Research Article

Domination Polynomials of k -Tree Related Graphs

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Let G be a simple graph of order n . The domination polynomial of G is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^n d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G . In this paper, we study the domination polynomials of several classes of k -tree related graphs. Also, we present families of these kinds of graphs, whose domination polynomials have no nonzero real roots.

1. Introduction

Throughout this paper, we will consider only simple graphs. Let $G = (V, E)$ be a simple graph. For $F \subseteq V(G)$ we use $\langle F \rangle$ for the subgraph induced by F . For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. For every vertex $v \in V(G)$, the degree of v is the number of edges incident with v and is denoted by $d_G(v) = |N(v)|$. Let d_i , $1 \leq i \leq n$, be the degrees of the vertices v_i of a graph in any order. The sequence $\{d_i\}_1^n$ is called the degree sequence of the graph. A clique in a graph is a subset of its vertices such that every two vertices in the subset are connected by an edge. We use K_n , P_n , C_n , and $S_{1,n-1}$ for a clique, a path, a cycle, and a star, all of order n , respectively.

A set $S \subseteq V(G)$ is a dominating set if $N[S] = V$ or, equivalently, every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G . A dominating set with cardinality $\gamma(G)$ is called a γ -set. For a detailed treatment of these parameters, the reader is referred to [1]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The domination polynomial $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G (see [2, 3]). Thus, $D(G, x)$

is the generating polynomial for the number of dominating sets of G of any cardinality. A root of $D(G, x)$ is called a domination root of G .

In [4], it is shown that computing the domination polynomial $D(G, x)$ of a graph G is NP-hard and some examples for graphs for which $D(G, x)$ can be computed efficiently are given. The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are pairwise joined to each other and then u is deleted (see [5]). The following theorem is useful for finding the recurrence relations for the domination polynomials of arbitrary graphs.

Theorem 1 (see [4, 6]). *Let G be a graph. For any vertex u in G , one has*

$$D(G, x) = xD\left(\frac{G}{u}, x\right) + D(G - u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x), \quad (1)$$

where $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in G .

Using Theorem 1 we are able to obtain an easier formula for a graph with at least one vertex of degree 1. Since every tree has at least two vertices of degree 1, we can use the following recurrence to obtain the domination polynomials of trees.

FIGURE 1: The 3-star and 3-path on 7 vertices, respectively.

Corollary 2 (see [4]). Let $G = (V, E)$ be a graph, let v be a vertex of degree 1 in G , and let u be its neighbor. Then,

$$D(G, x) = x \left(D\left(\frac{G}{u}, x\right) + D(G - u - v, x) + D(G - N[u], x) \right). \quad (2)$$

If G_1 and G_2 are disjoint graphs of orders n_1 and n_2 , respectively, then $D(G_1 \cup G_2, x) = D(G_1, x)D(G_2, x)$ and

$$D(G_1 + G_2, x) = ((1+x)^{n_1} - 1)((1+x)^{n_2} - 1) + D(G_1, x) + D(G_2, x), \quad (3)$$

where $G_1 + G_2$ is the join of G_1 and G_2 , formed from $G_1 \cup G_2$ by adding in all edges between a vertex of G_1 and a vertex of G_2 (see [2]).

The domination polynomials of trees, aside from path graph, have not been studied and there is no study for coefficients of $D(T, x)$ for trees T with n vertices. In this paper, k -trees, a generalization of trees, are considered. We study, similar to [7], k -tree related graphs and study their domination polynomials. An interesting study of the roots of domination polynomials is done in [8, 9]. Classification of the roots of domination polynomials is difficult to do, as well as finding graphs with no nonzero real roots. In this paper, we present k -tree related families which have this prementioned property.

In Section 2, we study the domination polynomials for some k -tree related graphs. In Section 3, we present some families of these kinds of graphs whose domination polynomials have no nonzero real roots.

2. Domination Polynomials of k -Tree Related Graphs

In this section, we study the domination polynomials for some k -tree related graphs. The class of k -trees is a very important subclass of triangulated graphs. Harary and Palmer [10] first introduced 2-trees in 1968. Beineke and Pippert [11] gave the definition of a k -tree in 1969. In the literature on k -trees, there are interesting applications to the study of computational complexity.

Definition 3. For a positive integer k , a k -tree, denoted by T_n^k , is defined recursively as follows. The smallest k -tree is the k -clique K_k . If G is a k -tree with $n \geq k$ vertices and a new vertex

v of degree k is added and joined to the vertices of a k -clique in G , then the larger graph is a k -tree with $n + 1$ vertices.

An *independent set* in a graph G is a set of pairwise nonadjacent vertices.

Definition 4. Let K_k be a k -clique and let S be an independent set of $n - k$ vertices. A (k, n) -star, denoted by $S_{k, n-k}$, is defined as $S_{k, n-k} = K_k + S$.

Definition 5. A (k, n) -path, denoted by P_n^k , begins with k -clique on $\{v_1, v_2, \dots, v_k\}$. For $i = k + 1$ to n , let vertex v_i be adjacent to vertices $\{v_{i-1}, v_{i-2}, \dots, v_{i-k}\}$ only (see Figure 1).

A helpful characteristic of the (k, n) -path P_n^k is that we may order the vertices $v_1, v_2, \dots, v_n$ such that $P_n^k \setminus \{v_1, \dots, v_i\}$ is a k -path on $n - i$ vertices for $1 \leq i \leq n - k - 1$; such a vertex ordering is referred to as a presentation.

Definition 6. A (k, n) -cycle, denoted by C_n^k , consists of a (k, n) -path on $\{v_1, v_2, \dots, v_n\}$ defined as above and an edge joining v_1 to v_n , where $n \geq k + 2$.

Definition 7. If G is a (k, n) -cycle of order n and v is a vertex not in G , then $G + v$ is called a (k, n) -wheel and is denoted by W_n^k .

Notice that P_n^1 , C_n^1 , W_n^1 , and $S_{1, n-1}$ are just the standard path, cycle, wheel, and star, respectively. It follows easily from the domination polynomial of join of two graphs that, for the star graph $S_{1, n-1}$, we have $D(S_{1, n-1}, x) = x(1+x)^{n-1} + x^{n-1}$. The following are recurrences for the domination polynomials of paths and cycles [12].

Theorem 8. For the natural number $n \geq 3$,

- (i) $D(P_{n+1}, x) = x(D(P_n, x) + D(P_{n-1}, x) + D(P_{n-2}, x))$, where $D(P_1, x) = x$, $D(P_2, x) = x^2 + 2x$, and $D(P_3, x) = x^3 + 3x^2 + x$;
- (ii) $D(C_{n+1}, x) = x(D(C_n, x) + D(C_{n-1}, x) + D(C_{n-2}, x))$, where $D(C_1, x) = x$, $D(C_2, x) = x^2 + 2x$, and $D(C_3, x) = x^3 + 3x^2 + 3x$.

Note that both (k, n) -cycles and (k, n) -wheels are not k -trees. But they are closely related to k -trees. We begin with a simple lemma which was proven in [7] as Proposition 2.

Lemma 9. For any k -tree T_n^k , $|E(T_n^k)| = kn - (1/2)k(k+1) = (2nk - k^2 - k)/2$.

The *independence number* is the size of a maximum independent set in a graph and is denoted by $\alpha(G)$. The following lemma gives independence numbers for k -tree related graphs.

Lemma 10 (see [7]). For each natural number $k \leq n$, one has

- (i) $\alpha(P_n^k) = \lfloor (n+k)/(k+1) \rfloor$,
- (ii) $\alpha(C_n^k) = \lfloor (n+k-1)/(k+1) \rfloor$,
- (iii) $\alpha(W_n^k) = \alpha(C_n^k)$,
- (iv) $\alpha(S_{k,n-k}) = n - k$.

Now, we present the following domination numbers for k -tree related graphs.

Theorem 11. For each natural number $k \leq n$, one has

- (i) $\gamma(P_n^k) = \lceil n/(2k+1) \rceil$,
- (ii) $\gamma(C_n^k) = \lceil n/(2k+1) \rceil$,
- (iii) $\gamma(W_n^k) = 1$,
- (iv) $\gamma(S_{k,n-k}) = 1$.

Proof. (i) Since $k \leq n$, by the definition of P_n^k , the degree sequence in this graph is

$$\{k, k+1, \dots, 2k-1, 2k, \dots, 2k, 2k-1, \dots, k+1, k\}; \quad (4)$$

we have $\gamma(P_n^k) = 1$ for $n \leq 2k+1$ and $\gamma(P_n^k) = 2$ for $2k+2 \leq n \leq 4k+2$. Thus, (i) holds for $k \leq n \leq 4k+2$. Now, assume $n \geq 4k+3$. We use induction on n . Since any γ -set of P_n^k contains only one vertex of the $\{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ and $P_n^k - \{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ is a k -path with $n - 2k - 1$ vertices, by induction, $\gamma(P_n^k) = 1 + \gamma(P_{n-2k-1}^k) = 1 + \lceil (n-2k-1)/(2k+1) \rceil = \lceil n/(2k+1) \rceil$. Hence, (i) holds.

(ii) Since $k \leq n$, by the definition of C_n^k , the degree sequence in this graph is

$$\{k+1, k+1, k+2, \dots, 2k-1, 2k, \dots, 2k, 2k-1, \dots, k+1, k+1\}; \quad (5)$$

we have $\gamma(C_n^k) = 1$ for $n \leq 2k+1$ and $\gamma(C_n^k) = 2$ for $2k+2 \leq n \leq 4k+2$. Thus, (ii) holds for $k \leq n \leq 4k+2$. Now, assume $n \geq 4k+3$ and use induction on n . Since any γ -set of C_n^k contains only one vertex of the $\{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ and $C_n^k - \{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ is a k -path with $n - 2k - 1$ vertices, by induction, $\gamma(C_n^k) = 1 + \gamma(P_{n-2k-1}^k) = 1 + \lceil (n-2k-1)/(2k+1) \rceil = \lceil n/(2k+1) \rceil$. Hence, (ii) holds.

(iii) Since the (k, n) -wheel W_n^k has a vertex v of degree $n - 1$, (iii) holds.

(iv) Since the k -star graph $S_{k,n-k}$ has k vertices of degree $n - 1$, (iv) holds. $\square$

The following theorem gives a recurrence formula for the domination polynomial of (k, n) -path graphs.

Theorem 12. If $n \leq k+1$, then $D(P_n^k, x) = D(K_n, x)$. For every $k+2 \leq n$,

$$D(P_n^k, x) = (1+x)D(P_{n-1}^k, x) + xD(P_{n-k-1}^k, x) - (1+x)p_u(P_n^k, x), \quad (6)$$

where

$$p_u(P_n^k, x) = \begin{cases} x(1+x)^{n-k-2}; & k+2 \leq n \leq 2k+2, \\ x((1+x)^{n-k-2} - (1+x)^{n-2k-3}); & 2k+3 \leq n \leq 2k+6, \\ p_u(P_n^k, x); & 2k+7 \leq n. \end{cases} \quad (7)$$

Proof. If $n \leq k+1$, then $P_n^k \cong K_n$. For every $k+2 \leq n$, we use Theorem 1 for the last vertex of P_n^k and since (by the definition of P_n^k) the first $k+1$ and the last $k+1$ vertices form two cliques, we have $P_n^k/u \cong P_n^k - u$. It is clear that $D(P_n^k - N[u], x) = D(P_{n-k-1}^k, x)$. Obviously, $p_u(P_n^k, x)$ is the polynomial counting the dominating sets of P_{n-k-1}^k and contains the vertex v_{n-k-1} . However, finding this polynomial involves complex calculations. We therefore give this polynomial only for $n \leq 2k+6$. Therefore, we have the result. $\square$

In general, finding the domination polynomial of a graph is a very difficult problem. In [4], Kotek et al. showed that there exist recurrence relations for the domination polynomial which allow for efficient schemes to compute the polynomial for some types of graphs. Consider the (k, n) -cycle graphs. If $n \leq k+2$, then $C_n^k \cong K_n$. Consequently, in this case, $D(C_n^k, x) = D(K_n, x)$. Until now, for every $k+3 \leq n$, all attempts to find formulas for $D(C_n^k, x)$ have failed.

The following theorem gives a formula for the domination polynomial of (k, n) -wheel graphs.

Theorem 13. For a (k, n) -wheel W_n^k , one has

$$D(W_n^k, x) = x(1+x)^n + D(C_n^k, x). \quad (8)$$

Proof. Since $W_n^k = C_n^k + K_1$, then

$$\begin{aligned} D(W_n^k, x) &= ((1+x) - 1)((1+x)^n - 1) + x + D(C_n^k, x) \\ &= x(1+x)^n + D(C_n^k, x). \end{aligned} \quad (9)$$

$\square$

The following theorem gives a formula for the domination polynomial of k -star graphs, which is derived from the fact that the k -star graph is the join of complete graph K_k and independent set S (empty graph O_{n-k}).

Theorem 14. For every $k \in \mathbb{N}$ and $n > k$,

$$D(S_{k,n-k}, x) = (1+x)^{n-k}((1+x)^k - 1) + x^{n-k}. \quad (10)$$

Proof. Let $S_{k,n-k}$ be the k -star graph with vertex set $V(S_{k,n-k}) = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$. It suffices to show that every dominating set of size $j = 1, \dots, n$ is accounted for exactly once in the above statement. Clearly, every nonempty subset of $\{v_1, v_2, \dots, v_k\}$ is a dominating set of k -star graphs. These sets can be extended with any number of vertices $\{v_{k+1}, \dots, v_n\}$. Also, obviously, the set $\{v_{k+1}, \dots, v_n\}$ is a dominating set of k -star graphs. It is easy to see that there is no other method to form a dominating set for k -star graphs. Therefore, we have the result. $\square$

Remark 15. It is easy to see that another approach for proving Theorem 14 is using the formula for the domination polynomial of join of two graphs.

The value of a graph polynomial at a specific point can give sometimes a surprising information about the structure of the graph [3, 13]. The following simple results give the domination polynomial of k -tree related graphs at -1 .

Corollary 16. *For each natural number $k \leq n$, the following hold:*

- (i) $D(P_n^k, -1) = (-1)^{\alpha(P_n^k)}$,
- (ii) $D(W_n^k, -1) = D(C_n^k, -1)$,
- (iii) $D(S_{k,n-k}, -1) = (-1)^{\alpha(S_{k,n-k})}$.

Proof. (i) Using the domination polynomial of a (k, n) -path in Theorem 12, for $k \leq n \leq k+1$, yields $D(P_n^k, -1) = D(K_n, -1) = -1$. For every $k+2 \leq n$, $D(P_n^k, -1) = -D(P_{n-k-1}^k, -1)$. We use induction on n . Suppose that the statement is true for every k -path with $n-k$ vertices; by induction and Lemma 10,

$$\begin{aligned} D(P_n^k, -1) &= -D(P_{n-k-1}^k, -1) \\ &= -(-1)^{\alpha(P_{n-k-1}^k)} = -(-1)^{\lfloor (n-1)/(k+1) \rfloor} \\ &= (-1)^{1+\lfloor (n-1)/(k+1) \rfloor} = (-1)^{\lfloor (n+k)/(k+1) \rfloor}. \end{aligned} \quad (11)$$

Hence, (i) holds.

(ii) Follows from Theorem 13.

(iii) Follows from Theorem 14 and Lemma 10. $\square$

3. Some Families of Graphs with No Nonzero Real Domination Roots

In [8], the authors asked the question: “Which graphs have no nonzero real domination roots?”

In this section, we obtain more results related to this question. We need some preliminaries.

For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H , by adding edges between the i th vertex of G and all vertices of i th copy of H [14]. It is easy to see that the corona operation of two graphs does not have the commutative property.

We need the following theorem which will be used to find the domination polynomial of the corona products of two graphs.

Theorem 17 (see [4, 15]). *Let $G = (V, E)$ and $H = (W, F)$ be nonempty graphs of orders n and m , respectively. Then,*

$$D(G \circ H, x) = (x(1+x)^m + D(H, x))^n. \quad (12)$$

A k -star, $S_{k,n-k}$, has vertex set $\{v_1, \dots, v_n\}$ where $\{v_1, v_2, \dots, v_k\} \cong K_k$ and $N(v_i) = \{v_1, \dots, v_k\}$ for $k+1 \leq i \leq n$.

Now, we will discuss the roots of the domination polynomial of the k -star graphs.

Theorem 18.

- (i) *For odd natural n and even natural k , no nonzero real number is the domination root of $S_{k,n-k}$.*
- (ii) *For even natural n and even natural k , there is at least one nonzero real domination root of $S_{k,n-k}$.*

Proof. By Theorem 14, for every $n > k$, $D(S_{k,n-k}, x) = (1+x)^{n-k}((1+x)^k - 1) + x^{n-k}$. If $D(S_{k,n-k}, x) = 0$, then we have

$$(1+x)^k - 1 = -\left(\frac{x}{1+x}\right)^{n-k}. \quad (13)$$

Now we prove the two cases of this theorem.

- (i) First, suppose that $x \geq 0$. Obviously, equality (13) is true just for 0, since for nonzero real number the left side of equality is positive but the right side is negative. If $-1 < x < 0$, then the left side of equality is negative but the right side is positive. Now, suppose that $x < -1$. In this case, the left side is greater than -1 and the right side $-(1/(1+1/x))^{n-k}$ is less than -1 , a contradiction. Therefore, in any cases, we have the result.
- (ii) Let $x \geq 0$. Similar to Case (i), equality (13) is true only for 0, since for nonzero real number the left side of equality is positive but the right side is negative. Now, suppose that $x < -1$. In this case, the right side is less than -1 and the left side is greater than -1 , a contradiction. It remains to consider $-1 < x < 0$. It is easy to see that $\lim_{x \rightarrow 0^-} D(S_{k,n-k}, x) < 0$, and $D(S_{k,n-k}, -1) = (-1)^{n-k} > 0$. So $D(S_{k,n-k}, x)$ has at least one real root in $(-1, 0)$. $\square$

Remark. Using Maple, we have shown the domination roots of $S_{4,n-4}$ for $5 \leq n \leq 44$ in Figure 2.

Here, we construct a sequence of graphs, of which the domination roots are the same as the domination roots of the $k+1$ -star graphs.

Theorem 19. *The domination roots of every graph H in the family*

$$\begin{aligned} \{G \circ S_{k,n-k}, (G \circ S_{k,n-k}) \circ S_{k,n-k}, \\ ((G \circ S_{k,n-k}) \circ S_{k,n-k}) \circ S_{k,n-k}, \dots\} \end{aligned} \quad (14)$$

have the same behavior as the domination roots of $k+1$ -star graphs.

FIGURE 2: Domination roots of graphs $S_{4,n-4}$ for $5 \leq n \leq 44$.

Proof. By Theorem 17, we can deduce that, for each arbitrary graph G ,

$$\begin{aligned}
 D(G \circ S_{k,n-k}, x) &= (x(1+x)^n + (1+x)^{n-k}((1+x)^k - 1) + x^{n-k})^{|V(G)|} \\
 &= ((1+x)^{n-k}((1+x)^{k+1} - 1) + x^{n-k})^{|V(G)|} \\
 &= (D(S_{k+1,n-k}, x))^{|V(G)|}.
 \end{aligned}
 \tag{15}$$

Therefore, we have the result. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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