(t) + \alpha_i q_i \frac{\partial \Pi_i(q_i, q^D)}{\partial q_i}, \quad i = 1, 2, \quad (7)$$

where $q^D = q_j^e(t+1) = \sum_{l=0}^T q_j(t-l)\omega_l$ and $\omega_l \geq 0$, $\sum_{l=0}^T \omega_l = 1$. The factors ω_l , $l = 0, 1, 2, \dots, T$ are the weights given to previous productions.

From (7), it is clear that the delay was put in the bounded rationality term for all players except the player i . This

argument is the basic difference between our paper and the other papers [29, 30].

In this work, we consider that each player has different bounded rationality method in duopoly game. We assume player 1 is a boundedly rational player without delay and player 2 is a boundedly rational player with delay. With above assumptions, we can express the process of duopoly game with heterogeneous bounded rationality defined as

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha_1 q_1 \frac{\partial \Pi_1(q_1(t), q_2(t))}{\partial q_1}, \\ q_2(t+1) &= q_2(t) + \alpha_2 q_2 \frac{\partial \Pi_2(q_1^D(t), q_2(t))}{\partial q_2}. \end{aligned} \quad (8)$$

3. Duopoly Game with Heterogeneous Bounded Rationality

For simplicity we take $T = 1$, and we consider the duopoly game with heterogeneous bounded rationality. The profit of i th firm is given by

$$\Pi_i = q_i (a - b(q_1 + q_2)) - c_i q_i, \quad i = 1, 2. \quad (9)$$

Under above assumption, the delayed duopoly game with heterogeneous bounded rationality (7) is given by

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha_1 q_1 \{a - c_1 - 2bq_1(t) - bq_2(t)\}, \\ q_2(t+1) &= q_2(t) + \alpha_2 q_2 \{a - c_2 - 2bq_2(t) - b \\ &\quad \times [\omega q_1(t) + (1 - \omega) q_1(t-1)]\}. \end{aligned} \quad (10)$$

To study the stability of dynamical system (10), rewrite it as a three-dimensional system in the form

$$\begin{aligned} p(t+1) &= q_1(t), \\ q_1(t+1) &= q_1(t) + \alpha_1 q_1(t) (a - c_1 - 2bq_1(t) - bq_2(t)), \\ q_2(t+1) &= q_2(t) + \alpha_2 q_2(t) (a - c_2 - 2bq_2(t) - b \\ &\quad \times [\omega q_1(t) + (1 - \omega) p(t)]). \end{aligned} \quad (11)$$

Game (11) is a three-dimensional map which depends on seven parameters. We are concerned with the qualitative changes of the asymptotic and/or long-run dynamics of the iterated map (11). We will study the asymptotic behavior of the dynamical model using the localization of fixed points of the dynamical system and the determination of the parameters sets for given local stable fixed points.

3.1. Equilibrium Points and Local Stability. It is clear that the system (11) has four fixed points in the following form:

$$\begin{aligned} E_0 &= (0, 0, 0), \\ E_1 &= \left(\frac{a - c_1}{2b}, \frac{a - c_1}{2b}, 0 \right), \quad \text{if } c_1 < a, \\ E_2 &= \left(0, 0, \frac{a - c_2}{2b} \right), \quad \text{if } c_2 < a, \\ E_* &= (q_1^*, q_1^*, q_2^*) \end{aligned} \quad (12)$$

such that

$$q_1^* = \frac{a + c_2 - 2c_1}{3b}, \quad q_2^* = \frac{a + c_1 - 2c_2}{3b}. \quad (13)$$

Obviously, E_0 , E_1 , E_2 are boundary equilibrium points. The fixed point E_* is a Nash equilibrium point and has economic meaning when

$$\begin{aligned} 2c_1 - c_2 &< a, \\ 2c_2 - c_1 &< a. \end{aligned} \quad (14)$$

To investigate the local stability of the equilibrium points E_0 , E_1 , E_2 and E_* we have found the Jacobian matrix of the system equations (11). The Jacobian matrix for the model system (11) at any point $J(p, q_1, q_2)$ takes the form

$$J(p, q_1, q_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & A & -\alpha_1 b q_1 \\ -\alpha_2 b (1 - \omega) q_2 & -\alpha_2 b \omega q_2 & B \end{bmatrix}, \quad (15)$$

where

$$\begin{aligned} A &= 1 + \alpha_1 (a - c_1 - 4bq_1 - bq_2), \\ B &= 1 + \alpha_2 (a - c_2 - 4bq_2 - b(\omega q_1 + (1 - \omega) p)). \end{aligned} \quad (16)$$

The stability of equilibrium points will be determined by the nature of the eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points.

Theorem 1. *The equilibrium point E_0 of system (11) is unstable equilibrium point.*

Proof. At the equilibrium point $E_0(0, 0, 0)$ the Jacobian matrix given by (15) takes the form

$$J(E_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 + \alpha_1(a - c_1) & 0 \\ 0 & 0 & 1 + \alpha_2(a - c_2) \end{bmatrix}. \quad (17)$$

The eigenvalues of $J(E_0)$ are given by $\lambda_1 = 0$, $\lambda_2 = 1 + \alpha_1(a - c_1)$ and $\lambda_3 = 1 + \alpha_2(a - c_2)$. From the conditions that a, α_i, c_i ($i = 1, 2$) are positive parameters and $c_i < a$, $i = 1, 2$. We have that $|\lambda_{2,3}| > 1$. Hence the equilibrium point E_0 is unstable. $\square$

Theorem 2. *If the Nash equilibrium point E_* is strictly positive, then the equilibrium points E_1, E_2 of system (11) are saddle points.*

Proof. At the boundary equilibrium point $E_1((a - c_1)/2b, (a - c_1)/2b, 0)$, the Jacobian matrix (15) takes the form of

$$J(E_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 - \alpha_1(a - c_1) & -\frac{\alpha_1(a - c_1)}{2} \\ 0 & 0 & 1 + \frac{\alpha(a - 2c_2 + c_1)}{2} \end{bmatrix} \quad (18)$$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1 + (\alpha_2/2)(a - 2c_2 + c_1)$ and $\lambda_3 = 1 - \alpha_1(a - c_1)$. If the Nash equilibrium point has positive coordinates, then $2c_2 - c_1 < a$; hence, $|\lambda_2| > 1$. Then E_1 is a saddle point. Similarly we can prove that E_2 is also saddle point. $\square$

Now we investigate the local stability of Nash equilibrium point E_* . The Jacobian matrix (15) at E_* is

$$J(E_*) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 - 2\alpha_1 b q_1^* & -\alpha_1 b q_1^* \\ -\alpha_2 b(1 - \omega) q_2^* & -\alpha_2 b \omega q_2^* & 1 - 2\alpha_2 b q_2^* \end{bmatrix}. \quad (19)$$

The Nash equilibrium point is given in (12) and stability conditions are that all roots of the equation $P(\lambda) = 0$ satisfy $|\lambda| < 1$, where

$$P(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 \quad (20)$$

such that

$$\begin{aligned} a_1 &= 2b\alpha_1 q_1^* + 2b\alpha_2 q_2^* - 2, \\ a_2 &= 1 - 2\alpha_1 b q_1^* - 2\alpha_2 b q_2^* \\ &\quad + 4\alpha_1 \alpha_2 b^2 q_1^* q_2^* - \alpha_1 \alpha_2 b^2 \omega q_1^* q_2^*, \\ a_3 &= \alpha_1 \alpha_2 b^2 q_1^* q_2^* (\omega - 1). \end{aligned} \quad (21)$$

A necessary and sufficient condition for (19) to have only roots of absolute value less than one is the following (see [37]):

$$\begin{aligned} 1 + a_1 + a_2 + a_3 &> 0, \\ 1 - a_1 + a_2 - a_3 &> 0, \\ |a_2 - a_1 a_3| &< 1 - a_3^2, \\ |a_3| &< 1. \end{aligned} \quad (22)$$

According to Jury criteria, the Nash equilibrium point is locally asymptotically stable if conditions (22) are satisfied. When ω is sufficiently small and for $0 < b < 1$ the Nash equilibrium point is stable; see Figures 8 and 9. Hence, we deduced from above analysis that delay has stabilization effect for Nash equilibrium point.

4. Numerical Simulations

To provide some numerical evidence for the dynamical behavior of model (11), we present various numerical results

FIGURE 1: Bifurcation diagram of game (11) with respect to α_1 when $\omega = 1$.

FIGURE 2: Bifurcation diagram of game (11) with respect to α_1 when $\omega = 0.8$.

here to show that the delay has effect to increase stability domain. In order to study local stability properties of the equilibrium points it is convenient to take the parameters values as follows: $a = 11$, $b = 0.5$, $c_1 = 3$, $c_2 = 5$.

To show the influence of delay weight ω on the dynamical behaviors of the delayed Cournot duopoly game, four cases of ω are considered. Figure 1 shows that the bifurcation diagram for the nondelay case $\omega = 1$: q_i ($i = 1, 2$) converges to the Nash equilibrium point as $\alpha_1 < 0.265$ approximately; when α_1 increases, the equilibrium point becomes unstable, period doubling bifurcations appear, and finally chaotic behaviors occur. Figure 2 (for $\omega = 0.8$) shows that the game keeps stable for α_1 taking its value up to nearly 0.29.

Also, Figure 3 (for $\omega = 0.6$) shows that the game keeps stable for α_1 taking its value up to nearly 0.32. Comparing these three diagrams, we can see that instability for a case of proper delay (Figures 2 and 3) occurs later than that for the nondelay case (Figure 1). The bifurcation diagram is illustrated in Figure 4, which shows that the stability loss is evidently due to a Neimark-Sacker bifurcation. The corresponding two-dimensional phase portraits for various values of ω are shown in Figures 5, 6, and 7.

FIGURE 3: Bifurcation diagram of game (11) with respect to α_1 when $\omega = 0.6$.

FIGURE 4: Bifurcation diagram of game (11) with respect to α_1 when $\omega = 0.5$.

FIGURE 5: Phase portrait for game (11) when $\omega = 1$.

FIGURE 6: Phase portrait for game (11) when $\omega = 0.8$.

FIGURE 7: Phase portrait for game (11) when $\omega = 0.5$.

Figure 8 shows the bifurcation diagram of q_1 with respect to the delay parameter ω , while other parameters being fixed ($a = 11$, $b = 0.5$, $c_1 = 3$, $c_2 = 5$, $\alpha_1 = 0.35$ and $\alpha_2 = 0.3$). It also shows that the bifurcation diagram of q_1 convergent to Nash equilibrium point as ω is small. It is clear that the Nash equilibrium is convergence as ω is small. This means that delay parameter ω has a stabilization effect for duopoly game.

A bifurcation diagram of q_2 with respect to ω , while other parameters are fixed ($a = 11$, $b = 0.5$, $c_1 = 3$, $c_2 = 5$, $\alpha_1 = 0.35$ and $\alpha_2 = 0.3$) is shown in Figure 9. Also, from the bifurcation diagram of q_2 the Nash equilibrium point is convergent as ω is small.

From Figures 8 and 9, every one can deduce that the delay has effect to convergent of the Nash equilibrium. So, the delay has a stabilization effect for the Nash equilibrium point.

Using the stability conditions (22) of the Nash equilibrium point we can draw the region of stability in the parameter plane (α_1, α_2) . Figure 10 shows the stability region of the Nash equilibrium point in nondelay case (i.e. $\omega = 1$). Also Figure 11 shows the stability region of the Nash

FIGURE 8: Bifurcation diagram of q_1 with respect to ω .

FIGURE 9: Bifurcation diagram of q_2 with respect to ω .

FIGURE 10: Stability region of the Nash equilibrium point without delay.

FIGURE 11: Stability region of the Nash equilibrium point with delay when $\omega = 0.1$.

equilibrium point in delay case when $\omega = 0.1$. Comparison between Figures 10 and 11, one can see that the stability region in delay case is larger than those in the nondelay case.

So, we can see that an intermediate delay weight can expand the stability region. Hence, we say that a proper delay has a stabilization effect on the delayed Cournot duopoly game.

5. Conclusion

In this paper, we established delayed Cournot duopoly game with heterogeneous bounded rationality. The stability conditions of the equilibrium points of this game are investigated. Basic properties of the game have been analyzed by means of bifurcation diagrams and stability regions. This paper shows that increasingly large delay speed act as an instability factor, which causes period doubling bifurcation and chaotic behavior. This means that the delay has a stabilization effect on the game, when it is sufficiently small and for $0 < b < 1$. We concluded that the delay weight can delay the occurrence of complex behaviors such as bifurcation and chaos and can expand the stability region for the system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] G. I. Bischi, C. Chiarella, M. Kopel, and F. Szidarovszky, *Non-linear Oligopolies: Stability and Bifurcations*, Springer, New York, NY, USA, 2009.
- [2] A. Cournot, *Research into the Principles of the Theory of Wealth*, Irwin Paper Back Classics in Economics, Chapter 7, 1963.
- [3] T. Puu, "Chaos in duopoly pricing," *Chaos, Solitons and Fractals*, vol. 1, no. 6, pp. 573–581, 1991.
- [4] G.-I. Bischi, L. Stefanini, and L. Gardini, "Synchronization, intermittency and critical curves in a duopoly game," *Mathematics and Computers in Simulation*, vol. 44, no. 6, pp. 559–585, 1998.
- [5] T. Puu, "The chaotic duopolists revisited," *Journal of Economic Behavior and Organization*, vol. 33, no. 3–4, pp. 385–394, 1998.
- [6] E. Ahmed and H. N. Agiza, "Dynamics of a cournot game with n-competitors," *Chaos, Solitons and Fractals*, vol. 9, no. 9, pp. 1513–1517, 1998.
- [7] H. N. Agiza, "On the analysis of stability, bifurcation, chaos and chaos control of Kopel map," *Chaos, Solitons and Fractals*, vol. 10, no. 11, pp. 1909–1916, 1999.
- [8] G. I. Bischi and A. Naimzada, "Global analysis of a dynamic duopoly game with bounded rationality," in *Advanced in Dynamic Games and Application*, vol. 5, Chapter 20, pp. 361–385, Birkhäuser, Boston, Mass, USA, 2000.
- [9] G. I. Bischi and M. Kopel, "Equilibrium selection in a nonlinear duopoly game with adaptive expectations," *Journal of Economic Behavior and Organization*, vol. 46, no. 1, pp. 73–100, 2001.
- [10] H. N. Agiza, A. S. Hegazi, and A. A. Elsadany, "The dynamics of Bowley's model with bounded rationality," *Chaos, Solitons and Fractals*, vol. 12, no. 9, pp. 1705–1717, 2001.
- [11] H. N. Agiza, A. S. Hegazi, and A. A. Elsadany, "Complex dynamics and synchronization of a duopoly game with bounded rationality," *Mathematics and Computers in Simulation*, vol. 58, no. 2, pp. 133–146, 2002.
- [12] F. Tramontana and A. E. A. Elsadany, "Heterogeneous triopoly game with isoelastic demand function," *Nonlinear Dynamics*, vol. 68, no. 1–2, pp. 187–193, 2012.
- [13] H. N. Agiza and A. A. Elsadany, "Nonlinear dynamics in the cournot duopoly game with heterogeneous players," *Physica A: Statistical Mechanics and Its Applications*, vol. 320, pp. 512–524, 2003.
- [14] H. N. Agiza and A. A. Elsadany, "Chaotic dynamics in nonlinear duopoly game with heterogeneous players," *Applied Mathematics and Computation*, vol. 149, no. 3, pp. 843–860, 2004.
- [15] J. Zhang, Q. Da, and Y. Wang, "Analysis of nonlinear duopoly game with heterogeneous players," *Economic Modelling*, vol. 24, no. 1, pp. 138–148, 2007.
- [16] T. Dubiel-Teleszynski, "Nonlinear dynamics in a heterogeneous duopoly game with adjusting players and diseconomies of scale," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 296–308, 2011.
- [17] N. Angelini, R. Dieci, and F. Nardini, "Bifurcation analysis of a dynamic duopoly model with heterogeneous costs and behavioural rules," *Mathematics and Computers in Simulation*, vol. 79, no. 10, pp. 3179–3196, 2009.
- [18] F. Tramontana, "Heterogeneous duopoly with isoelastic demand function," *Economic Modelling*, vol. 27, no. 1, pp. 350–357, 2010.
- [19] H.-X. Yao and F. Xu, "Complex dynamics analysis for a duopoly advertising model with nonlinear cost," *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 134–145, 2006.
- [20] J.-G. Du and T. Huang, "New results on stable region of Nash equilibrium of output game model," *Applied Mathematics and Computation*, vol. 192, no. 1, pp. 12–19, 2007.
- [21] A. K. Naimzada and L. Sbragia, "Oligopoly games with non-linear demand and cost functions: two boundedly rational adjustment processes," *Chaos, Solitons and Fractals*, vol. 29, no. 3, pp. 707–722, 2006.
- [22] L. Chen and G. Chen, "Controlling chaos in an economic model," *Physica A: Statistical Mechanics and Its Applications*, vol. 374, no. 1, pp. 349–358, 2007.
- [23] J.-G. Du, T. Huang, and Z. Sheng, "Analysis of decision-making in economic chaos control," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 4, pp. 2493–2501, 2009.
- [24] Z.-H. Sheng, T. Huang, J.-G. Du, Q. Mei, and H. Huang, "Study on self-adaptive proportional control method for a class of output models," *Discrete and Continuous Dynamical Systems B*, vol. 11, no. 2, pp. 459–477, 2009.
- [25] J. Ma and W. Ji, "Chaos control on the repeated game model in electric power duopoly," *International Journal of Computer Mathematics*, vol. 85, no. 6, pp. 961–967, 2008.
- [26] S. S. Askar, "Complex dynamic properties of Cournot duopoly games with convex and log-concave demand function," *Operations Research Letters*, vol. 42, no. 1, pp. 85–90, 2014.
- [27] L. Zhao and J. Zhang, "Analysis of a duopoly game with heterogeneous players participating in carbon emission trading," *Nonlinear Analysis: Modelling and Control*, vol. 19, no. 1, pp. 118–131, 2014.
- [28] E. Ahmed, H. N. Agiza, and S. Z. Hassan, "On modifications of Puu's dynamical duopoly," *Chaos, Solitons and Fractals*, vol. 11, no. 7, pp. 1025–1028, 2000.
- [29] M. T. Yassen and H. N. Agiza, "Analysis of a duopoly game with delayed bounded rationality," *Applied Mathematics and Computation*, vol. 138, no. 2–3, pp. 387–402, 2003.
- [30] S. Z. Hassan, "On delayed dynamical duopoly," *Applied Mathematics and Computation*, vol. 151, no. 1, pp. 275–286, 2004.
- [31] A. A. Elsadany, "Dynamics of a delayed duopoly game with bounded rationality," *Mathematical and Computer Modelling*, vol. 52, no. 9–10, pp. 1479–1489, 2010.
- [32] L. U. Yali, "Dynamics of a delayed duopoly game with increasing marginal costs and bounded rationality strategy," *Procedia Engineering*, vol. 15, pp. 4392–4396, 2011.
- [33] A. Matsumoto and F. Szidarovszky, "Nonlinear delay monopoly with bounded rationality," *Chaos, Solitons and Fractals*, vol. 45, no. 4, pp. 507–519, 2012.
- [34] A. Matsumoto and F. Szidarovszky, "Boundedly rational monopoly with continuously distributed single time delays," IERC Discussion Paper #180, 2012, http://www.chuo-u.ac.jp/chuo-u/ins_economics/pdf/f06_03_03_04/discussno180.pdf.
- [35] A. Matsumoto and F. Szidarovszky, "Discrete-time delay dynamics of boundedly rational monopoly," *Decisions in Economics and Finance*, vol. 37, no. 1, pp. 53–79, 2014.
- [36] A. Matsumoto and F. Szidarovszky, "Complex dynamics of monopolies with gradient adjustment," IERC Discussion Paper #209.
- [37] S. N. Elaydi, *An Introduction to Difference Equations*, Springer, New York, NY, USA, 1996.