Research Article

Stability of Hybrid Stochastic Systems with Time-Delay

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This paper develops some criteria for a kind of hybrid stochastic systems with time-delay, which improve existing results on hybrid systems without considering noises. The improved results show that the presence of noise is quite involved in the stability analysis of hybrid systems. New results can be used to analyze the stability of a kind of stochastic hybrid impulsive and switching neural networks (SHISNN). Therefore, stability analysis of SHISNN can be turned into solving a linear matrix inequality (LMI).

1. Introduction

With the development of social production, many practical systems cannot be modeled by linear time-invariant systems. In this case, hybrid systems are employed to model many practical systems. A hybrid system is a dynamical system with continuous dynamics, discrete dynamics, and the interaction between them (see, e.g., [1–3]). Hybrid systems are important from both the practical and theoretical point of view. In fact, hybrid systems naturally represent a wide class of practical systems which are subject to known or unknown abrupt parameter variations and which undergo sudden change of system structures due to the failure of a component. Another practical motivation for studying hybrid dynamical systems originates from the fact that the hybrid control scheme provides an effective approach for controlling highly nonlinear complex dynamical systems and systems with uncertain and/or unknown parameters. From theoretical viewpoints, the interactions between low-level continuous dynamics and high-level discrete logics, mainly governed by the switching and impulsive mechanisms, bring new challenges beyond the conventional system theory. An area of particular interest has been the analysis of stability of hybrid dynamical systems (see, e.g., [4–9]).

Impulsive stabilization of dynamical systems has attracted increasing interests in fields such as population dynamics, automatic control, drug administration, and

communication networks (see [4, 5, 8]) and references therein). In many cases, impulsive stabilization may give better performance than continuous stabilization since the former is implemented only at impulsive instants (see [10]) while the latter does so at every moment (see [5]). Even in some cases, only impulsive stabilization can be used for control purpose. For instance, a central bank cannot change its interest rate every day in order to regulate the money supply in a financial market. The fundamental theoretic and systemic method of impulsive dynamical systems have been established in the recent years (see [1, 5]).

On the other hand, many real systems are subject to stochastic effects, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. In the past decade years, the stability and stabilization of stochastic dynamic systems have been intensively investigated. Hence, the theory of hybrid stochastic systems has attracted increasing attention because the relating problems are not only academically challenging, but also of practical importance in many branches of science and engineering (see, e.g., [10, 11]). These systems are adequate mathematical models for many processes and phenomena studied in biology, physics technology, electric power systems, population dynamics, solar-powered systems, and so on (see, e.g., [12]). Consequently, the stability analysis of hybrid

stochastic systems has been studied by many researchers (see, e.g., [13–18]). For example, the classical stochastic analysis theory studies stability not only in moment sense but also in almost sure sense (see, e.g., [13]). In paper [14], some sufficient conditions for the existence and global p -exponential stability of periodic solution for impulsive stochastic neural networks with delays are given. In paper [15–18], stability of some kinds of stochastic equations is discussed.

However, among the existing results, up to now, very little is known about the stability of hybrid stochastic systems (HSS) with time-delay. In this paper, the stability of HSS with time-delay has been discussed. Some new results of the stability of HSS with time-delay have been attained in this paper. New results can be used to analyze the stability of a kind of stochastic hybrid impulsive and switching neural networks (SHISNN). Therefore, stability analysis of SHISNN can be turned into solving a linear matrix inequality.

This paper is organized as follows. In Section 2, the hybrid stochastic system model and some kinds of stability definition are defined. In Section 3, several criteria for p th moment global asymptotical and exponential stability are established. In Section 4, 2th moment global asymptotical and exponential stability for a kind of stochastic hybrid impulsive and switching neural networks is presented.

2. Preliminaries

Throughout this paper, unless otherwise specified, we will employ the following definitions. Let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a complete probability space with filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and F_0 contains all P -null sets) and let $E[\cdot]$ be the expectation operator with respect to the probability space. $B(t) = (B_1(t), \dots, B_n(t))^T$ is n -dimensional Brownian defined on $(\Omega, F, \{F_t\}_{t \geq 0}, P)$. $\|\cdot\|$ is Euclidean norm of R^n .

Consider the following HSS with time-delay:

$$\begin{aligned} dx(t) &= f_{\sigma(k)}(t, x(t), x(t-\tau)) dt \\ &\quad + g_{\sigma(k)}(t, x(t), x(t-\tau)) dB(t), \quad t \in [t_{k-1}, t_k), \\ \Delta x(t_k) &= x(t_k) - x(t_k^-) = J_k(t_k^-, x(t_k^-)), \quad t = t_k, \\ x(t_0, \sigma(k)) &= \xi(t_0 + s), \quad s \in [-\tau, 0), \\ x(t_k) &= Z_{\sigma(k)}, \end{aligned} \quad (1)$$

where $t \in R^+$, $x \in R^n$ is the state variable, $t_0 \geq 0$ is the initial time, $\sigma(k) : N \rightarrow I, I = \{1, 2, \dots, m\}, N$ is the positive integers, the time sequence $\{t_k\}$ satisfies

$$0 \leq t_0 < t_1 < \dots < t_k < \dots, \quad (2)$$

and $\lim_{k \rightarrow \infty} t_k = \infty$. Furthermore, we suppose

HSS (1) satisfies conditions (H₁)–(H₃) at any bounded interval $0 < t_k - t_{k-1} \leq T$:

(H₁) $J_k(t_k^-, x(t_k^-)) : R^+ \times R^n \rightarrow R^n (k \geq 1)$ are continuous functions with $J_1(t, 0) \equiv 0$ for $t \in R^+$;

(H₂) there exists nonnegative constant sequence $\{C_{\sigma(k)}\}, \{D_{\sigma(k)}\}$ such that

$$\begin{aligned} (i) \quad &\|f_{\sigma(k)}(t, x)\| + \|g_{\sigma(k)}(t, x)\| \leq C_{\sigma(k)}(1 + \|x\|); \\ (ii) \quad &\|f_{\sigma(k)}(t, x) - f_{\sigma(k)}(t, y)\| + \|g_{\sigma(k)}(t, x) - g_{\sigma(k)}(t, y)\| \leq D_{\sigma(k)}\|x - y\|; \end{aligned}$$

(H₃) let $x(t_k) = Z_{\sigma(k)}$ be a random variable which is independent of the σ -algebra $F_{\infty}^{(m)}$ generated by $B_s(\cdot)$ ($B_s(\cdot)$ is a m -dimensional normal Brownian motion), and $E[\|Z_{\sigma(k)}\|^2] \leq M < \infty$.

Let $k_0 = \min\{k \in Z, t_k \geq k\}, \Lambda = \{1, 2, \dots, k_0\} \cap \{k \in Z, \sigma(k) = i\}$. And $T_i(t_0, t) = \sum_{k \in \Lambda} [\min(t, t_k) - t_{k-1}]$ denotes the working time of the i th subsystem during the interval $[t_0, t]$. In addition, let $\mu(T_i(t_0, t))$ denote the Lebesgue measure of the set $T_i(t_0, t)$. Then HSS (1) can be written as

$$\begin{aligned} dx(s) &= f_i(s, x(s-\tau)) ds + g_i(s, x(s-\tau)) dB(s), \\ s &\in T_i(t_0, t), \\ \Delta x(s_k) &= x(s_k) - x(s_k^-) = J_k(s_k^-, x(s_k^-)), \quad s = t_k, \\ x(t_0, i) &= \xi_i(t_0 + s), \quad s \in [-\tau, 0), \quad k = 1, 2, \dots, n, \\ x(t_k) &= Z_{\sigma(k)}, \end{aligned} \quad (3)$$

where $i \in I$ and $\bigcup_{i=1}^m T_i(t_0, t) = [t_0, t]$; furthermore, $f_i(0, x(0)) = g_i(0, x(0)) = 0$ for all $i \in N$.

Let $\Gamma = \{t_i : i = 1, 2, \dots, \text{ and } \sigma \leq t_1 < \dots < t_i < \dots\}$, $R_{\sigma} = \{x \in R : x \geq \sigma\}$, where $\sigma \in R$ is a given constant and $C^{1,2}(R^n \times R_{\sigma} \times I; R^+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times R_{\sigma} \times I$ that are twice continuously differentiable in x and once in t . If $V(x, t, i) \in C^{2,1}(R^n \times R_{\sigma} \times I; R^+)$, define an operator L associated with HSS (3) from $R^n \times R_{\sigma} \times I$ to R by

$$\begin{aligned} LV(t, x(t), i) &= V_t(t, x(t), i) + V_x(t, x(t), i) f_i(t, x(t)) \\ &\quad + \frac{1}{2} \text{trace} [g_i^T(t, x(t)) V_{xx}(t, x(t), i) g_i(t, x(t))], \end{aligned} \quad (4)$$

where $V_t(t, x(t), i) = \partial V(t, x(t), i) / \partial t$,

$$V_{xx}(t, x(t), i) = \left(\frac{\partial^2 V(t, x(t), i)}{\partial x_i \partial x_j} \right)_{n \times n},$$

$$\begin{aligned} V_x(t, x(t), i) &= \left(\frac{\partial V(t, x(t), i)}{\partial x_1}, \frac{\partial V(t, x(t), i)}{\partial x_2}, \dots, \frac{\partial V(t, x(t), i)}{\partial x_n} \right). \end{aligned} \quad (5)$$

The purpose of this paper is to develop some stability criteria for systems (1) and (3).

Definition 1. (i) p th moment asymptotically stable: the trivial of HSS (1) and (3) is p th moment asymptotically stable if there exists $\delta = \delta(t_0) > 0$ such that

$$E(\|x_0\|^p) < \delta \quad \text{implies} \quad \lim_{t \rightarrow \infty} E\|x(t)\|^p = 0, \quad t > t_0. \quad (6)$$

(ii) p th moment exponentially stable: the trivial of HSS (1) and (3) is p th moment exponentially stable if there exist positive constants $\alpha > 0, K \geq 1$ such that

$$\lim_{t \rightarrow \infty} E \|x(t)\|^p \leq K e^{-\alpha(t-t_0)} E (\|x_0\|^p), \quad t \geq t_0 \geq 0. \quad (7)$$

(iii) Asymptotically stable in mean square: the trivial of HSS (1) and (3) is asymptotically stable in mean square if there exists $\delta = \delta(t_0) > 0$ such that

$$E (\|x_0\|^2) < \delta \quad \text{implies} \quad \lim_{t \rightarrow \infty} E \|x(t)\|^2 = 0, \quad t > t_0. \quad (8)$$

(iv) Exponentially stable in mean square: the trivial of HSS (1) and (3) is p th moment exponentially stable if there exist positive constants $\alpha > 0, K \geq 1$ such that

$$\lim_{t \rightarrow \infty} E \|x(t)\|^2 \leq K e^{-\alpha(t-t_0)} E (\|x_0\|^2), \quad t \geq t_0 \geq 0. \quad (9)$$

Definition 2. $D^+ f(x) = \lim_{\Delta x \rightarrow 0^+} ((f(x + \Delta x) - f(x))/\Delta x)$ if $f(x)$ is differentiable at its right side.

Moreover, in order to finish our results, we will introduce the following lemmas.

Lemma 3. Let ω be a nonnegative function defined on the interval $[t_0 - \tau, \infty)$ and be continuous on the interval $[t_0, \infty)$. Assume that

$$\dot{\omega}(t) \leq -a\omega(t) + b\omega(t - \tau), \quad t \geq t_0, \quad (10)$$

where a and b are nonnegative constants satisfying $a > b$. Then

$$\omega(t) \leq \bar{\omega}_0 \exp \{-\lambda(t - t_0)\}, \quad t \geq t_0, \quad (11)$$

where $\bar{\omega}_0 = \sup_{t_0 - \tau \leq \theta \leq t_0} \omega(\theta)$ and $\lambda > 0$ satisfies

$$\lambda - a + b e^{\lambda\tau} = 0. \quad (12)$$

Lemma 4. If $V \in C^{2,1}(R^n \times R_\sigma \times S; R^+)$, then

$$\begin{aligned} EV(t_2, x(t_2), \sigma(t_2)) &= EV(t_1, x(t_1), \sigma(t_1)) \\ &+ E \int_{t_1}^{t_2} LV(s, x(s), \sigma(s)) ds \end{aligned} \quad (13)$$

for $\sigma \leq t_1 < t_2 < \dots < t_i < \dots$.

3. Main Results

In this section, we will present some criteria for the stability of systems (3).

Theorem 5. If there exist switching Lyapunov functions $V_i(t) \in C^{1,2}(R^n \times R_\sigma \times I; R^+)$ and positive number $c_i, d_i (i = 1, 2, \dots, n)$ such that

$$\begin{aligned} (i) \quad E [LV_i(t)] &\leq -a_i E [V_i(t, x(t))] \\ &+ b_i E [V_i(t - \tau(t))], \quad i \in N, \end{aligned} \quad (14)$$

$$\sum_{i=\sigma(k)} \left[\ln \left(\frac{d_i}{c_i} \right) + p \ln \beta_k \right] + \sum_{k=1}^n [\lambda_i u(T_i(t_0, t))] \leq \psi(t_0, t), \quad (15)$$

where $a_i > b_i > 0$, and λ_k such that

$$\lambda_i - a_i + b_i e^{\lambda_i \tau} = 0, \quad (16)$$

$$(ii) \quad c_i E [\|x(t)\|^p] \leq EV_i(t) \leq d_i E [\|x(t)\|^p], \quad (17)$$

$$(iii) \quad \|x(t_k^-) + J_k(t_k^-, x(t_k^-))\| \leq \beta_k \|x(t_k^-)\|, \quad k = 1, 2, \dots, \quad (18)$$

$\psi(t_0, t)$ is a continuous function on R^+ . Then

$$\lim_{t \rightarrow +\infty} \psi(t_0, t) = -\infty \quad (19)$$

implies that the trivial solution of (3) is p th moment globally asymptotically stable, and

$$\psi(t_0, t) \leq M - c(t - t_0), \quad t \geq t_0, \quad (20)$$

where $M > 0$ and $c > 0$ are constants, implies that the trivial solution of (3) is p th moment globally exponentially stable.

Proof. For any $x(t_0, i) \in PC_{F_0}^b([-\tau, 0]; R^n)$, we denote the solution $x(t, t_0, x(t_0, i))$ of (3) by $x(t)$. Let $V_i(t)$ be the switching Lyapunov function $t \in \mu_i(t_0, t), i = \sigma(k)$. From Ito differential formula, for $t \in [t_{k-1}, t_k], i = \sigma(k) \in N$, we have

$$dV_i(t) = LV_i(t) dt + g_i(t, x(t)) \frac{\partial V_i(t)}{\partial t} dw(t). \quad (21)$$

Let $\Delta t > 0$ be small enough such that $t + \Delta t \in [t_{i-1}, t_i]$. Then

$$EV_i(t + \Delta t) - EV_i(t) = \int_t^{t+\Delta t} LV_i(s) ds. \quad (22)$$

Hence, $D^+ V_i(t) = ELV_i(t), t \in [t_{k-1}, t_k], i = \sigma(k) \in N$.

It follows from condition (14) that we have

$$\begin{aligned} D^+ E [V_i(t)] &= ELV_i(t) \\ E [LV_i(t)] &\leq -a_i E [V_i(t, x(t))] \\ &+ b_i E [V_i(t - \tau(t))], \quad i \in N. \end{aligned} \quad (23)$$

And it follows from Lemma 3 that there exists λ_i such that

$$E [V_i(t)] \leq [EV_i(t_k)] e^{-\lambda_i(t_k - t_{k-1})}. \quad (24)$$

λ_i satisfies the following equation:

$$\lambda_i - a_i + b_i e^{\lambda_i \tau} = 0. \quad (25)$$

Hence, it follows from condition (17) and (18) that we have

$$\begin{aligned} E [V_i(t)] &\leq [EV_i(t_k)] e^{-\lambda_i(t_k - t_{k-1})} \\ &\leq d_i E [\|x(t_k)\|^p] e^{-\lambda_i(t_k - t_{k-1})} \\ &\leq d_i (\beta_k)^p E [\|x(t_k^-)\|^p] e^{-\lambda_i(t_k - t_{k-1})}. \end{aligned} \quad (26)$$

Therefore, when $t \in (t_{k-1}, t_k]$,

$$\begin{aligned}
 E[\|x(t)\|^p] &\leq \frac{d_i}{c_i} (\beta_k)^p EV_{\sigma(k-1)}(t_k^-) e^{-\lambda_i(t_k - t_{k-1})} \\
 &\leq \dots \\
 &\leq \prod_{i=\sigma(k)} \left[\frac{d_i}{c_i} (\beta_k)^p \right] EV_{\sigma(0)}(t_0^-) e^{\sum_{i=\sigma(k)} -\lambda_{\sigma(k)}(t_k - t_{k-1})}, \\
 &\prod_{i=\sigma(k)} \left[\frac{d_i}{c_i} (\beta_k)^p \right] EV_{\sigma(0)}(t_0^-) e^{\sum_{i=\sigma(k)} -\lambda_{\sigma(k)}(t_k - t_{k-1})} \\
 &= e^{\sum_{i=\sigma(k)} [\ln(d_i/c_i) + p \ln \beta_k] + \sum_{k=1}^n [-\lambda_i u(T_i(t_0, t))]} \\
 &\quad \times E[V_{\sigma(0)}(x(t_0^-))] \\
 &\leq E[V_{\sigma(0)}(t_0)] e^{\psi(t_0, t)}.
 \end{aligned} \tag{27}$$

It follows from the above discussion that

$$E[\|x(t)\|^p] \leq E[V_{\sigma(0)}(t_0)] e^{\psi(t_0, t)}, \quad t \in (t_{k-1}, t_k], \tag{28}$$

which implies the conclusions of the theorem. The proof is thus completed. $\square$

Using the proof techniques in Theorem 5, we can obtain Corollaries 6–8.

Corollary 6. Assume that conditions in Theorem 5 hold. Then, the trivial solution of (3) is p th moment globally exponentially stable if one of the following conditions is satisfied.

(i) $\lambda_i < -\alpha < 0$ for all $i \in N$, and there exists a constant $0 < \varepsilon < \alpha$ such that

$$\ln \frac{d_i}{c_i} + p \ln \beta_k - \varepsilon(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \dots \tag{29}$$

(ii) $t_k - t_{k-1} \leq T < \infty$, and there exist two positive constants α and $\bar{\alpha}$ such that $\bar{\alpha} > \alpha \geq |\lambda_i|$ and

$$\ln \frac{d_i}{c_i} + p \ln \beta_k - \bar{\alpha}(t_k - t_{k-1}) \leq 0, \quad k = 1, 2, \dots \tag{30}$$

Proof. When $\lambda_i < -\alpha < 0$, it follows from (29) that

$$\begin{aligned}
 &\sum_{i=\sigma(k)} \left[\ln \frac{d_i}{c_i} + p \ln \beta_k \right] + \sum_{k=1}^n [\lambda_i u(T_i(t_0, t))] \\
 &\leq -(\alpha - \varepsilon)(t - t_0).
 \end{aligned} \tag{31}$$

Letting $\psi(t_0, t) = -(\alpha - \varepsilon)(t - t_0)$ and noting $(\alpha - \varepsilon) > 0$, we can obtain from Theorem 5 that the trivial solution of (3) is p th moment globally exponentially stable.

When $\bar{\alpha} > \alpha \geq |\lambda_i|$, it follows from (30) that

$$\begin{aligned}
 &\sum_{i=\sigma(k)} \left[\ln \frac{d_i}{c_i} + p \ln \beta_k \right] + \sum_{k=1}^n [\lambda_i u(T_i(t_0, t))] \\
 &\leq \bar{\alpha}T - (\bar{\alpha} - \alpha)(t - t_0).
 \end{aligned} \tag{32}$$

Letting $\psi(t_0, t) = \bar{\alpha}T - (\bar{\alpha} - \alpha)(t - t_0)$ and noting that $\bar{\alpha} - \alpha > 0$, we conclude the proof. $\square$

Corollary 7. In Theorem 5, if there exist switching Lyapunov functions $V_i(t) = x^T(t)P_i x(t)$, P_i is a positive-definite matrix ($i = 1, 2, \dots, n$) and $\rho_i = \lambda_{\max}(P_i)/\lambda_{\min}(P_i)$, $\lambda_{\max}(P_i)$ ($\lambda_{\min}(P_i)$) denote the maximum (minimum) eigenvalue of positive-definite matrices P_i and P_i such that

$$\begin{aligned}
 (i) \quad E[LV_i(t)] &\leq -a_i E[V_i(t, x(t))] \\
 &\quad + b_i E[V_i(t - \tau(t))], \quad i \in N
 \end{aligned} \tag{33}$$

$$\sum_{i=\sigma(k)} [\ln \rho_i + 2 \ln \beta_k] + \sum_{k=1}^n [\lambda_i u(T_i(t_0, t))] \leq \psi(t_0, t), \tag{34}$$

where $a_i > b_i > 0$, and λ_k such that

$$\lambda_i - a_i + b_i e^{\lambda_i \tau} = 0,$$

$$(ii) \quad \|x(t_k^-) + J_k(t_k^-, x(t_k^-))\| \leq \beta_k \|x(t_k^-)\|, \quad k = 1, 2, \dots, \tag{35}$$

$\psi(t_0, t)$ is a continuous function on R^+ . Then

$$\lim_{t \rightarrow +\infty} \psi(t_0, t) = -\infty \tag{36}$$

implies that the trivial solution of (3) is globally asymptotically stable in mean square, and

$$\psi(t_0, t) \leq M - c(t - t_0), \quad t \geq t_0, \tag{37}$$

where $M > 0$ and $c > 0$ are constants, implies that the trivial solution of (3) is globally exponentially stable in mean square.

Proof. When P_i is a positive-definite matrix, there exists an orthogonal matrix $U_i (U_i^T U_i = I)$ (I is an identity matrix) such that

$$U_i^T P_i U_i = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \tag{38}$$

$\lambda_i > 0$ ($i = 1, 2, \dots, n$) is eigenvalue of positive-definite matrix P_i , and

$$\begin{aligned}
 x^T(t) P_i x(t) &= x^T(t) U_i^T P_i U_i x(t) \\
 &= y^T(t) \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} y(t) \\
 &= \sum_{i=1}^n (\lambda_i y_i^2).
 \end{aligned} \tag{39}$$

Here, $y(t) = U_i x(t)$, and

$$\begin{aligned}
 \|y(t)\|^2 &= y^T(t) y(t) \\
 &= [U_i x(t)]^T U_i x(t) \\
 &= x(t)^T U_i^T U_i x(t) \\
 &= x(t)^T x(t) = \|x(t)\|^2, \\
 \lambda_{\min}(P_i) \sum_{i=1}^n (y_i^2) &\leq \sum_{i=1}^n (\lambda_i y_i^2) \leq \lambda_{\max}(P_i) \sum_{i=1}^n (y_i^2).
 \end{aligned} \tag{40}$$

Hence, we have

$$\lambda_{\min}(P_i) \|x(t)\|^2 \leq V_i(t) = x^T(t) P_i x(t) \leq \lambda_{\max}(P) \|x(t)\|^2. \tag{41}$$

Let $c_i = \lambda_{\min}(P_i)$ and $d_i = \lambda_{\max}(P_i)$; in this situation, it is similar to condition (17) in Theorem 5 and $p = 2$; furthermore, condition (34) is similar to condition (15) in Theorem 5 ($\rho_i = d_i/c_i = \lambda_{\max}(P_i)/\lambda_{\min}(P_i)$, $\rho = \max_{i \in N}(\rho_i)$).

Similar to the proof of Theorem 5, we can easily complete the proof of Corollary 7. $\square$

Corollary 8. In Theorem 5, if $p = 2$ in (15) and $c_i E[\|x(t)\|^2] \leq EV_i(t) \leq d_i E[\|x(t)\|^2]$ in (17), then

$$\lim_{t \rightarrow +\infty} \psi(t_0, t) = -\infty \tag{42}$$

implies that the trivial solution of (3) is asymptotically stable in mean square, and

$$\psi(t_0, t) \leq M - c(t - t_0), \quad t \geq t_0, \tag{43}$$

where $M > 0$ and $c > 0$ are constants, implies that the trivial solution of (3) is exponentially stable in mean square.

4. Application to a Kind of Stochastic Hybrid Impulsive and Switching Neural Networks (SHISNN)

Similar to paper [9], due to the stochastic distribution, a kind of stochastic hybrid impulsive and switching neural networks (SHISNN) can be described as follows:

$$\begin{aligned}
 dx(t) &= [-C_{\sigma(k)} x(t) + A_{\sigma(k)} f_{\sigma(k)}(x(t)) + B_{\sigma(k)} \\
 &\quad f_{\sigma(k)}(x(t - \tau))] dt \\
 &\quad + g_{\sigma(k)}(t, x(t), x(t - \tau)) dw(t), \\
 t_{k-1} - \tau &\leq t < t_k, \\
 \Delta x(t_k) &= J_k(t_k^-, x(t_k^-)), \quad t = t_k \\
 x(\theta) &= \phi(\theta), \quad \theta \in [t_0 - \tau, t_0].
 \end{aligned} \tag{44}$$

Similarly, systems (44) can be written as follows:

$$\begin{aligned}
 dx(s) &= [-C_i x(s) + A_i f_i(x(s)) + B_i \\
 &\quad f_i(x(s - \tau))] ds + g_i(s, x(s), x(s - \tau)) dw(s), \\
 s \in T_i(t_0, t), \quad i \in I, \quad \bigcup_{i=1}^m T_i(t_0, t) &= [t_0, t], \\
 \Delta x(t_k) &= J_k(t_k^-, x(t_k^-)), \quad t = t_k, \\
 x(\theta) &= \phi(\theta), \quad \theta \in [t_0 - \tau, t_0].
 \end{aligned} \tag{45}$$

To simplify, we always suppose that the solution of (45) always exists.

In order to obtain the according result, we give out Lemmas 9 and 10.

Lemma 9. Let Y be the real matrix of $n \times m$ dimensions and let P be a positive matrix; then

$$\text{trace}[Y^T P Y] \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \text{trace}[Y^T Y]. \tag{46}$$

Proof. Since P is a positive matrix, there exists an orthogonal matrix U such that

$$U^T P U = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}. \tag{47}$$

Then

$$P = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} U^T,$$

$$\text{trace}[y^T P Y]$$

$$\begin{aligned}
 &\leq \text{trace} \left[y^T U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} U^T Y \right] \\
 &\quad \times \lambda_{\min}(P) \text{trace}[Y^T Y] = \lambda_{\min}(P) Y^T Y \\
 &\leq \text{trace} \left[y^T U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} U^T Y \right] \\
 &\leq \lambda_{\max}(P) y^T Y = \lambda_{\max}(P) \text{trace}[y^T Y].
 \end{aligned} \tag{48}$$

$\square$

Lemma 10. Let $P \in R^{n \times n}$ be a symmetric and positive-definite matrix and let $Q \in R^{n \times n}$ be a symmetric matrix; then $\lambda_{\min}(P^{-1}Q)x^T Px \leq x^T Qx \leq \lambda_{\max}(P^{-1}Q)x^T Px$.

Theorem 11. If SHISNN (18) satisfies the following conditions:

- (H₁) activation functions are bounded; that is, for $i \in N, j = 1, 2, \dots, n$; there exist positive constants $M_j^{(i)}$ such that $|f_j^{(i)}(\alpha)| \leq M_j^{(i)}, \forall \alpha \in R^n$;
- (H₂) activation functions are globally Lipschitzian, for $i \in N, j = 1, 2, \dots, n$; there exist positive constants $L_j^{(i)}$ such that $|f_j^{(i)}(\alpha) - f_j^{(i)}(\beta)| \leq L_j^{(i)}|\alpha - \beta|$;
- (H₃) for $k = 1, 2, \dots$, there exist positive constants β_k such that $\|x(t_k^-) + J_k(t_k^-, x(t_k^-))\| \leq \beta_k \|x(t_k^-)\|$;
- (H₄) for SHISNN (45), there exists a constant $\delta \geq 1$ such that $t_k - t_{k-1} \geq \delta\tau$ for all k ;
- (H₅) $\text{trace}[g_i^T(t, x(t), x(t - \tau))g_i(t, x(t), x(t - \tau))] \leq \lambda_i^{(1)}\|x(t)\|^2 + \lambda_i^{(2)}\|x(t - \tau)\|^2$;
- (H₆) there exist symmetric and positive-definite matrices P_i , positive-definite diagonal matrices Q_i and S_i , and positive constant scalars $\alpha_i (\alpha_i > v_i)$, $i \in I$, such that

$$\begin{aligned} (a) \quad & \Omega_i + \alpha_i P_i \leq 0, \\ (b) \quad & \sum_{i=\sigma(k)} [\ln \rho_i + 2 \ln \beta_k] \\ & + \sum_{k=1}^n [-\lambda_i u(T_i(t_0, t))] \leq \psi(t_0, t), \end{aligned} \quad (49)$$

where $\psi(t_0, t)$ and ρ_i are defined in Theorem 5, and

$$\begin{aligned} \Omega_i = & -P_i C_i - C_i P_i + P_i A_i Q_i^{-1} A_i^T P_i + L_i Q_i L_i \\ & + P_i B_i S_i^{-1} B_i^T P_i + \rho_i \lambda_i^{(1)} P_i + \alpha_i P_i, \\ v_i = & \lambda_{\max} [P_i^{-1} (L_i S_i L_i + \rho_i \lambda_i^{(2)} P_i)], \end{aligned} \quad (50)$$

and λ_i is the unique positive root of the equation

$$-\alpha_i + v_i e^{\lambda_i \tau} + \lambda_i = 0, \quad (51)$$

then $\lim_{t \rightarrow +\infty} \psi(t_0, t) = -\infty$ implies that the trivial solution of (45) is globally mean square asymptotically stable and $\psi(t_0, t) = -c(t - t_0), t \geq t_0$, implies that the trivial solution of (45) is globally asymptotically mean square exponentially stable.

Proof. Construct the switching Lyapunov function

$$V(t) = x^T(t) P_i x(t), \quad t \in [t_{k-1}, t_k], \quad i = \sigma(k). \quad (52)$$

Using Ito formula to (52) yields

$$dV_i(t) = LV_i(t) dt + g_i(t, x(t)) \frac{\partial V_i(t)}{\partial t} dw(t), \quad (53)$$

where

$$\begin{aligned} LV_i(t) &= 2x^T(t) P_i \\ &\quad \times [-C_i x(t) + A_i f_i(x(t)) + B_i \cdot f_i(x(t - \tau))] \\ &\quad + \text{trace} [g_i^T(t, x(t), x(t - \tau)) P_i g_i(t, x(t), x(t - \tau))]. \end{aligned} \quad (54)$$

Applying conditions (H₅) and Lemmas 9 and 10 to (54) yields

$$\begin{aligned} LV_i(t) &\leq x^T(t) (-P_i C_i - C_i P_i) x^T(t) \\ &\quad + x^T(t) P_i A_i Q_i^{-1} A_i^T P_i x^T(t) + f_i^T(x(t)) Q_i f_i(x(t)) \\ &\quad + x^T(t) P_i B_i S_i^{-1} B_i^T P_i x^T(t) \\ &\quad + f_i^T(x(t - \tau)) Q_i f_i(x(t - \tau)) \\ &\quad + \text{trace} [g_i^T(t, x(t), x(t - \tau)) g_i(t, x(t), x(t - \tau))] \\ &\leq x^T(t) (-P_i C_i - C_i P_i) x^T(t) \\ &\quad + x^T(t) P_i A_i Q_i^{-1} A_i^T P_i x^T(t) \\ &\quad + f_i^T(x(t)) Q_i f_i(x(t)) + x^T(t) P_i B_i S_i^{-1} B_i^T P_i x^T(t) \\ &\quad + f_i^T(x(t - \tau)) S_i f_i(x(t - \tau)) \\ &\quad + \rho_i \lambda_i^{(1)} x^T(t) P_i x(t) + \rho_i \lambda_i^{(2)} x^T(t - \tau) P_i x(t - \tau) \\ &\leq x^T(t) (-P_i C_i - C_i P_i) x^T(t) \\ &\quad + x^T(t) P_i A_i Q_i^{-1} A_i^T P_i x^T(t) \\ &\quad + x^T(t) L_i Q_i L_i x(t) + x^T(t) P_i B_i S_i^{-1} B_i^T P_i x^T(t) \\ &\quad + x^T(t - \tau) L_i S_i L_i x^T(t - \tau) \\ &\quad + \rho_i \lambda_i^{(1)} x^T(t) P_i x(t) + \rho_i \lambda_i^{(2)} x^T(t - \tau) P_i x(t - \tau) \\ &\leq x^T(t) (-P_i C_i - C_i P_i + P_i A_i Q_i^{-1} A_i^T P_i + L_i Q_i L_i \\ &\quad + P_i B_i S_i^{-1} B_i^T P_i + \rho_i \lambda_i^{(1)} P_i) x^T(t) \\ &\quad + \lambda_{\max} [P_i^{-1} (L_i S_i L_i + \rho_i \lambda_i^{(2)} P_i)] x^T(t - \tau) P_i x^T(t - \tau) \\ &\leq -\alpha_i V(t) + v_i V(t - \tau). \end{aligned} \quad (55)$$

Applying Theorem 5 to (55) yields

$$E[V_i(t)] \leq [EV_i(t_k)] e^{\lambda_i(t_k - t_{k-1})}. \quad (56)$$

λ_i satisfies the following equation:

$$-\alpha_i + v_i e^{\lambda_i \tau} + \lambda_i = 0. \quad (57)$$

Similar to the proof of Theorem 5,

$$\begin{aligned}
 \lambda_{\min}(P_i) E\|x(t)\|^2 &\leq E[V_i(t)] \\
 &\leq [EV_i(t_k)] e^{-\lambda_i(t_k-t_{k-1})} \\
 &\leq \lambda_{\max}(P_i) e^{-\lambda_i(t_k-t_{k-1})} E[\|x(t_k)\|^2] \\
 &\leq \lambda_{\max}(P_i) \beta_k^2 e^{-\lambda_i(t_k-t_{k-1})} E[\|x(t_k^-)\|^2].
 \end{aligned} \tag{58}$$

Therefore, when $t \in [t_k, t_{k+1})$,

$$\begin{aligned}
 E\|x(t)\|^2 &\leq E[V_i(t)] \\
 &\leq \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)} \beta_k^2 e^{-\lambda_i(t_k-t_{k-1})} E[\|x(t_k^-)\|^2] \\
 &= \rho_i \beta_k^2 e^{-\lambda_i(t_k-t_{k-1})} E[\|x(t_k^-)\|^2] \\
 &\leq \dots \\
 &\leq \prod_{i=\sigma(k)} [\rho_i \beta_k^2] EV_{\sigma(0)}(t_0) \\
 &\quad \times e^{\sum_{i=\sigma(k)} -\lambda_{\sigma(k)}(t_k-t_{k-1})}, \\
 &\quad \prod_{i=\sigma(k)} \left[\frac{d_i}{c_i} (\beta_k)^2 \right] EV_{\sigma(0)}(t_0) e^{\sum_{i=\sigma(k)} -\lambda_{\sigma(k)}(t_k-t_{k-1})} \\
 &= e^{\sum_{i=\sigma(k)} [\ln(d_i/c_i) + 2 \ln \beta_k] + \sum_{k=1}^n [-\lambda_i u(T_i(t_0, t))]} \\
 &\quad \times E[V_{\sigma(0)}(t_0)] \\
 &\leq E[V_{\sigma(0)}(t_0)] e^{\psi(t_0, t)}.
 \end{aligned} \tag{59}$$

When $\lim_{t \rightarrow +\infty} \psi(t_0, t) = -\infty$, $E[\|x(t)\|^2] \rightarrow 0$, this implies that the trivial solution of (45) is globally asymptotically stable. When $\psi(t_0, t) = -c(t - t_0)$, $t \geq t_0$, $E\|x(t)\|^2 \leq E[V_{\sigma(0)}(t_0)] e^{-c(t-t_0)}$, this implies that the trivial solution of (45) is globally asymptotically mean square exponentially stable.

It follows from (17) that

$$E[\|x(t)\|^2] \leq EV_{\sigma(0)}(t_0) e^{\psi(t_0, t)}, \quad t \in (t_{k-1}, t_k]. \tag{60}$$

The proof is completed. $\square$

Via Schur complement, it is easy to show that (49) is equivalent to the LMI

$$\begin{bmatrix}
 -P_i C_i - C_i P_i + L_i Q_i L_i + \rho_i \lambda_i^{(1)} P_i + \alpha_i P & -P_i A_i & -P_i B_i \\
 A_i^T P_i & -Q_i & 0 \\
 B_i^T P & 0 & -S_i
 \end{bmatrix} \leq 0. \tag{61}$$

5. Conclusion

The hybrid impulsive and switching stochastic systems model has been discussed based on the existing models of hybrid impulsive and switching systems. Some general criteria for the asymptotic and exponential stability analysis of the new model have been established by using switching Lyapunov functions and stochastic analysis techniques. The new result can be used to analyze mean square global asymptotical and exponential stability for a kind of stochastic hybrid impulsive and switching neural networks. Finally, this paper points out that this kind of stability analysis problem can turn into solving a linear matrix inequality (LMI).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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