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Research Article

On the Genus of the Zero-Divisor Graph of $\mathbb{Z}_n$

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Let R be a commutative ring with identity. The zero-divisor graph of R, denoted $\Gamma(R)$, is the simple graph whose vertices are the nonzero zero-divisors of R, and two distinct vertices x and y are linked by an edge if and only if $xy = 0$. The genus of a simple graph G is the smallest integer g such that G can be embedded into an orientable surface S_g. In this paper, we determine that the genus of the zero-divisor graph of $\mathbb{Z}_n$, the ring of integers modulo n, is two or three.

1. Introduction

This paper concerns the zero-divisor graphs of rings. For a commutative ring R, define a simple graph called *zero-divisor graph*, denoted by $\Gamma(R)$, whose vertices are the nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$ in R. This definition was first introduced by Beck in [1]. However, he let all elements of R be the vertices of the graph and mainly considered the coloring of this graph. Here our definition is the same as in [2], where some basic properties of $\Gamma(R)$ are established. The zero-divisor graph, as well as other graphs of rings, is an active research topic in the last two decades (see, e.g., [3–11]).

Let us first recall some needed notions in graph theory. Let G be a simple graph, that is, no loops and no multiedges. The *degree* of a vertex $v \in G$, denoted by $\deg(v)$, is the number of edges of G incident with v. If $V' \subseteq V(G)$, then $G - V'$ is the subgraph of G obtained by deleting the vertices in V' and all edges incident with them. If $V' = \{x \in V \mid \deg(x) = 0 \text{ or } 1\}$, then we use $\widetilde{G}$ for the subgraph $G - V'$ and call it the *reduction* of G. A *complete bipartite graph* is a bipartite graph (i.e., a set of graph vertices decomposed into two disjoint sets such that no two vertices within the same set are adjacent) such that every pair of vertices in the two sets are adjacent. The complete bipartite graph with partitions of sizes m and n is denoted by $K_{m,n}$. The *complete graph* on n vertices, denoted K_n, is the graph in which every pair of distinct vertices is joined by an edge. A surface is said to be of *genus* g if it is topologically homomorphic to a sphere with g handles. A graph G that can be drawn without crossings on a compact surface of genus g, but not on one of genus $g - 1$, is called a graph of *genus* g. We write $\gamma(G)$ for the genus of the graph G. It is clear that $\gamma(G) = \gamma(\widetilde{G})$, where $\widetilde{G}$ is the reduction of G, and $\gamma(H) \leq \gamma(G)$ for any subgraph H of G.

Determining the genus of a graph is one of the most fundamental problems in topological graph theory. It has been shown to be NP-complete by Thomassen in [12]. Several papers focus on the genera of zero-divisor graphs. For instance, in [6, 7, 13, 14], the authors studied the planar zero-divisor graphs (genus equals to 0); Wang et al. investigated the genus one zero-divisor graphs in [11, 15, 16], respectively; and Bloomfield and Wickham determined all local rings whose zero-divisor graphs have genus two in [8]. In this paper, we study the zero-divisor graph of $\mathbb{Z}_n$, the ring of integers modulo n. In particular, we determine when $\gamma(\Gamma(\mathbb{Z}_n)) = 2$ or 3. Here we first summarize the results about the genus of $\Gamma(\mathbb{Z}_n)$ from [5, Theorem 5.1(a)], [8, Theorem 1], and [16, Section 5].

Theorem 1. *Let $\Gamma(\mathbb{Z}_n)$ be not empty. Then the following hold.*

(1) $\gamma(\Gamma(\mathbb{Z}_n)) = 0$ *if and only if* $n \in \{8, 12, 16, 18, 25, 27, 2p, 3p\}$, *where p is prime.*

(2) $\gamma(\Gamma(\mathbb{Z}_n)) = 1$ *if and only if* $n \in \{20, 24, 28, 32, 49\}$.

(3) $\gamma(\Gamma(\mathbb{Z}_{p^t})) = 2$ *if and only if* $p^t = 81$.

All rings considered in this paper will be commutative rings with identity. Let a be an element of a ring R. Then the principal ideal generated by a is denoted by $\langle a\rangle$. For a set A, $|A|$ means the order of A.

2. The Genus of $\Gamma(\mathbb{Z}_n)$

The following two lemmas are frequently used in the proofs of our main results.

Lemma 2 ([17, Theorem 6.38]). $\gamma(K_n) = \{(1/12)(n-3)(n-4)\}$, *where $\{x\}$ is the least integer that is greater than or equal to x.*

Lemma 3 ([17, Theorem 6.37]). $\gamma(K_{m,n}) = \{(1/4)(m-2)(n-2)\}$, *where $\{x\}$ is the least integer that is greater than or equal to x.*

Lemma 4 ([17, Corollary 6.15]). *Suppose a simple graph G is connected with $v \geq 3$ vertices and e edges. If G has no triangles, then $\gamma(G) \geq (e/4) - (v/2) + 1$.*

Lemma 5. *Let G be a graph with vertex set $\{u_1, u_2, u_3, u_4; v_1, \ldots, v_7; w_1, w_2, w_3, w_4\}$ and the edge set $\{(u_i, v_j) \mid 1 \leq i \leq 4, 1 \leq j \leq 7\} \cup \{(v_i, w_j) \mid 1 \leq i \leq 3, 1 \leq j \leq 3\}$. Then $\gamma(G) \geq 4$.*

Proof. Note that there are no triangles in G. As G has 40 edges and 15 vertices, $\gamma(G) \geq 4$ by Lemma 4. □

We first consider the case that n has only one prime divisor.

Theorem 6. *Let $n = p^t$, where p is prime and $t \geq 2$. Then $\gamma(\Gamma(\mathbb{Z}_n)) = 3$ if and only if $n = 64$.*

Proof. ($\Rightarrow$). Let $I = \langle p^{t-1}\rangle - \{0\}$. Then $|I| = p - 1$. As any two vertices in I are adjacent, there exists a complete subgraph K_{p-1} in $\Gamma(\mathbb{Z}_n)$. It follows that if $p \geq 11$, then $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(\Gamma(K_{10})) = 4$ by Lemma 2. Therefore, $p \leq 7$. We proceed with three cases.

Case 1 ($p = 5$ or 7). By Theorem 1, $\gamma(\Gamma(\mathbb{Z}_{5^2})) = 0$ and $\gamma(\Gamma(\mathbb{Z}_{7^2})) = 1$. So we can further assume $t \geq 3$. Let $I = \langle p^{t-2}\rangle - \langle p^{t-1}\rangle$ and $J = \langle p^{t-1}\rangle - \{0\}$. Then $I \cap J$ is empty and $|I| = p^2 - p \geq 20$, $|J| = p - 1 \geq 4$. As each vertex of I is adjacent to every vertex of J, there exists a complete bipartite subgraph $K_{4,20}$ in $\Gamma(\mathbb{Z}_n)$, which implies that $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(\Gamma(K_{4,20})) = 9$ by Lemma 3.

Case 2 ($p = 3$). From Theorem 1, we have $\gamma(\Gamma(\mathbb{Z}_{3^2})) = 0$, $\gamma(\Gamma(\mathbb{Z}_{3^3})) = 0$, and $\gamma(\Gamma(\mathbb{Z}_{81})) = 2$. So we may assume $t \geq 5$. Let $I = \langle 3^{t-3}\rangle - \langle 3^{t-2}\rangle$ and $J = \langle 3^{t-2}\rangle - \{0\}$. Then $I \cap J$ is empty and $|I| = 3^3 - 3^2 = 18$, $|J| = 3^2 - 1 = 8$. Since each vertex of I is adjacent to every vertex of J, $\Gamma(\mathbb{Z}_n)$ contains a complete bipartite subgraph $K_{8,18}$, which implies that $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(\Gamma(K_{8,18})) = 24$ by Lemma 3.

Case 3 ($p = 2$). By Theorem 1, $\gamma(\Gamma(\mathbb{Z}_{2^t})) = 0$ if $t = 2, 3, 4$, and $\gamma(\Gamma(\mathbb{Z}_{2^5})) = 1$.

If $n \neq 64$, we can assume $t \geq 7$. We let $I = \langle 2^{t-4}\rangle - \langle 2^{t-3}\rangle$, $J = \langle 2^{t-3}\rangle - \{0\}$. Then $|I| = 2^4 - 2^3 = 8$, $|J| = 2^3 - 1 = 7$.

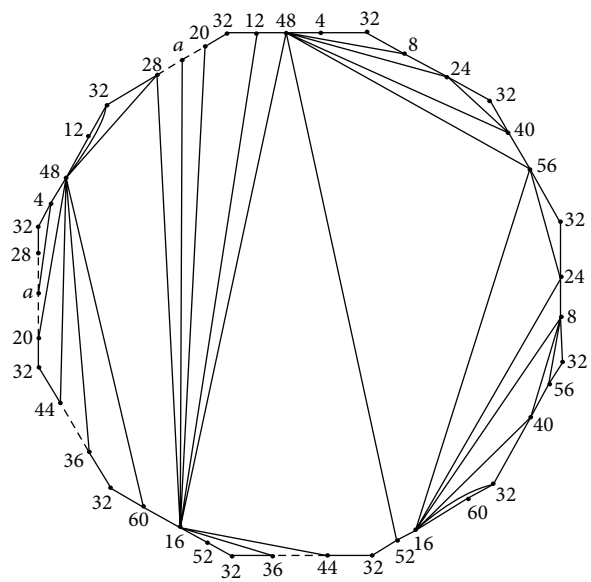

Figure 1: The reduction of $\Gamma(\mathbb{Z}_{64})$.

Note that each vertex of I is adjacent to each vertex of J and $I \cap J$ is empty, so there exists a complete bipartite subgraph $K_{7,8}$ in $\Gamma(\mathbb{Z}_n)$, which implies that $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{7,8}) = 8$ by Lemma 3. Therefore, $n = 64$.

($\Leftarrow$). For $n = 64$, let $I = \langle 4\rangle - \langle 16\rangle$ and $J = \langle 16\rangle - \{0\}$. Then $|I| = 12$, $|J| = 3$, and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there exists a complete bipartite subgraph $K_{3,12}$ in $\Gamma(\mathbb{Z}_{64})$, which implies that $\gamma(\Gamma(\mathbb{Z}_{64})) \geq \gamma(K_{3,12}) = 3$ by Lemma 3. On the other hand, we can embed the reduction of $\Gamma(\mathbb{Z}_{64})$ into S_3 as shown in Figure 1. Therefore, $\gamma(\Gamma(\mathbb{Z}_{64})) = 3$.

This completes our proof. □

We now consider the case that n has exactly two prime divisors.

Theorem 7. *Let $n = p^s q^t$, where $p < q$ are primes and $s, t \geq 1$. Then $\gamma(\Gamma(\mathbb{Z}_n)) = 2$ if and only if $n \in \{35, 36, 44, 50\}$, and $\gamma(\Gamma(\mathbb{Z}_n)) = 3$ if and only if $n \in \{45, 52, 54\}$.*

Proof. We first prove that if $2 \leq \gamma(\Gamma(\mathbb{Z}_n)) \leq 3$ then $n \in \{35, 36, 44, 50, 45, 52, 54\}$. We then determine $\gamma(\Gamma(\mathbb{Z}_n))$ for each $n \in \{35, 36, 44, 50, 45, 52, 54\}$. We proceed with four cases to get the result.

Case 1 ($s = t = 1$). It is clear that $\Gamma(\mathbb{Z}_n)$ is a complete bipartite graph $K_{p-1,q-1}$. In this case, if $p = 5$ and $q \geq 11$, then $K_{4,10}$ is a subgraph of $\Gamma(\mathbb{Z}_n)$; it follows that $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{4,10}) = 4$ by Lemma 3. If $p = 2$ or $p = 3$, by Theorem 1, we have $\gamma(\Gamma(\mathbb{Z}_n)) = 0$. So $p = 5$, $q = 7$; that is, $n = 35$.

Case 2 ($s = 1$ and $t \geq 2$). If $t \geq 3$, set $I = \langle pq^{t-2}\rangle - \langle pq^{t-1}\rangle$ and $J = \langle q^2\rangle - \{0\}$. Then $I \cap J$ is empty and each vertex in I is adjacent to each vertex in J. If $q \geq 5$, then $|I| = q^2 - q \geq 20$, $|J| = pq^{t-2} - 1 \geq 9$, which implies that $\Gamma(\mathbb{Z}_n)$ contains a complete bipartite subgraph $K_{9,20}$. Therefore $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{9,20}) = 32$ by Lemma 3. If $q = 3$ and $t \geq 4$,

then $|I| = q^2 - q = 6$, $|J| = pq^{t-2} - 1 \geq 17$, which implies that there exists a complete bipartite subgraph $K_{6,17}$ in $\Gamma(\mathbb{Z}_n)$. Therefore $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{6,17}) = 15$ by Lemma 3. Hence, in this situation, we have $n = 2 \cdot 3^3 = 54$.

Consider now $t = 2$; that is, $n = pq^2$. Let $I = \langle q \rangle - \langle pq \rangle$ and $J = \langle pq \rangle - \{0\}$. Then $I \cap J$ is empty and $|I| = pq - q$, $|J| = q - 1$. Notice that each vertex in I is adjacent to each vertex in J, so there exists a complete bipartite subgraph $K_{pq-q,q-1}$ in $\Gamma(\mathbb{Z}_n)$. If $q = 7$, then $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{7p-7,6}) \geq 5$. If $q = 5$ and $p = 3$ then $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{4,10}) = 4$. By Theorem 1, we know $\gamma(\Gamma(\mathbb{Z}_{18})) = 0$. So $n = 50$.

Case 3 ($s \geq 2$ and $t = 1$). Let $I = \langle p^s \rangle - \{0\}$ and $J = \langle q \rangle - \{0\}$. Then $|I| = q - 1$, $|J| = p^s - 1$, and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there exists a complete bipartite subgraph K_{q-1,p^s-1} in $\Gamma(\mathbb{Z}_n)$. Therefore $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{q-1,p^s-1})$. Simply checking, we can see that there are only seven cases satisfying the inequality $\gamma(K_{q-1,p^s-1}) \leq 3$; that is, $n \in \{2^s \times 3, 20, 40, 45, 28, 44, 52\}$. By Theorem 1, we have $\gamma(\Gamma(\mathbb{Z}_{20})) = 1$ and $\gamma(\Gamma(\mathbb{Z}_{28})) = 1$. So $n \in \{2^s \times 3, 40, 45, 44, 52\}$.

For $n = 40$, we set $I = \{8, 16, 24, 32\}$, $J = \{5, 10, 15, 20, 25, 30, 35\}$, and $K = \{4, 12, 28, 36\}$. Note that each vertex in I is adjacent to each vertex in J, and the vertices 10, 20, and 30 are adjacent to each vertex in K, so by Lemma 5, $\gamma(\Gamma(\mathbb{Z}_{40})) \geq 4$.

For the case $n = 2^s \cdot 3$, by Theorem 1, we have $\gamma(\Gamma(\mathbb{Z}_{12})) = 0$ and $\gamma(\Gamma(\mathbb{Z}_{24})) = 1$. If $s = 4$, that is, $n = 48$, we set $I = \{8, 16, 24, 32\}$, $J = \{6, 12, 18, 24, 30, 36, 42\}$, and $K = \{4, 20, 28, 44\}$. Note that each vertex in I is adjacent to each vertex in J, and the vertices 12, 24, and 36 are adjacent to each vertex in K, so by Lemma 5, $\gamma(\Gamma(\mathbb{Z}_{48})) \geq 4$. If $s \geq 5$, we set $I = \langle 2^{s-1} \rangle - \langle 6 \rangle$ and $J = \langle 6 \rangle - \{0\}$. Then $|I| = 4$, $|J| = 2^{s-1} - 1$ and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there is a complete bipartite subgraph $K_{4,2^{s-1}-1}$ in $\Gamma(\mathbb{Z}_n)$. It then follows that $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{4,2^{s-1}-1}) \geq 7$ by Lemma 3.

Case 4 ($s \geq 2$ and $t \geq 2$). We set $I = \langle p^{s-1}q^{t-1} \rangle - \{0\}$. Then $|I| = pq - 1$. Since any two vertices in I are adjacent, there exists a complete subgraph K_{pq-1} in $\Gamma(\mathbb{Z}_n)$. If $pq \geq 11$, then $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{10}) = 4$ by Lemma 2. So $n = 2^s \cdot 3^t$ or $n = 2^s \cdot 5^t$. If $n = 2^s \cdot 3^t$, set $I = \langle 2^s \rangle - \{0\}$ and $J = \langle 3^t \rangle - \{0\}$; then $|I| = 3^t - 1$, $|J| = 2^s - 1$, and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there exists a complete bipartite subgraph $K_{2^s-1,3^t-1}$ in $\Gamma(\mathbb{Z}_n)$. If $e_1 \geq 3$ or $e_2 \geq 3$, then $\gamma(\Gamma(\mathbb{Z}_n)) \geq 4$. So $s = 2$, $t = 2$; that is $n = 36$. If $n = 2^s \cdot 5^t$, similarly, there exists a complete bipartite subgraph $K_{2^s-1,5^t-1}$ in $\Gamma(\mathbb{Z}_n)$. Therefore, $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{2^s-1,5^t-1}) \geq 6$ by Lemma 3.

Now we have proved that if $2 \leq \gamma(\Gamma(\mathbb{Z}_n)) \leq 3$, then $n \in \{35, 36, 44, 50, 45, 52, 54\}$.

In the following, we determine $\gamma(\Gamma(\mathbb{Z}_n))$ for each $n \in \{35, 36, 44, 50, 45, 52, 54\}$.

It is easy to see that $\Gamma(\mathbb{Z}_{35}) = K_{4,6}$. So $\gamma(\Gamma(\mathbb{Z}_{35})) = 2$ by Lemma 3.

For $n = 36$, as mentioned in Case 4 above, $\Gamma(\mathbb{Z}_{36})$ contains a subgraph $K_{3,8}$. So $\gamma(\Gamma(\mathbb{Z}_{36})) \geq \gamma(K_{3,8}) = 2$. $\Gamma(\mathbb{Z}_{36})$ can be embedded into S_2 as shown in Figure 2. So $\gamma(\Gamma(\mathbb{Z}_{36})) = 2$.

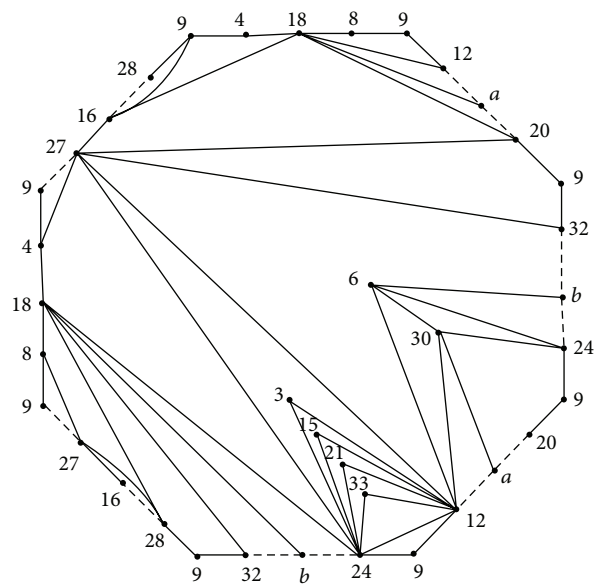

Figure 2: The reduction of $\Gamma(\mathbb{Z}_{36})$.

Since the reductions of the graphs $\Gamma(\mathbb{Z}_{44})$ and $\Gamma(\mathbb{Z}_{52})$ are $K_{3,10}$ and $K_{3,12}$, respectively, we have $\gamma(\Gamma(\mathbb{Z}_{44})) = \gamma(K_{3,10}) = 2$ and $\gamma(\Gamma(\mathbb{Z}_{52})) = \gamma(K_{3,12}) = 3$, respectively.

For $n = 50$, as mentioned in Case 2 above, we have $\gamma(\Gamma(\mathbb{Z}_n)) \geq \gamma(K_{4,5}) = 2$ by Lemma 3. We can embed the reduction of the graph $\Gamma(\mathbb{Z}_{50})$ into S_2 as shown in Figure 3. Thus, $\gamma(\Gamma(\mathbb{Z}_{50})) = 2$.

For $n = 45$, we set $I = \langle 9 \rangle - \{0\}$ and $J = \langle 5 \rangle - \{0\}$. Then $|I| = 4$, $|J| = 8$, and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there is a complete bipartite subgraph $K_{4,8}$ in $\Gamma(\mathbb{Z}_{45})$. It then follows that $\gamma(\Gamma(\mathbb{Z}_{45})) \geq \gamma(K_{4,8}) = 3$. On the other hand, we can embed the reduction of the graph $\Gamma(\mathbb{Z}_{45})$ into S_3 as shown in Figure 4. Thus, $\gamma(\Gamma(\mathbb{Z}_{45})) = 3$.

For $n = 54$, let $I = \langle 6 \rangle - \langle 18 \rangle$ and $J = \langle 9 \rangle - \{0\}$. Then $|I| = 6$, $|J| = 5$, and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there exists a complete bipartite subgraph $K_{5,6}$ in $\Gamma(\mathbb{Z}_{54})$. Therefore $\gamma(\Gamma(\mathbb{Z}_{54})) \geq \gamma(K_{5,6}) = 3$. On the other hand, we can embed the graph $\Gamma(\mathbb{Z}_{54})$ into S_3 as shown in Figure 5.

This completes the proof. □

The final case is that n has more than two prime divisors.

Theorem 8. *Let $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ ($s \geq 3$), where $p_1 < p_2 < \cdots < p_s$ are primes. Then $\gamma(\Gamma(\mathbb{Z}_n)) = 2$ if and only if $n = 30$, and $\gamma(\Gamma(\mathbb{Z}_n)) = 3$ if and only if $n = 42$.*

Proof. Let $I = \langle p_1^{e_1} p_2^{e_2} \rangle - \{0\}$ and $J = \langle n/p_1^{e_1} p_2^{e_2} \rangle - \{0\}$. Then $|I| = p_3^{e_3} \cdots p_s^{e_s} - 1$, $|J| = p_1^{e_1} p_2^{e_2} - 1 \geq 5$ and $I \cap J$ is empty; moreover, every vertex in I is adjacent to each vertex in J. Thus, there exists a complete bipartite subgraph $K_{|I|,|J|}$ in $\Gamma(\mathbb{Z}_n)$. It then follows that $p_3^{e_3} \cdots p_s^{e_s} \leq 7$ as $\gamma(\Gamma(\mathbb{Z}_n)) \leq 3$. So either $n = 2^{e_1} 3^{e_2} \cdot 5$ or $n = 2^{e_1} 3^{e_2} \cdot 7$.

For the former case, set $I = \langle 2^{e_1} 3^{e_2} \rangle - \{0\}$ and $J = \langle 5 \rangle - \{0\}$; then $|I| = 4$, $|J| = 2^{e_1} 3^{e_2} - 1$ and $I \cap J$ is empty. Note that each vertex in I is adjacent to each vertex in J, so there exists a

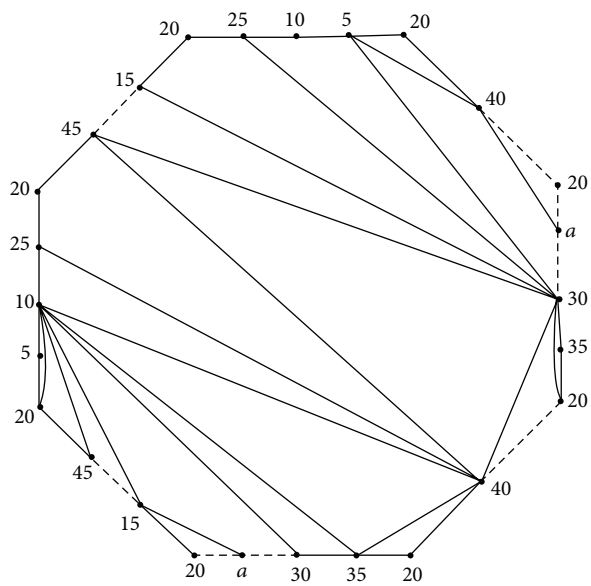

Figure 3: The reduction of $\Gamma(\mathbb{Z}_{50})$.

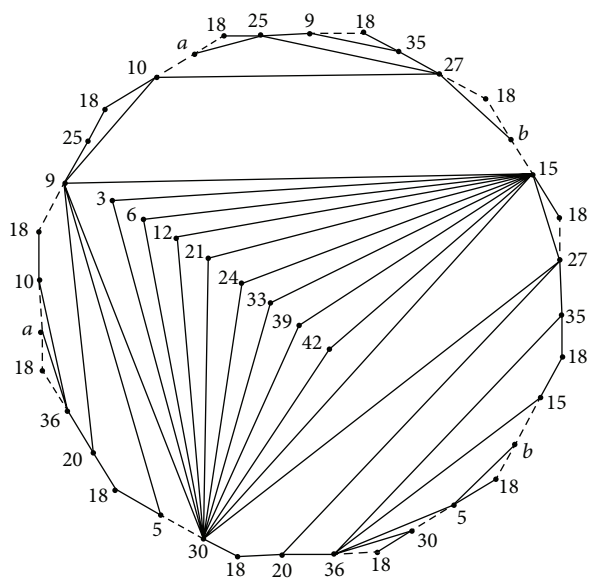

Figure 4: The reduction of $\Gamma(\mathbb{Z}_{45})$.

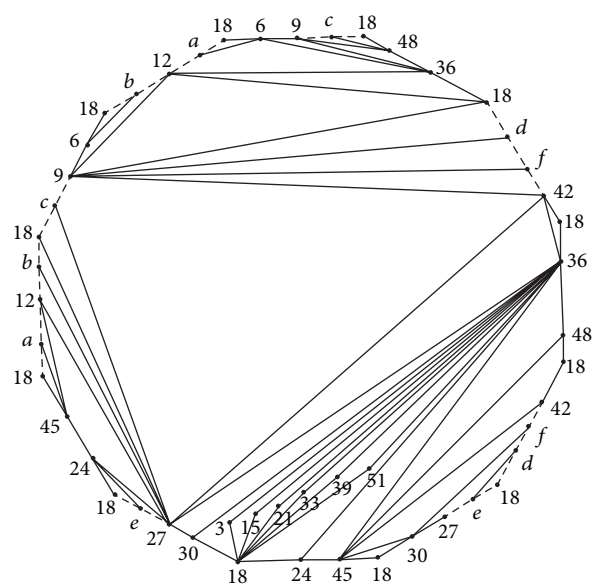

Figure 5: The reduction of $\Gamma(\mathbb{Z}_{54})$.

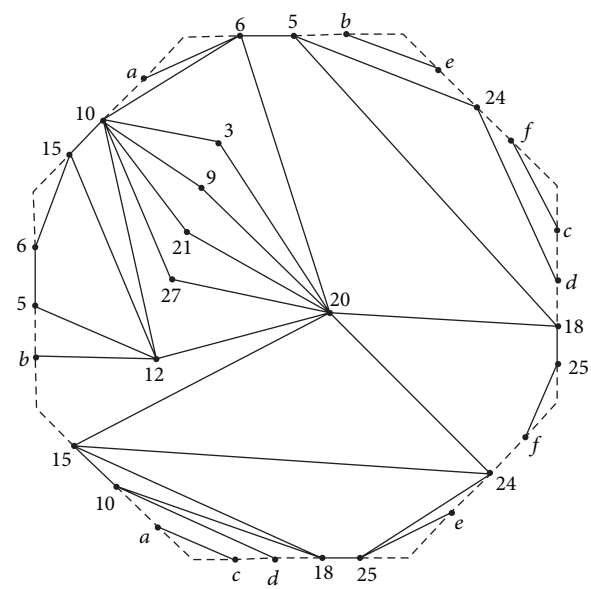

Figure 6: The reduction of $\Gamma(\mathbb{Z}_{30})$.

complete bipartite subgraph $K_{4,2^{e_1}3^{e_2}-1}$ in $\Gamma(\mathbb{Z}_n)$. If $2^{e_1}3^{e_2}-1 \geq 9$, then $\gamma(\Gamma(\mathbb{Z}_n)) \geq 4$ by Lemma 3. So $e_1 = e_2 = 1$; that is, $n = 30$.

Let $I = \langle 5\rangle - \{0\}$ and $J = \langle 6\rangle - \{0\}$. Then $|I| = 5$, $|J| = 4$, and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there exists a complete bipartite subgraph $K_{4,5}$ in $\Gamma(\mathbb{Z}_{30})$. It then follows that $\gamma(\Gamma(\mathbb{Z}_{30})) \geq \gamma(K_{4,5}) = 2$. On the other hand, we can embed $\Gamma(\mathbb{Z}_{30})$ into S_2 as shown in Figure 6, so $\gamma(\Gamma(\mathbb{Z}_{30})) = 2$.

For the latter case, with a similar argument above, we have $n = 42$. Let $I = \langle 7\rangle - \{0\}$ and $J = \langle 6\rangle - \{0\}$. Then $|I| = 5$, $|J| = 6$, and $I \cap J$ is empty. Since each vertex in I is adjacent to each vertex in J, there exists a complete bipartite subgraph $K_{5,6}$ in $\Gamma(\mathbb{Z}_{42})$, which implies that $\gamma(\Gamma(\mathbb{Z}_{42})) \geq \gamma(K_{5,6}) = 3$. On the other hand, we can embed $\Gamma(\mathbb{Z}_{42})$ into S_3 as shown in Figure 7, so $\gamma(\Gamma(\mathbb{Z}_{42})) = 3$.

This completes our proof. □

Now we have completely determined when $\gamma(\Gamma(\mathbb{Z}_n)) = 2$ or 3. We summarize the result by the following theorem.

Theorem 9. *(1) $\gamma(\Gamma(\mathbb{Z}_n)) = 2$ if and only if $n \in \{30, 35, 36, 44, 50, 81\}$.*

(2) $\gamma(\Gamma(\mathbb{Z}_n)) = 3$ if and only if $n \in \{42, 45, 52, 54, 64\}$.

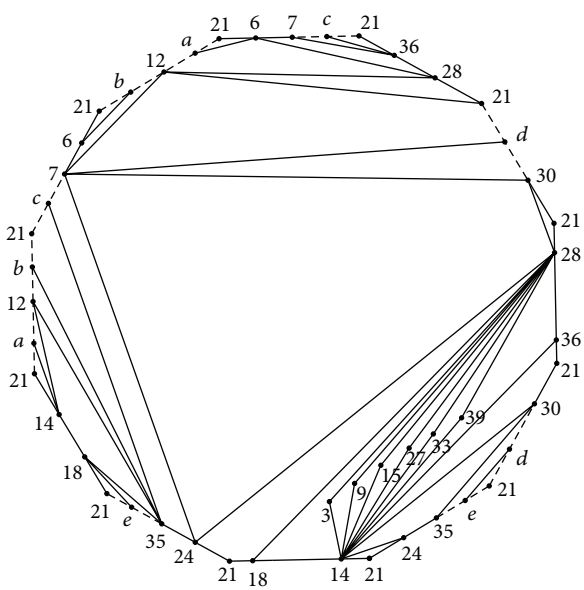

Figure 7: The reduction of $\Gamma(\mathbb{Z}_{42})$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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