Research Article

Unions of Parafree Lie Algebras

Naime Ekici¹ and Zehra Velioglu²

¹ Department of Mathematics, Çukurova University, Adana, Turkey

² Department of Mathematics, Harran University, Şanlıurfa, Turkey

Correspondence should be addressed to Naime Ekici; nekici@cu.edu.tr

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We consider unions of parafree Lie algebras and we prove that such unions are again parafree under some conditions.

1. Introduction

The study of Lie algebras which share many properties with a free Lie algebra has begun with Baur [1]. This extraordinary class of Lie algebras is called the class of parafree Lie algebras. These Lie algebras arose from Baumslag's works about parafree groups. In [2–4], Baumslag has introduced parafree groups and he obtained some interesting results about these groups. In his doctoral dissertation [1], Baur has defined parafree Lie algebras as in the group case and he proved the existence of a nonfree parafree Lie algebra [5]. Baumslag's and Baur's works have given a start for studies in the theory of parafree Lie algebras. Although there are some works about parafree Lie algebras in the literature, many questions about them have remained unanswered. This fact and paucity of studies about parafree Lie algebras motivated us for this work.

The objective of this work is to investigate the ascending unions of parafree Lie algebras. Our main theorem is as follows.

Theorem 1. *Let $L_1 \subset L_2 \subset \dots \subset L = \bigcup_{n=1}^{\infty} L_n$ be a properly ascending series of parafree Lie algebras of the same finite rank r . Then,*

- (i) L is residually nilpotent;
- (ii) L is residually finite;
- (iii) L has the same lower central sequence as some free Lie algebra;
- (iv) L is not free.

2. Preliminaries

Let K be a field and L be a Lie algebra over K . By $\gamma_n(L)$, we denote the n th term of the lower central series of L . If $\bigcap_{n=1}^{\infty} \gamma_n(L) = \{0\}$, then L is called residually nilpotent; equivalently, given any $0 \neq u \in L$, there exists an ideal J of L such that $u \notin J$ with L/J being nilpotent. More generally, if P is a property or a class of Lie algebras, then L is called residually P if, given any nontrivial element $u \in L$, there exists an ideal J of L such that $u \notin J$ with $L/J \in P$.

We say that two Lie algebras L and G have the same lower central sequence if $L/\gamma_n(L) \cong G/\gamma_n(G)$ for every $n \geq 1$.

Definition 2. Let L be a Lie algebra. L is called a parafree Lie algebra over a set X if

- (i) L is residually nilpotent;
- (ii) L has the same lower central sequence as a free Lie algebra generated by the set X .

The cardinality of X is called the rank of L .

Definition 3. Let L be a parafree Lie algebra and let Y be a subset of L . Y is called a paragenerating set if it freely generates L modulo $\gamma_2(L)$.

We will use some functorial properties of Lie algebras.

Definition 4. A directed set I is a partially ordered set such that, for each pair $i, j \in I$, there exists a $k \in I$ for which $i \leq k$ and $j \leq k$.

Definition 5. A system $(\{L_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ of Lie algebras L_i is called a directed system, if

- (i) I is a directed set.
- (ii) If $i, j, k \in I$, $i \leq j \leq k$, then $\varphi_{ij} : L_i \rightarrow L_j$ is a Lie algebra homomorphism such that

$$\varphi_{ik} = \varphi_{jk} \varphi_{ij} \quad (1)$$

and for each $i \in I$

$$\varphi_{ii} = id_{L_i}. \quad (2)$$

Definition 6. The direct limit of a directed system $(\{L_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ of Lie algebras is a pair $(D, \{\phi_\alpha\}_{\alpha \in I})$, where D is a Lie algebra and each $\phi_\alpha : L_\alpha \rightarrow D$ is a Lie algebra homomorphism such that

- (i) $\phi_\beta \varphi_{\alpha\beta} = \phi_\alpha$, for $\alpha \leq \beta$, $\alpha, \beta \in I$;
- (ii) if there is a Lie algebra M together with maps $\pi_i : L_i \rightarrow M$ such that $\pi_j \varphi_{ij} = \pi_i$, for each $i \leq j$, then there exists a unique Lie algebra homomorphism $\pi : D \rightarrow M$ such that $\pi \phi_i = \pi_i$.

The direct limit, if it exists, is unique up to isomorphism. We denote it by $\varinjlim L_i$.

Let $(\{L_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})$ be a directed system of Lie algebras and $P = \bigcup_{i=1}^\infty L_i$. Now, we will construct the direct limit $\varinjlim L_i$ [6].

Definition 7. One defines a relation $\sim$ on P by $u_i \sim u_j$, $u_i \in L_i$, $u_j \in L_j$, if there exists $k \in I$, $i, j \leq k$ such that $\varphi_{ik}(u_i) = \varphi_{jk}(u_j)$.

Let $u_i \in L_i$ and $\bar{u}_i$ be the equivalence class of u_i and let L be the set of equivalence classes. A short calculation shows that L is a Lie algebra.

It is not difficult to see that L has the same mapping property as does the direct limit. So we can obtain that the direct limit exists and is equal to L . Hence, direct limit of the Lie algebras L_i may be viewed as the union of these algebras. That is,

$$\varinjlim L_i = L = \bigcup_{i=1}^\infty L_i. \quad (3)$$

For a background of the theory of systems of objects and their limits, we refer to [7].

Let X be a nonempty set and let $F(X)$ be the free Lie algebra freely generated by X over K . Let G be a Lie algebra and let Y be any subset of G . By $\langle Y \rangle$, we denote the subalgebra of G generated by the set Y . Throughout this work, all Lie algebras will be considered over a fixed field K .

Any relation among free generators of a free Lie algebra of a variety is an identical relation in this variety. The following proposition is an obvious corollary of Proposition 4.1.2.2 of [8].

Proposition 8. Let X, Y be two nonempty sets. Then,

$$\frac{F(X)}{\gamma_n(F(X))} \cong \frac{F(Y)}{\gamma_n(F(Y))}, \quad (4)$$

if and only if X, Y have the same cardinality.

It is easy to see that the following theorem is an immediate consequence of Theorem 9 of the Section 4.2.4 of [8].

Theorem 9. Let $G = F(X)/\gamma_n(F(X))$ and let Y be a subset of G which is linearly independent modulo the derived algebra $\gamma_2(G)$. Then,

$$\langle Y \rangle \cong \frac{F(Y)}{\gamma_n(F(Y))}. \quad (5)$$

Using Theorem 8 and Lemma 2.10.1 of [9], we obtain the following well-known result.

Lemma 10. Suppose that a set Y freely generates $F = F(X)$ modulo $\gamma_2(F)$. Then, Y freely generates F modulo $\gamma_n(F)$ for $n = 2, 3, \dots$

3. Parafree Unions

Theorem 11. Let $L_1 \subset L_2 \subset \dots \subset L = \bigcup_{n=1}^\infty L_n$ be a properly ascending series of parafree Lie algebras of the same finite rank r . Then,

- (i) L is residually nilpotent;
- (ii) L is residually finite;
- (iii) L has the same lower central sequence as some free Lie algebra;
- (iv) L is not free.

Proof. (i) For $i \geq 1$, let X_i be a paragenerating set for L_i . Then, X_i freely generates modulo $\gamma_2(L_i)$, a free abelian Lie algebra. Since all the algebras L_i have the same finite rank r and for $j \geq i$, $L_i \subseteq L_j$, then X_i freely generates modulo $\gamma_2(L_j)$, a free abelian Lie algebra. Therefore, by Lemma 10 X_i freely generates modulo $\gamma_n(L_j)$, a free nilpotent Lie algebra for every $n = 2, 3, \dots$. Clearly, the set X_i is linearly independent modulo $\gamma_2(L_j)$. By Lemma 10, the subalgebra $\langle X_i \rangle$ of L_j is isomorphic to $L_j/\gamma_n(L_j)$. Hence, we obtain

$$L_j = \langle X_i \rangle + \gamma_n(L_j). \quad (6)$$

Thus,

$$L_j = L_i + \gamma_n(L_j), \quad \text{where } i \leq j. \quad (7)$$

By (7) and the second isomorphism theorem, it follows that

$$\frac{L_j}{\gamma_n(L_j)} = \frac{(L_i + \gamma_n(L_j))}{\gamma_n(L_j)} \cong \frac{L_i}{L_i \cap \gamma_n(L_j)}. \quad (8)$$

Since the algebras L_j are parafree Lie algebras of rank r , then there exist free Lie algebras F_j of rank r such that

$$\frac{L_j}{\gamma_n(L_j)} \cong \frac{F_j}{\gamma_n(F_j)}, \quad j \geq 1. \quad (9)$$

By Proposition 8, we have

$$\frac{F_j}{\gamma_n(F_j)} \cong \frac{F_i}{\gamma_n(F_i)}, \quad i, j \geq 1. \quad (10)$$

Hence,

$$\frac{L_j}{\gamma_n(L_j)} \cong \frac{L_i}{\gamma_n(L_i)}. \quad (11)$$

Using (8) and (11), we obtain

$$\frac{L_i}{\gamma_n(L_i)} \cong \frac{L_i}{L_i \cap \gamma_n(L_j)}. \quad (12)$$

It is clear that $\gamma_n(L_i) \subseteq L_i \cap \gamma_n(L_j)$. By hopfcity of $L_i/\gamma_n(L_i)$, it follows that

$$\gamma_n(L_i) = L_i \cap \gamma_n(L_j). \quad (13)$$

Now, let $\varphi_{ij} : L_i \rightarrow L_j$ be the inclusion map for all i, j such that $i \leq j$. Then, the system of Lie algebras L_i and the homomorphisms φ_{ij} provide us with a direct system. Let P be the direct limit of this system. P may be viewed as the union of its subalgebras L_i :

$$P = \bigcup_{i=1}^{\infty} L_i = L. \quad (14)$$

By definition of the direct limit, L is a Lie algebra and $\gamma_n(L) = \bigcup_{i=1}^{\infty} \gamma_n(L_i)$.

We now compute $L_i \cap \gamma_n(L)$. Consider

$$\begin{aligned} L_i \cap \gamma_n(L) &= L_i \cap \left(\bigcup_{k=1}^{\infty} \gamma_n(L_k) \right) \\ &= (L_i \cap \gamma_n(L_1)) \cup \cdots \cup (L_i \cap \gamma_n(L_i)) \\ &\quad \cup (L_i \cap \gamma_n(L_{i+1})) \cup \cdots \\ &= \gamma_n(L_1) \cup \cdots \cup \gamma_n(L_i) \cup \gamma_n(L_i) \cup \cdots \\ &= \gamma_n(L_i). \end{aligned} \quad (15)$$

Hence, we get

$$\gamma_n(L_i) = L_i \cap \gamma_n(L). \quad (16)$$

Now, suppose that $0 \neq l \in L$. Then, $l \in L_i$ for some i . Therefore, $l \notin \gamma_n(L_i)$ for some n since L_i is residually nilpotent. By (16), this implies $l \notin \gamma_n(L)$. Hence, L is residually nilpotent.

(ii) Since L_i is parafree, there exists a free Lie algebra F_i of rank r such that

$$\frac{L_i}{\gamma_n(L_i)} \cong \frac{F_i}{\gamma_n(F_i)}, \quad (17)$$

where $n \geq 2$, $i \geq 1$. It is well known that free nilpotent Lie algebras of finite rank are finite-dimensional algebras. Thus, by (17), we have that $L_i/\gamma_n(L_i)$ is a finite-dimensional algebra. It follows immediately that L_i is residually finite.

Now, we are going to prove the equality

$$L = L_i + \gamma_n(L), \quad (18)$$

where $i \geq 1$. Clearly $L_i + \gamma_n(L) \subseteq L$. Now, suppose that $l \in L$. Then, $l \in L_j$ for some j . By virtue of (7), l can be written as

$$l = l_1 + l_2, \quad (19)$$

where $l_1 \in L_i$, $l_2 \in \gamma_n(L_j)$. Since, for each j , $n \geq 1$ $\gamma_n(L_j) \subseteq \gamma_n(L)$, it follows that $l_2 \in \gamma_n(L)$ and, hence, $L \subseteq L_i + \gamma_n(L)$. Therefore,

$$L = L_i + \gamma_n(L). \quad (20)$$

Using (20) and the second isomorphism theorem, we obtain

$$\frac{L}{\gamma_n(L)} \cong \frac{L_i + \gamma_n(L)}{\gamma_n(L)} \cong \frac{L_i}{L_i \cap \gamma_n(L)}. \quad (21)$$

Hence, by (16),

$$\frac{L}{\gamma_n(L)} \cong \frac{L_i}{\gamma_n(L_i)}, \quad i \geq 1, \quad n \geq 2. \quad (22)$$

Finite dimensionality of $L_i/\gamma_n(L_i)$ implies that $L/\gamma_n(L)$ is a finite-dimensional algebra. Now, let $0 \neq l \in L$. Then, $l \in L_i$ for some i . Therefore, $l \notin \gamma_n(L_i)$ for some n . By (16), $l \notin \gamma_n(L)$. Hence, L is residually finite.

(iii) From the proof of (ii) (using (17) and (22)), we have

$$\frac{L}{\gamma_n(L)} \cong \frac{L_i}{\gamma_n(L_i)} \cong \frac{F_i}{\gamma_n(F_i)}, \quad (23)$$

where F_i is a free Lie algebra of rank r , $n \geq 2$. Thus, L has the same lower central sequence with the free Lie algebra F_i . So if we combine this result with (i), we observe that L is parafree of rank r .

(iv) Since L is the direct limit of the parafree Lie algebras L_i , it is not finitely generated. Hence, L is not free since it is not finitely generated, but $L/\gamma_2(L)$ is. $\square$

The following corollary is an immediate consequence of Theorem 11.

Corollary 12. *Union of a properly ascending series of parafree Lie algebras of the same finite rank is a parafree Lie algebra.*

Corollary 13. *For any integer $i \geq 1$, let F_i be a free Lie algebra of rank two. Then, the union $\bigcup_{i=1}^{\infty} F_i$ of F_i is a parafree Lie algebra of rank two. Moreover, $\bigcup_{i=1}^{\infty} F_i$ is not free.*

Proof. For $i \geq 1$, consider the free Lie algebra F_i generated by the free generating set $X_i = \{u_i, v_i\}$, where $i \geq 1$. Let ϕ_{ii+1} be the map defined by

$$\begin{aligned} \phi_{ii+1} : u_i &\longrightarrow [u_{i+1}, v_{i+1}], \\ v_i &\longrightarrow [[u_{i+1}, v_{i+1}], v_{i+1}], \end{aligned} \quad (24)$$

where $[\cdot, \cdot]$ is the Lie product. The subalgebra generated by the elements $[u_{i+1}, v_{i+1}]$ and $[[u_{i+1}, v_{i+1}], v_{i+1}]$ of F_{i+1} is a free Lie algebra of rank 2. Therefore, the map ϕ_{ii+1} can be extended to an embedding

$$\bar{\phi}_{ii+1} : F_i \longrightarrow F_{i+1}. \quad (25)$$

Clearly, the subalgebra $\bar{\phi}_{ii+1}(F_i)$ is generated by

$$\{[u_{i+1}, v_{i+1}], [[u_{i+1}, v_{i+1}], v_{i+1}]\} \quad (26)$$

and it is a proper subalgebra of F_{i+1} . Furthermore, the algebras F_i and the monomorphisms ϕ_{ii+1} form a directed system. Let P be the direct limit of this system. Then, P may be viewed as the properly ascending union of the algebras F_i . That is,

$$P = \bigcup_{i=1}^{\infty} F_i. \quad (27)$$

Hence, by Theorem 11, P is parafree. From the proof of Theorem 11 (ii), we have

$$\frac{P}{\gamma_n(P)} \cong \frac{F_i}{\gamma_n(F_i)}, \quad n \geq 2. \quad (28)$$

This shows that the rank of P is two. Now, recall the construction of the direct limit and the relation $\sim$. It is easy to prove that for all $w \in L_i$ $w \sim \phi_{ii+1}(w)$. Therefore, $P = [P, P]$. Hence, P is not free. $\square$

Corollary 14. *There exists a parafree Lie algebra of rank two such that every finitely generated quotient algebra of P can be generated by two elements.*

Proof. Let the algebras F_i be as in Corollary 13. Then, $P = \bigcup_{i=1}^{\infty} F_i$ is a parafree Lie algebra of rank two. Let J be any ideal of P such that P/J is finitely generated. Let $a + J \in P/J$. Then, $a \in F_i$ for some i . Thus $a + J \in F_i + J/J$ for some i . So $a + J \in \bigcup_{i=1}^{\infty} (F_i + J/J)$. This shows that $P/J \subseteq \bigcup_{i=1}^{\infty} (F_i + J/J)$. Therefore,

$$\frac{P}{J} = \bigcup_{i=1}^{\infty} \left(F_i + \frac{J}{J} \right). \quad (29)$$

Assume that P/J is finitely generated. Since every finite subset of P/J is included in a suitable subalgebra $(F_i + J)/J$, then

$$\frac{P}{J} = \frac{(F_i + J)}{J} \quad (30)$$

for some i . Hence, P/J is generated by two elements. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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