Research Article

Some Properties of Multiplicative H_v -Rings of Polynomials over Multiplicative Hyperrings

Utpal Dasgupta

Department of Mathematics, Sree Chaitanya College, Habra-Prafullanagar, 24 Parganas (North), West Bengal 743268, India

Correspondence should be addressed to Utpal Dasgupta; dasgupta_utpal@yahoo.co.in

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The set of all polynomials $R[x]$, over a multiplicative hyperring $(R, +, \cdot)$, form a commutative group with respect to the component-wise addition $(+)$ of the polynomials. For polynomials f, g in $R[x]$, $f * g$ is a set of polynomials whose $(k+1)$ th components $(k \in \mathbb{N} \cup \{0\})$ are chosen from the set $\sum_{i+j=k} a_i \cdot b_j$, where a_i and b_j are the $(i+1)$ th and the $(j+1)$ th components of f and g , respectively. A multiplicative hyperring is polynomially structured if the hyperstructure $(R[x], +, *)$ is a multiplicative H_v -ring. The purpose of the paper is to study the properties of the multiplicative H_v -ring $(R[x], +, *)$, corresponding to those of a polynomially structured multiplicative hyperring R .

1. Introduction

H_v -structures [1] are introduced by Vougiouklis, at the Fourth AHA congress in the year of 1990. Since then, the study of H_v -structure theory has been approached in several directions by many researchers (see [2–5]). The essence of the notion of H_v -structures is to generalize the well-known algebraic hyperstructures (such as hypergroup, hyperring, and hypermodule), simply by replacing some or all axioms of the respective hyperstructures by the corresponding weak axioms. The H_v -structure of our initial concern is multiplicative H_v -ring, studied in [6, 7], which is a commutative group $(R, +)$ along with a hyperoperation $\circ$ such that (i) $(R, \circ)$ is an H_v -semigroup [3, 8] (i.e., a hyperstructure in which $\circ$ is weak associative in the sense that $a \circ (b \circ c) \cap (a \circ b) \circ c \neq \emptyset$, for all $a, b, c \in R$) and (ii) $\circ$ is weak distributive with respect to $+$ (i.e., $a \circ (b + c) \cap (a \circ b + a \circ c) \neq \emptyset$ and $(a + b) \circ c \cap (a \circ c + b \circ c) \neq \emptyset$, for all $a, b, c \in R$). A multiplicative H_v -ring $(R, +, \circ)$ is commutative if $a \circ b = b \circ a$, for all $a, b \in R$. The identity element 0_R of the group $(R, +)$ is said to be absorbing in the multiplicative H_v -ring $(R, +, \circ)$ if $a \circ 0_R = 0_R \circ a = \{0_R\}$, for all $a \in R$. A nonempty finite subset $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ of a multiplicative H_v -ring $(R, +, \circ)$ is called an *identity set* (or *i-set*, in short) [9]

of R if (i) $e_i \notin \langle 0_R \rangle$ for at least one $i = 1, 2, \dots, n$ and (ii) for any $a \in R$, $a \in (\sum_{i=1}^n e_i \circ a) \cap (\sum_{i=1}^n a \circ e_i)$. An element e of R is called a *hyperidentity* of R if the set $\{e\}$ is an *i-set* of R .

Unlike a ring, the equality of the set-expressions $(-a) \circ b$, $a \circ (-b)$, and $-(a \circ b)$ does not hold in general on a multiplicative H_v -ring $(R, +, \circ)$ for any $a, b \in R$. In fact, if $(\mathbb{Z}, +, \cdot)$ is the ring of integers and if $\circ$ is a hyperoperation on $\mathbb{Z}$, defined by $a \circ b = \{a + b, -a - b, a, b\}$, for all $a, b \in \mathbb{Z}$, then $(\mathbb{Z}, +, \circ)$ is a commutative multiplicative H_v -ring, in which $1 \circ (-2) \neq (-1) \circ 2$, $(-1) \circ 2 \neq -(1 \circ 2)$, and $-(1 \circ 2) \neq 1 \circ (-2)$.

A multiplicative H_v -ring $(R, +, \circ)$ is said to satisfy the condition $(\mathcal{R})$ if the set equality $(-a) \circ b = a \circ (-b) = -(a \circ b)$ (called the condition $(\mathcal{R})$ [9]) holds true for any two elements a and b of R . Let $(R, +, \cdot)$ be a ring and $\circ$ a hyperoperation on R , defined by $a \circ b = \{a, b, -a, -b\}$, for all $a, b \in R$. Then, $(R, +, \circ)$ is a multiplicative H_v -ring with condition $(\mathcal{R})$.

We consider now an H_v -structure $(R, +, \circ)$ in which (i) $(R, +)$ is a commutative group, (ii) $(R, \circ)$ is an H_v -semigroup, and (iii) $\circ$ is semidistributive across the operation $+$ (i.e., $a \circ (b + c) \subseteq a \circ b + a \circ c$ and $(a + b) \circ c \subseteq a \circ c + b \circ c$, for all $a, b, c \in R$). This H_v -structure is clearly a multiplicative H_v -ring and we thus call it a semidistributive multiplicative H_v -ring. Henceforth, throughout the paper, a multiplicative

H_v -ring wherever considered will always be assumed to be a semidistributive multiplicative H_v -ring with the condition $(\mathcal{R})$.

In 1982, the notion of multiplicative hyperring is inducted in the study on hyperring theory by Rota, which is subsequently investigated in [10–14]. A commutative group $(R, +)$ endowed with an *associative* hyperoperation $\circ$ is called a multiplicative hyperring [15] if (i) $\circ$ is semidistributive across the operation $+$ on R and (ii) $\circ$ satisfies the condition $(\mathcal{R})$ for elements in R . An associative hyperoperation is eventually weakly associative and thus a multiplicative hyperring is, eventually, a semidistributive multiplicative H_v -ring with condition $(\mathcal{R})$.

Procesi Ciampi and Rota define in [16] polynomials over multiplicative hyperring as follows: let $(R, +, \cdot)$ be a multiplicative hyperring with absorbing zero 0_R and let x be any symbol out of R . Then, a polynomial in x is an expression of the form $f(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots = \sum a_kx^k$, ($k \in \mathbb{N}$, $a_k \in R$), in which $+$ is a connective and only a finite number of the a_k 's (called the coefficients of x^k in $f(x)$) are different from zero (0_R) of R . The degree of a polynomial $f(x) = \sum a_kx^k$ (in short $\deg f(x)$) is a nonnegative integer n such that $a_n \neq 0_R$ and $a_k = 0_R$, for all $k > n$. A polynomial $f(x) = \sum a_kx^k$ over a multiplicative hyperring R will be written as $f(x) = \sum_{k=1}^n a_kx^k$ when and only when $\deg f(x) = n$. For an integer $m \in \mathbb{N}$ and any $a, a_k, b \in R$ ($a \neq 0_R$, $b \neq 0_R$, $k = 1, 2, \dots, m-1$), we write the polynomials $a(x) = 0_Rx^0 + \cdots + 0_Rx^{m-1} + ax^m$ and $f(x) = ax^0 + a_1x^1 + \cdots + a_{m-1}x^{m-1} + bx^m$ simply as ax^m and $a + a_1x + \cdots + a_{m-1}x^{m-1} + bx^m$, respectively. Denote by $R[x]$ the set of all polynomials in x over R and define on $R[x]$ a binary operation $+$ and a hyperoperation $*$ as follows: for any two polynomials $f(x) = \sum a_kx^k$ and $g(x) = \sum b_kx^k$, from $R[x]$, $f(x) + g(x) = \sum (a_k + b_k)x^k$ and $f(x) * g(x) = \{\sum c_kx^k : c_k \in \sum_{i+j=k} a_ib_j\}$, where, for any $a, b \in R$, the juxtaposition ab means the set $a \cdot b$. The purpose of the present paper is to study the properties of the hyperstructure $(R[x], +, *)$, in connection to those of a particular class of multiplicative hyperrings, called polynomially structured multiplicative hyperrings which we describe formally in the following section.

2. Polynomially Structured Multiplicative Hyperring

It is asserted in [16] that for a multiplicative hyperring R , the hyperstructure $(R[x], +, *)$ is always a multiplicative hyperring. But we note here that, given a multiplicative hyperring R with absorbing zero, the hyperoperation $*$ (as is defined in Section 1) does not necessarily induce a multiplicative hyperring structure over the group of polynomials $(R[x], +)$. In fact, we have the following example.

Example 1. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers and $(\mathbb{Z}, +, \circ)$ the multiplicative hyperring where, for any $a, b \in \mathbb{Z}$, $a \circ b = \{2ab, 3ab\}$ (denoting the product $a \cdot b$ of elements in the ring $(\mathbb{Z}, +, \cdot)$ simply by the juxtaposition ab). Consider three

polynomials $f(x) = 1 + 2x$, $g(x) = 2 + 1x$, and $h(x) = 1 + 3x$ over the multiplicative hyperring $(\mathbb{Z}, +, \circ)$.

Then, the set of coefficients of x in the polynomials belonging to $(f(x) * g(x)) * h(x)$ is $M = \{4, 6\} \circ 3 + \{10, 11, 14, 15\} \circ 1$. Again, the set of coefficients of x in the polynomials belonging to $f(x) * (g(x) * h(x))$ is $N = 1 \circ \{14, 15, 20, 21\} + 2 \circ \{4, 6\}$. By a tedious but routine calculation, one can see that $\{57, 74, 82\} \subseteq M$, whereas $\{57, 74, 82\} \cap N = \emptyset$. So, there are some polynomials in $(f(x) * g(x)) * h(x)$, the coefficient of x in each of which is an element of the set $\{57, 74, 82\}$. These polynomials do not belong to $f(x) * (g(x) * h(x))$ (since $\{57, 74, 82\} \cap N = \emptyset$). Thus, $(f(x) * g(x)) * h(x) \not\subseteq f(x) * (g(x) * h(x))$. So, there is no question of claiming $(\mathbb{Z}[x], +, *)$ to be a multiplicative hyperring.

However, we observe that it is possible to construct a multiplicative hyperring R corresponding to which the hyperstructure $(R[x], +, *)$ turns out to be a multiplicative H_v -ring, if not a multiplicative hyperring at all.

Let us consider a commutative group $(G, +)$. Suppose that $\phi \neq A \subseteq \text{Hom}(G, \text{Hom}(G))$ is such that, for any $a, b \in A$ and $x, y \in G$, $(x_a(y))_b = x_a y_b$, where x_a denotes the image of $x \in G$ under $a \in A$ and $x_a y_b$ is simply the mapping composition. Define a hyperoperation $\circ_A$ on G by stating that $x \circ_A y = \{x_a(y) : a \in A\}$. Then we have the following.

Lemma 2. $(G, +, \circ_A)$ is a multiplicative hyperring with absorbing zero.

Proof. Let $x, y, z \in G$. Then, $p \in x \circ_A (y \circ_A z) \Rightarrow p \in x \circ_A s$ (for some $s \in y \circ_A z$) $\Rightarrow p = x_a(s)$ and $s = y_b(z)$ (for some $a, b \in A$) $\Rightarrow p = x_a(y_b(z)) = (x_a y_b)(z) = (x_a(y))_b(z) \Rightarrow p \in (x_a(y)) \circ_A z \Rightarrow p \in (x \circ_A y) \circ_A z$ (since $x_a(y) \in x \circ_A y$). So, $x \circ_A (y \circ_A z) \subseteq (x \circ_A y) \circ_A z$. The reverse inclusion can also be shown to be true by adopting similar arguments. Hence, $x \circ_A (y \circ_A z) = (x \circ_A y) \circ_A z$. Now, $p \in x \circ_A (y + z) \Rightarrow p \in x_a(y + z)$ (for some $a \in A$) $= x_a(y) + x_a(z) \in x \circ_A y + x \circ_A z$. So, $x \circ_A (y + z) \subseteq x \circ_A y + x \circ_A z$. Again, $p \in (y + z) \circ_A x \Rightarrow p \in (y + z)_a x$ (for some $a \in A$) $= (y_a + z_a)x$ (since $a \in \text{Hom}(G, \text{Hom}(G))$) $= y_a(x) + z_a(x) \subseteq y \circ_A x + z \circ_A x$. So, $(y + z) \circ_A x \subseteq y \circ_A x + z \circ_A x$. Moreover, $x \circ_A (-y) = \{x_a(-y) : a \in A\} = \{-(x_a(y)) : a \in A\}$ (since $x_a \in \text{Hom}(G) = -\{x_a(y) : a \in A\} = -(x \circ_A y)$) and $(-x) \circ_A y = \{(-x)_a(y) : a \in A\} = \{(-(x_a))(y) : a \in A\}$ (since $a \in \text{Hom}(G, \text{Hom}(G))$) $= \{x_a(-y) : a \in A\} = x \circ_A (-y)$. Thus, $(G, +, \circ_A)$ is a multiplicative hyperring. Finally, if 0_G denotes the identity element of the group $(G, +)$, then $x \circ 0_G = \{x_a(0_G) : a \in A\} = \{0_G\}$ (since for $a \in A$, $x_a \in \text{Hom}(G)$) and also $0_G \circ x = \{(0_G)_a(x) : a \in A\} = \{0_G\}$ (since, for $a \in \text{Hom}(G, \text{Hom}(G))$, $(0_G)_a$ is the zero homomorphism from G to G). Thus, 0_G is absorbing in the multiplicative hyperring $(G, +, \circ_A)$.

The multiplicative hyperring $(G, +, \circ_A)$, defined in Lemma 2, is called a multiplicative A -hyperring (of course if such a set A exists for the group $(G, +)$). That a multiplicative A -hyperring exists is evident in the following example. $\square$

Example 3. Let $(R, +, \cdot)$ be a ring. Then, as is shown in [9, 13], $(R, +, P)$ is a multiplicative hyperring with absorbing zero,

where $P \in P(R)$ with $|P| \geq 2$ and $\overset{*}{P}$ is the P -hyperoperation [8, 17] on the semigroup $(R, \cdot)$; that is, $x \overset{*}{P} y = \{x \cdot a \cdot y : a \in P\}$ for all $x, y \in R$. Now, for each $a \in P$ and $x \in R$, we define a mapping $x_a : R \rightarrow R$ by stating that $x_a(y) = x \cdot a \cdot y$, for all $y \in R$. Then $x_a \in \text{Hom}(R, +)$. Thus, corresponding to each $a \in P$, we have a mapping $a : (R, +) \rightarrow \text{Hom}(R, +)$, given by $a(x) = x_a$, for any $x \in R$. Then, $P \subseteq \text{Hom}(R, \text{Hom}(R))$. Moreover, for any $x, y, z \in R$ and $a, b \in P$, $(x_a(y))_b(z) = (x \cdot a \cdot y) \cdot b \cdot z = x \cdot a \cdot (y \cdot b \cdot z) = x \cdot a \cdot (y_b(z)) = x_a(y_b(z)) = (x_a y_b)(z)$; that is, $(x_a(y))_b = x_a y_b$. Thus, the hyperoperation $\circ_A$ (for $A = P$) is defined on the group $(R, +)$. Note that, for any $x, y \in R$, $x \circ_A y = \{x_a(y) : a \in A\} = \{x \cdot a \cdot y : a \in P\} = a \overset{*}{P} y$. Thus, $(R, +, \overset{*}{P})$ is a multiplicative A -hyperring, for $A = P$.

Proposition 4. *For a multiplicative A -hyperring $(R, +, \circ_A)$, the hyperoperation $*$ induces a multiplicative H_v -ring structure on the group $(R[x], +)$ of polynomials over R .*

Proof. Let $f(x) = \sum a_k x^k$, $g(x) = \sum b_k x^k$, and $h(x) = \sum d_k x^k$ be three polynomials in $R[x]$. Then, $(f(x) * g(x)) * h(x) = \{\sum t_k x^k : t_k \in \sum_{u+v=k} (\sum_{i+j=u} a_i \circ_A b_j) \circ_A d_v\}$ and $f(x) * (g(x) * h(x)) = \{\sum s_k x^k : s_k \in \sum_{u+v=k} a_u \circ_A (\sum_{i+j=v} b_i \circ_A d_j)\}$. Now, we choose and fix an element $\alpha \in A$. Then, for each $k \in \mathbb{N}_0$,

$$\begin{aligned} p_k &= \sum_{u+v=k} \left(\sum_{i+j=u} (a_i)_\alpha (b_j)_\alpha \right)_\alpha (d_v)_\alpha \\ &\in \sum_{u+v=k} \left(\sum_{i+j=u} a_i \circ_A b_j \right) \circ_A d_v, \\ q_k &= \sum_{u+v=k} (a_u)_\alpha \left(\sum_{i+j=v} (b_i)_\alpha (d_j)_\alpha \right) \\ &\in \sum_{u+v=k} a_u \circ_A \left(\sum_{i+j=v} b_i \circ_A d_j \right). \end{aligned} \quad (1)$$

Again, $p_k = \sum_{u+v=k} (\sum_{i+j=u} (a_i)_\alpha (b_j)_\alpha)_\alpha (d_v)_\alpha = \sum_{u+v=k} \sum_{i+j=u} ((a_i)_\alpha (b_j)_\alpha)_\alpha (d_v)_\alpha$ (since α is a homomorphism) $= \sum_{u+v=k} \sum_{i+j=u} ((a_i)_\alpha (b_j)_\alpha)_\alpha (d_v)_\alpha$ (since $(x_\alpha(y))_\alpha = x_\alpha y_\alpha$), $q_k = \sum_{u+v=k} (a_u)_\alpha (\sum_{i+j=v} (b_i)_\alpha (d_j)_\alpha) = \sum_{u+v=k} \sum_{i+j=v} (a_u)_\alpha ((b_i)_\alpha (d_j)_\alpha)$ (since $(a_u)_\alpha$ is a homomorphism) $= \sum_{u+v=k} \sum_{i+j=v} ((a_u)_\alpha (b_i)_\alpha)_\alpha (d_j)_\alpha$. Clearly then, for each $k \in \mathbb{N}_0$,

$$\begin{aligned} p_k &= q_k \in \left(\sum_{u+v=k} \left(\sum_{i+j=u} a_i \circ_A b_j \right) \circ_A d_v \right) \\ &\cap \left(\sum_{u+v=k} a_u \circ_A \left(\sum_{i+j=v} b_i \circ_A d_j \right) \right). \end{aligned} \quad (2)$$

Hence, $(f(x) * g(x)) * h(x) \cap f(x) * (g(x) * h(x)) \neq \emptyset$. It is shown in [16] that, for any multiplicative hyperring R the hyperoperation $*$ defined on $R[x]$ is semidistributive over the operation $+$ on $R[x]$ and also satisfies the condition $(\mathcal{R})$ for any two polynomials in $R[x]$. Thus, for the multiplicative A -hyperring $(R, +, \circ_A)$, the hyperstructure $(R[x], +, *)$ is a multiplicative H_v -ring (with absorbing zero $0_{R[x]} = 0_R x^0 + 0_R x^1 + 0_R x^2 + \dots$).

We call a multiplicative hyperring $(R, +, \cdot)$ with absorbing zero polynomially structured if $(R[x], +, *)$ is a multiplicative H_v -ring. The class of multiplicative A -hyperrings is a subclass of the class of polynomially structured multiplicative hyperrings (by Proposition 4). Throughout the rest of the paper, R will stand for a polynomially structured multiplicative hyperring. $\square$

3. Polynomials over Integral Hyperrings

An element a ($\neq 0_R$) of a multiplicative H_v -ring $(S, +, \circ)$ is a left (resp., right) divisor of zero in S if there exists $b \in S^*$ (resp., $c \in S^* = S \setminus \{0_S\}$) such that $0_S \in a \circ b$ (resp., $0_S \in c \circ a$) and a *divisor of zero* in S if it is either a left or a right divisor of zero in S . An element a ($\neq 0_S$) of S is a left (resp., right) strong divisor of zero in S if there exists $b \in S^* = S \setminus \{0_S\}$ (resp., $c \in S^*$) such that $a \circ b = \{0_S\}$ (resp., $c \circ a = \{0_S\}$) and a *strong divisor of zero* in S if it is either a left or a right strong divisor of zero in S .

Definition 5. A multiplicative H_v -ring is called an integral H_v -ring if there is no strong divisor of zero in it. A commutative integral H_v -ring is an H_v -domain. A strong integral H_v -ring is a multiplicative H_v -ring in which there is no divisor of zero. A strong H_v -domain is a commutative strong integral H_v -ring. We call an integral H_v -ring (resp., an H_v -domain) S simply an integral hyperring (resp., a hyperdomain) [13], when the H_v -ring S is a multiplicative hyperring.

Before entering into the study of the multiplicative H_v -ring $R[x]$ of polynomials over integral hyperring and hyperdomain, let us go through the following useful observations.

Remark 6. (a) For any polynomially structured multiplicative hyperring R , the identity element $0_{R[x]} = 0_R x^0 + 0_R x^1 + 0_R x^2 + \dots$ of the group $(R[x], +)$ is absorbing in the multiplicative H_v -ring $(R[x], +, *)$.

(b) $R[x]$ is commutative if R is a commutative multiplicative hyperring.

(c) The mapping $\psi : R \rightarrow R[x]$, defined by, for all $r \in R^*$, $\psi(r) = rx^0$ and $\psi(0_R) = 0_{R[x]}$, is a strong monomorphism. In fact, for any $r, s \in R$, $\psi(rs) = \{\psi(c) : c \in rs\} = \{cx^0 : c \in rs\} = (rx^0) * (sx^0) = \psi(r) * \psi(s)$. Thus R can be identified with its isomorphic image in $R[x]$ and for any $r \in R^*$, we can write the polynomial rx^0 simply as r and the zero polynomial $0_{R[x]}$ as 0_R .

(d) If $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ is an i -set of R , then the set $\mathcal{E}_x \subseteq R[x]$ is an i -set in the multiplicative H_v -ring $R[x]$, where, for any $A \in P^*(R)$, A_x denotes the set $\{ax^0 : a \in A\}$. In

(ii) Now suppose that R is a strong hyperdomain with absorbing zero. Consider two polynomials $f(x), g(x) \in R[x] \setminus \{0_{R[x]}\}$. Let $\text{ord} f(x) = p$ and $\text{ord} g(x) = m$. Then, $f(x) = \sum a_k x^k$, where $a_p \neq 0_R$, $a_k = 0_R$ for $k < p$ and

$g(x) = \sum b_k x^k$, where $b_m \neq 0_R$, $b_k = 0_R$ for $k < m$. So, $0_R \notin a_p b_m$ (since R is a strong hyperdomain) and also $a_p b_m = \sum_{i+j=p+m} a_i b_j$ (since $a_k = 0_R$ for $k < p$ and $b_k = 0_R$ for $k < m$). Thus, for any $h(x) = \sum c_k x^k \in f(x) * g(x)$, $c_{p+m} \neq 0_R$ (since $c_{p+m} \in \sum_{i+j=p+m} a_i b_j = a_p b_m$) and $c_k = 0_R$ for all k with $0 \leq k < p+m$. Thus, $\text{ord} h(x) = p+m$ for any $h(x) \in f(x) * g(x)$ and so $\text{ord}(f(x) * g(x)) = p+m$. Hence, $\text{ord} f(x) + \text{ord} g(x) = \text{ord}(f(x) * g(x)) = \text{ord} h(x)$, for all $h(x) \in f(x) * g(x)$. So, the implication is true for $n = 2$ (noting that $[f(x) * g(x)]_r = f(x) * g(x)$). Suppose that, for some integer $k \geq 2$, the implication holds true for each value of n ranging from 2 to $k-1$ and take any $f_i(x) \in R[x] \setminus \{0_{R[x]}\}$ for $i = 1, 2, \dots, k$. Then, $\sum_{i=1}^{k-1} \text{ord} f_i(x) = \text{ord}([f_1(x) * f_2(x) * \dots * f_{k-1}(x)]_r) = \text{ord} h(x)$, for all $h(x) \in [f_1(x) * f_2(x) * \dots * f_{k-1}(x)]_r$. Now, let $g(x) \in [f_1(x) * f_2(x) * \dots * f_k(x)]_r$ be arbitrary. Then, $g(x) \in h(x) * f_k(x)$ for some $h(x) \in [f_1(x) * f_2(x) * \dots * f_{k-1}(x)]_r$. So, $\text{ord} g(x) = \text{ord}(h(x) * f_k(x)) = \text{ord} h(x) + \text{ord} f_k(x) = \sum_{i=1}^{k-1} \text{ord} f_i(x) + \text{ord} f_k(x)$ (since $h(x) \in [f_1(x) * f_2(x) * \dots * f_{k-1}(x)]_r = \sum_{i=1}^k \text{ord} f_i(x)$). Thus, $\sum_{i=1}^k \text{ord} f_i(x) = \text{ord}([f_1(x) * f_2(x) * \dots * f_k(x)]_r) = \text{ord} g(x)$, for all $g(x) \in [f_1(x) * f_2(x) * \dots * f_k(x)]_r$.

Hence, by strong induction the implication follows for any $n \in \mathbb{N}$ with $n \geq 2$. $\square$

Corollary 12. *If R is a strong hyperdomain then the implication*

$$\begin{aligned} f_i(x) \in R[x] \setminus \{0_{R[x]}\} &\implies \sum_{i=1}^n \text{ord} f_i(x) \\ &= \text{ord}([f_1(x) * f_2(x) * \dots * f_n(x)]_l) = \text{ord} g(x), \end{aligned} \quad (6)$$

for all $g(x) \in [f_1(x) * f_2(x) * \dots * f_n(x)]_l$ holds true for any $n \in \mathbb{N}$ with $n \geq 2$.

Proof. Since the multiplicative hyperring R is a hyperdomain, it is commutative and so $R[x]$ is also a commutative multiplicative H_v -ring. Hence, for any $f_i(x) \in R[x] \setminus \{0_{R[x]}\}$ ($i = 1, 2, \dots, k$), we have that $[f_1(x) * f_2(x) * f_3(x) * \dots * f_k(x)]_l = [f_k(x) * f_{k-1}(x) * \dots * f_2(x) * f_1(x)]_r$. Hence, the assertion follows straight from Proposition 11. $\square$

Proposition 13. *If the multiplicative hyperring R is a strong hyperdomain, the multiplicative H_v -ring $R[x]$ is a strong H_v -domain.*

Proof. R being a strong hyperdomain is a commutative multiplicative hyperring. Thus $R[x]$ is a commutative multiplicative H_v -ring. Again, since 0_R is absorbing in R , $0_{R[x]}$ is also absorbing in $R[x]$. Thus we take $f(x), g(x) \in R[x] \setminus \{0_{R[x]}\}$. Then, by Proposition 11, $\text{ord}(f(x) * g(x)) = \text{ord} f(x) + \text{ord} g(x) > 0$. Hence $0_{R[x]} \notin f(x) * g(x)$. So, $R[x]$ is a strong H_v -domain. $\square$

4. $\mathcal{C}$ -Ideals in $R[x]$

A subgroup I of the group $(S, +)$ is called a left (resp., right) H_v -ideal of a multiplicative H_v -ring $(S, +, \circ)$ if, for any $s \in S$

and $a \in I$, $s \circ a \subseteq I$ (resp., $a \circ s \subseteq I$). I is an H_v -ideal of S if it is both a left and a right H_v -ideal of S .

We call an H_v -ideal of a multiplicative H_v -ring S simply a hyperideal when S is a multiplicative hyperring. The notion of a typical hyperideal in a multiplicative hyperring, called $\mathcal{C}$ -ideal, is introduced in [18] to study prime and primary hyperideals of multiplicative hyperrings. A hyperideal I of a multiplicative hyperring S is a $\mathcal{C}$ -ideal if for any $A \in \mathcal{C}$, $A \cap I \neq \emptyset \implies A \subseteq I$, where $\mathcal{C} = \{s_1 \circ s_2 \circ \dots \circ s_n : s_i \in S, n \in \mathbb{N}\} \subseteq P^*(S)$. Following is the definition of a $\mathcal{C}$ -ideal in an arbitrary multiplicative H_v -ring.

Definition 14. A left (resp., right) H_v -ideal I of a multiplicative H_v -ring $(S, +, \circ)$ is called a left (resp., right) $\mathcal{C}$ -ideal if for any type l hyperproduct $[s_1 \circ s_2 \circ \dots \circ s_n]_l$ (resp., type r hyperproduct $[s_1 \circ s_2 \circ \dots \circ s_n]_r$) of elements $s_i \in S$, we have that $[s_1 \circ s_2 \circ \dots \circ s_n]_l \cap I \neq \emptyset \implies [s_1 \circ s_2 \circ \dots \circ s_n]_l \subseteq I$ (resp., $[s_1 \circ s_2 \circ \dots \circ s_n]_r \cap I \neq \emptyset \implies [s_1 \circ s_2 \circ \dots \circ s_n]_r \subseteq I$). An H_v -ideal I of a multiplicative H_v -ring S is called a $\mathcal{C}$ -ideal if it is a left as well as a right $\mathcal{C}$ -ideal in S .

We write $\mathcal{C}\text{-ideal}(S)$ (resp., $\mathcal{C}\text{-ideal}^*(S)$) to denote the set of all (resp., proper) $\mathcal{C}$ -ideals of a multiplicative H_v -ring S . In a commutative multiplicative H_v -ring S , every left $\mathcal{C}$ -ideal is a right $\mathcal{C}$ -ideal and vice versa, since commutativity implies the equality $[s_1 \circ s_2 \circ \dots \circ s_n]_r = [s_n \circ s_{n-1} \circ \dots \circ s_1]_l$, for any $s_i \in R$. Note that in a multiplicative hyperring (even if it is not commutative) a hyperideal is a left $\mathcal{C}$ -ideal (as an H_v -ideal) if and only if it is a right $\mathcal{C}$ -ideal.

Proposition 15. *If R is a strong hyperdomain, then the set $\mathcal{C}\text{-Ideal}^*(R[x])$ is nonempty.*

Proof. For each $n \in \mathbb{N} \cup \{0\}$, let $\mathfrak{C}_n = \{f(x) \in R[x] : \text{ord} f(x) \geq n\} \cup \{0_{R[x]}\}$. Then $\mathfrak{C}_n$ is a subgroup of the group $(R[x], +)$ (since for any $f(x), g(x) \in R[x]$, $\text{ord}(-f(x)) = \text{ord} f(x)$ and $\text{ord}(f(x) + g(x)) = \min\{\text{ord} f(x), \text{ord} g(x)\}$, when $f(x) + g(x) \neq 0_{R[x]}$). Let $g(x) \in R[x]$ and $f(x) \in \mathfrak{C}_n$. If $g(x) = 0_{R[x]}$ or $f(x) = 0_{R[x]}$, then $g(x) * f(x) = \{0_{R[x]}\} \subseteq \mathfrak{C}_n$ (since $0_{R[x]}$ is absorbing in $R[x]$). So, let $g(x) \neq 0_{R[x]}$ and $f(x) \neq 0_{R[x]}$. Then, by Proposition 11(ii), for all $h(x) \in g(x) * f(x)$, $\text{ord} h(x) = \text{ord}(g(x) * f(x)) = \text{ord} f(x) + \text{ord} g(x) > n$. Thus, $g(x) * f(x) \subseteq \mathfrak{C}_n$. Hence, $\mathfrak{C}_n$ is an H_v -ideal of $R[x]$ (since $R[x]$ is a commutative multiplicative H_v -ring). Let $[f_1(x) * f_2(x) * \dots * f_n(x)]_l$ be a type- l hyperproduct of elements of the H_v -ring $R[x]$, such that $[f_1(x) * f_2(x) * \dots * f_n(x)]_l \notin \mathfrak{C}_n$. Then, $f_i(x) \neq 0_{R[x]}$ for each i (since $0_{R[x]} \in \mathfrak{C}_n$ is absorbing in $R[x]$). Thus, $\text{ord}([f_1(x) * f_2(x) * \dots * f_n(x)]_l) = \text{ord} g(x)$, for all $g(x) \in [f_1(x) * f_2(x) * \dots * f_n(x)]_l$. Now, since $[f_1(x) * f_2(x) * \dots * f_n(x)]_l \notin \mathfrak{C}_n$, so there exists $h(x) \in [f_1(x) * f_2(x) * \dots * f_n(x)]_l$ such that $h(x) \notin \mathfrak{C}_n$. Then, for any $g(x) \in [f_1(x) * f_2(x) * \dots * f_n(x)]_l$, $\text{ord} g(x) = \text{ord} h(x) < n$. Hence, by definition of $\mathfrak{C}_n$, $[f_1(x) * f_2(x) * \dots * f_n(x)]_l \cap \mathfrak{C}_n = \emptyset$ and so $\mathfrak{C}_n$ is a left $\mathcal{C}$ -ideal and thus a $\mathcal{C}$ -ideal of $R[x]$ (since $R[x]$ is commutative). Hence, $\mathfrak{C}_n \in \mathcal{C}\text{-Ideal}^*(R[x])$, for all $n > 0$.

Since the intersection of left $\mathcal{C}$ -ideals of a multiplicative H_v -ring S is also a left $\mathcal{C}$ -ideal of S and S is itself a left $\mathcal{C}$ -ideal, so the smallest left $\mathcal{C}$ -ideal containing a subset A of S ,

being naturally called the left $\mathcal{C}$ -ideal generated by A , exists and is, in fact, the intersection of all left $\mathcal{C}$ -ideals containing A . The left $\mathcal{C}$ -ideal generated by a left H_v -ideal I of S is called the left $\mathcal{C}$ -closure of I and is denoted by $\mathcal{C}_l(I)$. Clearly, $\mathcal{C}_l(\mathcal{C}_l(I)) = \mathcal{C}_l(I)$ for a left H_v -ideal I . $\mathcal{C}_r(I)$ and $\mathcal{C}(I)$, respectively, denote the right $\mathcal{C}$ -closure of a right H_v -ideal I and the $\mathcal{C}$ -closure of an H_v -ideal I of S . For an H_v -ideal I of a commutative multiplicative H_v -ring S , $\mathcal{C}_l(I) = \mathcal{C}_r(I) = \mathcal{C}(I)$. The following lemma presents a description of the set $\mathcal{C}_l(I)$, for a left H_v -ideal I of a multiplicative H_v -ring S . The set $\mathcal{C}_r(I)$ for a right H_v -ideal I can be described dually. $\square$

Lemma 16. *Let S be a multiplicative H_v -ring with an i -set $\mathcal{E}$ and let $\mathcal{L}$ denote the set of all left H_v -ideals of S . $F : P(S) \rightarrow P(S)$ is a mapping defined by*

$$F(A) = \bigcup \left\{ \sum_{i=1}^n [r_{i1} \circ r_{i2} \circ \cdots \circ r_{im_i}]_l : n, m_i \in \mathbb{N}, r_{i1} \in S \right. \\ \left. \text{satisfying } [r_{i1} \circ r_{i2} \circ \cdots \circ r_{im_i}]_l \cap A \neq \emptyset \right\}, \quad (7)$$

for any $A \in P(S)$. Then for any $I, I_1, I_2 \in \mathcal{L}$, we have the following: (i) $F(I) \in \mathcal{L}$, (ii) $I \subseteq F(I)$, (iii) $I_1 \subseteq I_2 \Rightarrow F(I_1) \subseteq F(I_2)$, (iv) $F(I) = I$ if and only if I is a left $\mathcal{C}$ -ideal of S , and (v) left $\mathcal{C}$ -closure of I is $\mathcal{C}_l(I) = \bigcup_{k=1}^{\infty} F^k(I)$, where for any $k \in \mathbb{N}$, F^k denotes the $(k-1)$ -times mapping composition of F .

Proof. All the assertions made in this lemma can be established by adopting the arguments that are applied in proving well-known analogous results on “complete closure of a set” in semihypergroup theory (see [3, 8]). $\square$

Proposition 17. *Let R be a strong hyperdomain with an i -set. Then, for any H_v -ideal I of the multiplicative H_v -ring $R[x]$, $\text{ord}(\mathcal{C}(I)^*) \geq \text{ord} I^*$, where for any H_v -ideal J of $R[x]$, $J^* = J \setminus \{0_{R[x]}\}$.*

Proof. Since R is a (strong) hyperdomain, the multiplicative H_v -ring is commutative. So, for any H_v -ideal I of $R[x]$, $\mathcal{C}(I) = \mathcal{C}_l(I) = \bigcup_{k=1}^{\infty} F^k(I)$ (by Lemma 16(v)). For any $n \in \mathbb{N}$, $f(x) \in F^n(I)^* \Rightarrow f(x) \in \sum_{i=1}^m A_i$, for some type- l hyperproducts $A_i = [f_{i1}(x) * f_{i2}(x) * \cdots * f_{im_i}(x)]_l$ of elements of $R[x]$, satisfying $A_i \cap F^{n-1}(I) \neq \emptyset$. Since here $R[x]$ is a strong H_v -domain (by Proposition 13) with absorbing zero, we may assume that $0_{R[x]} \notin A_i$ for each i (since $\{0_{R[x]}\} + A = A$ for any $A \in P^*(R[x])$). Then, for each i ($i = 1, 2, \dots, m$) and j_i ($j_i = 1, 2, \dots, m_i$), $f_{ij_i}(x) \neq 0_{R[x]}$. Thus, by Corollary 12, $\text{ord} A_i = \text{ord} g_i(x)$ for any $g_i(x) \in A_i$. Now since, for each i , $A_i \cap F^{n-1}(I) \neq \emptyset$, so there exists $g_i(x) \in A_i$ such that $g_i(x) \in F^{n-1}(I)^*$. Then, for any $f(x) \in F^n(I)^*$, $\text{ord} f(x) \geq \sum_{i=1}^m \text{ord} A_i = \min\{\text{ord} g_i(x) : i\} \geq \text{ord}(F^{n-1}(I)^*)$. Consequently, $\text{ord}(F^n(I)^*) \geq \text{ord}(F^{n-1}(I)^*)$, for any $n \in \mathbb{N}$. Now, let $f(x) \in \mathcal{C}(I)^*$ be arbitrary. Then $f(x) \in F^n(I)^*$ for some $n \in \mathbb{N}$. So, we have that $\text{ord} f(x) \geq \text{ord}(F^n(I)^*) \geq \text{ord}(F^{n-1}(I)^*) \geq \cdots \geq \text{ord}(F(I)^*) \geq \text{ord} I^*$. Thus, $\text{ord}(\mathcal{C}(I)^*) \geq \text{ord} I^*$. $\square$

5. Polynomials over Multiplicative Hyperfield

A nonzero element a of a multiplicative H_v -ring $(S, +, \circ)$ with an i -set $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ is referred to be an $\mathcal{E}$ -invertible element (or an $\mathcal{E}$ -unit) of S if, for each $i = 1, 2, \dots, n$, ($n \in \mathbb{N}$), there exist $A_{ij} \in \mathcal{H}_l(a) = \{[s_p \circ s_{p-1} \circ \cdots \circ s_1 \circ a]_l : s_i \in S, p \in \mathbb{N}\}$ and $B_{ik} \in \mathcal{H}_r(a) = \{[a \circ s_1 \circ s_2 \circ \cdots \circ s_q]_r : s_i \in S, q \in \mathbb{N}\}$ such that $e_i \in (\sum_{j=1}^m A_{ij}) \cap (\sum_{k=1}^l B_{ik})$. An element a ($\neq 0_S$) of the multiplicative H_v -ring S with a hyperidentity e is said to be e -hyperinvertible (or an e -hyperunit) in S if there exist $A \in \mathcal{H}_l(a)$ and $B \in \mathcal{H}_r(a)$ such that $e \in A \cap B$.

If $a \in S$ is an $\mathcal{E}$ -unit (resp., e -hyperunit) in a multiplicative H_v -ring $(S, +, \circ)$ with two i -sets $\mathcal{E}$ and $\mathcal{E}'$ (resp., with two hyperidentities e and e'), then one can easily verify that a is also an $\mathcal{E}'$ -unit (resp., an e' -hyperunit) in S . We thus call an $\mathcal{E}$ -unit (resp., an e -hyperunit) of a multiplicative H_v -ring S simply a unit (resp., a hyperunit). Denote by $U(S)$ and $U_h(S)$, respectively, the sets of units and hyperunits of a multiplicative H_v -ring S .

An H_v -ideal I ($\neq S$) of a multiplicative H_v -ring S is maximal in S if, for any H_v -ideal J of S , $I \subsetneq J \subseteq S \Rightarrow J = S$. For a commutative multiplicative H_v -ring S with an i -set, this is immediate to observe that $a \in U(S)$ if and only if $a \notin M$, for any maximal H_v -ideal M of S .

Proposition 18. *Let the multiplicative hyperring R be commutative and contain a hyperidentity e . Then, for a polynomial $f(x) = \sum a_k x^k \in R[x]$, $f(x) \in U_h(R[x])$ if and only if $a_0 \in U_h(R)$.*

Proof. Since e is a hyperidentity in the multiplicative hyper-ring R , $ex^0 \in R[x]$ is a hyperidentity in the multiplicative H_v -ring $R[x]$. Now let $f(x) = \sum a_k x^k \in U_h(R[x])$. Then, there exist $g_i(x) = \sum b_{i,k} x^k \in R[x]$ ($i = 1, 2, \dots, n$; $n \in \mathbb{N}$) such that $ex^0 \in [f(x) * g_1(x) * \cdots * g_n(x)]_r$. So, $e \in a_0(b_{1,0}b_{2,0} \cdots b_{n,0})$ and thus there exists $b \in b_{1,0}b_{2,0} \cdots b_{n,0} \subseteq R$ such that $e \in a_0b$ whence $a_0 \in U_h(R)$ (since R is commutative).

Conversely, let a_0 be hyperinvertible in R with respect to the hyperidentity e . Then there exists $b_0 \in R$ such that $e \in a_0b_0$. We assert that there is a sequence $\{b_k\}_{k \in \mathbb{N}}$ in R whose n th term b_n ($n \in \mathbb{N}$) is inductively defined so as to satisfy the relation that

$$0_R \in a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0 \cdots (\#). \quad (8)$$

In fact, we see that $0_R \in 0_Rb_0 = (-a_1 + a_1)b_0 \subseteq (e(-a_1) + a_1)b_0 \subseteq ((a_0b_0)(-a_1) + a_1)b_0 \subseteq (a_0b_0)(-a_1)b_0 + a_1b_0 = a_0((-b_0)a_1b_0) + a_1b_0 \Rightarrow$ there exist $b_1 \in (-b_0)a_1b_0$ such that $0_R \in a_0b_1 + a_1b_0$ (the relation $\#$ for $n = 1$). Suppose, for some $m \geq 1$, the terms b_k ($1 \leq k \leq m-1$) are defined in such a way that each b_k ($1 \leq k \leq m-1$) satisfies the relation $(\#)$ for k . Then $(a_1b_{m-1} + a_2b_{m-2} + \cdots + a_{m-1}b_1 + a_mb_0)$ is defined to be a nonempty subset of R . Let $t \in a_1b_{m-1} + a_2b_{m-2} + \cdots + a_{m-1}b_1 + a_mb_0$. Then $0_R = -t + t \in e(-t) + t \subseteq (a_0b_0)(-t) + t \subseteq a_0(-b_0)(a_1b_{m-1} + a_2b_{m-2} + \cdots + a_{m-1}b_1 + a_mb_0) + (a_1b_{m-1} + a_2b_{m-2} + \cdots + a_{m-1}b_1 + a_mb_0) \Rightarrow$ there exists $b_m \in (-b_0)(a_1b_{m-1} + a_2b_{m-2} + \cdots + a_{m-1}b_1 + a_mb_0)$ such that $0_R \in a_0b_m + a_1b_{m-1} + a_2b_{m-2} + \cdots + a_mb_0$ (the relation $\#$ for $k = m$). Hence the assertion is true for all $n \in \mathbb{N}$. Thus

consider the polynomial $g(x) = \sum b_k x^k \in R[x]$. Then from the definition of $*$, $f(x) * g(x) = \{\sum c_k x^k : c_0 \in a_0 b_0 \text{ and } c_k \in a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \dots + a_k b_0 \text{ for } k \in \mathbb{N}\}$. Thus, $ex^0 \in f(x) * g(x)$ (due to relation $\#$ and since $e \in a_0 b_0$). So, $f(x)$ is hyperinvertible in $R[x]$; that is, $f(x) \in U_h(R[x])$.

The (left, right) H_v -ideal of a multiplicative H_v -ring S generated by $A \in P(S)$ is the smallest (resp., left, right) H_v -ideal of S containing A which is denoted by (resp., $\langle A \rangle_l$, $\langle A \rangle_r$) $\langle A \rangle$. The principal (left, right) H_v -ideal of the multiplicative H_v -ring S generated by an element a of S , denoted by (resp., $\langle a \rangle_l$, $\langle a \rangle_r$) $\langle a \rangle$, is the (resp., left, right) H_v -ideal (resp., $\langle a \rangle_l$, $\langle a \rangle_r$) $\langle a \rangle$ of the multiplicative H_v -ring S .

If the multiplicative H_v -ring S has an i -set, then for any $a \in S$, $\langle a \rangle_l = \overline{\mathcal{H}_l(a)} = \bigcup \{\sum_{i=1}^n A_i : A_i \in \mathcal{H}_l(a), n \in \mathbb{N}\}$ and $\langle a \rangle_r = \overline{\mathcal{H}_r(a)} = \bigcup \{\sum_{i=1}^n A_i : A_i \in \mathcal{H}_r(a), n \in \mathbb{N}\}$. $\square$

Definition 19. A commutative multiplicative H_v -ring S with an i -set $\mathcal{E}$ is called a multiplicative H_v -field (resp., an inversive multiplicative H_v -field) if $S \setminus \langle 0_S \rangle \subseteq U(S)$ (resp., $S \setminus \langle 0_S \rangle \subseteq U_h(S)$). If a multiplicative H_v -field (resp., an inversive multiplicative H_v -field) is a multiplicative hyperring, then we call it a multiplicative hyperfield (resp., an inversive multiplicative hyperfield).

Proposition 20. Let F be a polynomially structured inversive multiplicative hyperfield. Then a polynomial $f(x) = \sum a_k x^k \in F[x] \setminus \{0_{F[x]}\}$ is hyperinvertible in $F[x]$ if and only if $\text{ord } f = 0$.

Proof. If F is an inversive multiplicative hyperfield, then by Proposition 18, any polynomial $f(x) = \sum a_k x^k \in F[x] \setminus \{0_{F[x]}\}$ is hyperinvertible in $F[x]$ if and only if $a_0 \neq 0_F$. Hence, the result follows. $\square$

Definition 21. A commutative multiplicative H_v -ring S is called a principal $\mathcal{E}$ -ideal H_v -ring if every $\mathcal{E}$ -ideal of S is a principal H_v -ideal. A principal $\mathcal{E}$ -ideal H_v -ring which is a (strong) H_v -domain is called a principal $\mathcal{E}$ -ideal (strong) H_v -domain.

Proposition 22. Let F be a polynomially structured inversive multiplicative hyperfield. Then, the multiplicative H_v -ring $F[x]$ of polynomials over F is a principal $\mathcal{E}$ -ideal H_v -ring.

Proof. Let e be a hyperidentity of the inversive multiplicative hyperfield F . Then, the polynomial ex^0 is a hyperidentity in $F[x]$ and thus $F[x] = \langle ex^0 \rangle$ (since F is commutative). So, let I be any proper $\mathcal{E}$ -ideal of $F[x]$. If $I = \{0_{F[x]}\}$, then I is the principal hyperideal $\langle 0_{F[x]} \rangle$ (since, for F having absorbing zero, $0_{F[x]}$ is absorbing in $F[x]$). Suppose that $I \neq \{0_{F[x]}\}$. Then, take a nonzero polynomial $f(x) = \sum a_k x^k \in I$ such that $\text{ord } f(x) \leq \text{ord } t(x)$ for any $t(x) \in I^*$. Let us write $\text{ord } f(x) = m$. Then, $a_m \neq 0_F$ and $a_k = 0_F$ for any $0 \leq k < m$. Consider then the polynomial $g(x) = \sum_{k \in \mathbb{N}_0}^* b_k x^k \in F[x]$, where $b_k = a_{m+k}$. Then, clearly $f(x) \in (ex^m) * g(x)$. Also, by Proposition 18, $g(x)$ is hyperinvertible in $F[x]$. Thus there exists $h(x) \in F[x]$ such that $ex^0 \in g(x) * h(x)$. Now, $f(x) \in (ex^m) * g(x) \Rightarrow f(x) * h(x) \subseteq [ex^m * g(x) * h(x)]_r$.

Again, $f(x) * h(x) \subseteq I$ (since $f(x) \in I$). So, $[ex^m * g(x) * h(x)]_r \cap I \neq \emptyset$. Hence, $[ex^m * g(x) * h(x)]_r \subseteq I$ (since I is a $\mathcal{E}$ -ideal and every $\mathcal{E}$ -ideal is a right $\mathcal{E}$ -ideal). Now, $[ex^m * g(x) * h(x)]_l \cap [ex^m * g(x) * h(x)]_r \neq \emptyset$, whence $[ex^m * g(x) * h(x)]_l \cap I \neq \emptyset$ (since $[ex^m * g(x) * h(x)]_r \subseteq I$). Consequently, $[ex^m * g(x) * h(x)]_l \subseteq I$ (since I is a left $\mathcal{E}$ -ideal). Then, $ex^m \in (ex^m) * (ex^0) \subseteq ex^m * (g(x) * h(x)) = [ex^m * g(x) * h(x)]_l \subseteq I$. Thus, $\langle ex^m \rangle \subseteq I$.

Now, let $h(x) = \sum b_k x^k \in I^*$ be arbitrary. Suppose that $\text{ord } h = n$. Then, $b_n \neq 0_F$ and $b_k = 0_F$ for any $0 \leq k < n$. By choice of $f(x)$ from I , here $m \leq n$. So, one can define a polynomial $g(x) = \sum c_k x^k \in F[x]$, where $c_k = 0_F$ for all $0 \leq k \leq n - m - 1$ and $c_k = b_{m+k}$ for all $k \geq n - m$. Clearly then $h(x) \in ex^m * g(x) \subseteq \langle ex^m \rangle$, whence $I \subseteq \langle ex^m \rangle$. Thus, $I = \langle ex^m \rangle$. $\square$

Remark 23. In a ring, an invertible element can never be a divisor of zero. This not true in general for a multiplicative hyperring. In fact, on the commutative group of integers $(\mathbb{Z}, +)$, if we define a hyperoperation $\circ$ by stating that $x \circ y = \{0, xy\}$, for all $x, y \in \mathbb{Z}$, then $(\mathbb{Z}, +, \circ)$ is a commutative multiplicative hyperring with a hyperidentity 1. Every nonzero element of $(\mathbb{Z}, +, \circ)$ is a zero divisor and $1, -1 \in \mathbb{Z}$ are, in particular, hyperunits of $(\mathbb{Z}, +, \circ)$. To get a parity with the ring theory in this regard, we perceive the notion of strong hyperinvertibility of an element of a multiplicative hyperring.

Definition 24. A hyperinvertible element of a multiplicative hyperring with a hyperidentity is said to be strongly hyperinvertible (or a strong hyperunit), if it is not a zero divisor in that multiplicative hyperring. A commutative multiplicative hyperring with absorbing zero and a hyperidentity e is said to be a strongly inversive multiplicative hyperfield if each of its nonzero elements is a strong hyperunit.

Example 25. Let $R = \{a\sqrt{2} + b\sqrt{3} : a, b \in \mathbb{Q}\}$ and $A = \{\sqrt{2}, \sqrt{3}\}$. Then, with respect to usual addition $+$ of reals, $(R, +)$ is a commutative group, with identity 0. On R , $\circ$ is a hyperoperation defined by

$$\begin{aligned} (a\sqrt{2} + b\sqrt{3}) \circ (c\sqrt{2} + d\sqrt{3}) \\ = \{(a\sqrt{2} + b\sqrt{3}) \cdot t \cdot (c\sqrt{2} + d\sqrt{3}) : t \in A\} \quad (9) \\ = \{p\sqrt{2} + 2q\sqrt{3}, 3q\sqrt{2} + p\sqrt{3}\}, \end{aligned}$$

where $p = 2ac + 3bd$ and $q = bc + ad$.

Then, $(R, +, \circ)$ is a strongly inversive multiplicative hyperfield which is polynomially structured.

Definition 26. A local ($\mathcal{E}$ -local) multiplicative H_v -ring is a commutative multiplicative H_v -ring with an i -set, which has a unique maximal H_v -ideal (resp., $\mathcal{E}$ -ideal).

Proposition 27. Let F be a polynomially structured strongly inversive multiplicative hyperfield. Then, (i) $F[x]$ is a principal $\mathcal{E}$ -ideal strong H_v -domain, (ii) for any $f \in F[x] \setminus \{0_{F[x]}\}$, there exist a hyperinvertible element $g \in F[x]$ and $m \in \mathbb{N} \cup \{0\}$ such that $f \in (ex^m) * g$, where e is a hyperidentity

in F , and (iii) $F[x]$ is a local as well as a $\mathcal{C}$ -local multiplicative H_v -ring.

Proof. (i) Here F is a strongly inversive multiplicative hyperfield. So, F is a strong hyperdomain. Thus, by Proposition 13, the multiplicative H_v -ring $F[x]$ is a strong H_v -domain. Again, by Proposition 22, $F[x]$ is a principal $\mathcal{C}$ -ideal H_v -ring (since F is an inversive multiplicative hyperfield). So, by Definition 21, $F[x]$ is a principal $\mathcal{C}$ -ideal strong H_v -domain.

(ii) Let $f \in F[x] \setminus \{0_{F[x]}\}$ and $J = \langle f \rangle$. Then, $J = \overline{\mathcal{H}_r(f)}$. So, for any $h \in J^*$, there exist $A_i = [f * f_{i1} * f_{i2} * \cdots * f_{im_i}]_r \in \mathcal{H}_r(f)$ ($i = 1, 2, \dots, k$; $k, m_i \in \mathbb{N}$) such that $h \in \sum_{i=1}^k A_i$. Thus, $h = \sum_{i=1}^k t_i$ for some $t_i \in A_i$. Since $F[x]$ is a strong H_v -domain (by Proposition 13) with absorbing zero, so $0_{F[x]} \in A_i = [f * f_{i1} * f_{i2} * \cdots * f_{im_i}]_r \Rightarrow [f * f_{i1} * f_{i2} * \cdots * f_{im_i}]_r = \{0_{F[x]}\}$. Hence, without any loss of generality (since $\{0_{F[x]}\} + A = A$ for any $A \in P^*(F[x])$), we may assume that $0_{F[x]} \notin A_i$ for any i (since $h \neq 0_{F[x]}$). Then, for any $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, m_i$, $f_{ij} \neq 0_{F[x]}$. So, by Proposition 11(ii), $\text{ord } t_i = \text{ord}(A_i) \geq \text{ord } f$ and so $\text{ord } h = \min\{\text{ord } t_i : i\} \geq \text{ord } f$. Thus $\text{ord } J^* \geq \text{ord } f$. So, by Proposition 17, f is a nonzero polynomial in the $\mathcal{C}$ -ideal $\mathcal{C}(J)$, such that $\text{ord } f \leq \text{ord } h$ for all $h \in \mathcal{C}(J)^*$. So, there exists an invertible element $g \in F[x]$ and an integer $m \geq 0$ such that $f \in (ex^m) * g$, where e is a hyperidentity in F (see proof of Proposition 22).

(iii) For any integer $m \geq 1$, since $ex^m \in (ex) * (ex^{m-1})$, so $\langle ex^m \rangle \subseteq \langle ex \rangle$. Again, for any H_v -ideal I of the multiplicative H_v -ring $F[x]$, if $f \in I^*$ is such that $\text{ord } f \leq \text{ord } g$ for all $g \in I^*$, then $I \subseteq \langle ex^m \rangle$, where $m = \text{ord } f$. So, $\langle ex \rangle$ is the unique maximal H_v -ideal in $F[x]$. Thus, $F[x]$ is a local multiplicative H_v -ring. Now, $ex \in \mathfrak{C}_1$ (as defined in Proposition 15). So, $\text{ord}(ex) \leq \text{ord } f$ for any $f \in \mathfrak{C}_1^*$. Thus, $\mathfrak{C}_1$ being a $\mathcal{C}$ -ideal of $F[x]$, we have $\mathfrak{C}_1 = \langle ex \rangle$ (see proof of Proposition 22). Hence, $F[x]$ is a $\mathcal{C}$ -local multiplicative H_v -ring. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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