Research Article

Existence of Solutions for a Class of Quasilinear Parabolic Equations with Superlinear Nonlinearities

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Working in a weighted Sobolev space, this paper is devoted to the study of the boundary value problem for the quasilinear parabolic equations with superlinear growth conditions in a domain of R^N . Some conditions which guarantee the solvability of the problem are given.

1. Introduction

In this paper, we deal with the existence of solutions for the quasilinear parabolic problem:

$$\begin{aligned} \rho D_t u + \mathcal{M}u &= \lambda_1 u \rho + f(x, u) \rho - G, \\ (x, t) &\in \tilde{\Omega}, \quad u \in \tilde{H}(\tilde{\Omega}, \Gamma), \end{aligned} \quad (1)$$

where $\tilde{\Omega} = \Omega \times T$, Ω is an open set in R^N ($N \geq 1$), $T = (-\pi, \pi)$, $\tilde{H}(\tilde{\Omega}, \Gamma)$ is a weighted Sobolev space, λ_1 is the first eigenvalue of $\mathcal{L}$, and $\mathcal{M}$ is a singular quasilinear operator defined by

$$\begin{aligned} \mathcal{M}u &= - \sum_{i,j=1}^N D_i \left(p_i^{1/2}(x) p_j^{1/2}(x) \sigma_i^{1/2}(u) \sigma_j^{1/2}(u) b_{ij}(x) D_j u \right) \\ &\quad + b_0(x) \sigma_0(u) q u, \end{aligned} \quad (2)$$

where $u(x, t) : \tilde{\Omega} \rightarrow R$. The nonlinear part $f(x, s)$ satisfies the superlinear growth condition

$$|f(x, s)| \leq h_0(x) + K |s|^{\theta-1}, \quad \forall s \in R, \text{ a.e. } x \in \Omega, \quad (3)$$

where $h_0(x) \in L_{\rho}^{\theta^*}$, K is a nonnegative constant, $2 < \theta < 2N/(N-1)$, and $\theta^* = \theta/(\theta-1)$.

There are a number of results concerning solvability of different boundary problems for quasilinear equations

(elliptic and parabolic) in which the nonlinearities satisfy sublinear or linear conditions in the weighted Sobolev space, for example, [1–6].

In [1], Shapiro established a new weighted compact Sobolev embedding theorem and proved a series of existence problems for weighted quasilinear elliptic equations and parabolic equations.

In [2], working in Sobolev space $H_{p,\rho}^1(\Omega, \Gamma)$ only for the first eigenvalue, Rumbos and Shapiro on the basis of [3] by using the generalized Landesman-Lazer conditions discussed the existence of the solutions for weighted quasilinear elliptic equations

$$\begin{aligned} \mathcal{P}u - \lambda_1 u \rho &= -a(x, u) u^- \rho + g(x, u) \rho + h, \quad x \text{ in } \Omega, \\ u &= 0, \quad x \text{ on } \Omega, \end{aligned} \quad (4)$$

where

$$\mathcal{P}u = - \sum_{i,j=1}^N D_i \left(p_i^{1/2} p_j^{1/2} b_{ij}(x) D_j u \right) + \rho(x) c(x) u. \quad (5)$$

In [4], Jia and Zhao obtained the existence of a nontrivial solution for a class of singular quasilinear elliptic equations in weighted Sobolev spaces.

However, past research results regarding this kind of parabolic equations on superlinearity in the weighted Sobolev space like (1) are very limited. Two notable exceptions are

found in [7, 8], where they discuss the periodic solutions for quasilinear parabolic equations when the nonlinearity may grow superlinearly.

Our goal here is to extend these results to the case of quasilinear parabolic operators.

In fact, (1) is one of the most useful sets which describe the motion of viscous fluid substances. They are widely used in the design of aircrafts and cars, the study of blood flow, the design of power stations, and so forth. Furthermore, coupled with Maxwell's equations, the Navier-Stokes equations can also be used to model and study magnetohydrodynamics.

The main tools applied in our approaches consist of Galerkin method, Brouwer's theorem, and a new weighted compact Sobolev-type embedding theorem due to Shapiro.

This paper is organized as follows. In Section 2, we introduce some necessary assumptions and basic results. In Section 3, five fundamental lemmas are established. The subsequent Section 4 contains proofs of the main results.

2. Basic Assumptions and Main Theorem

In this section, we introduce some assumptions and give the main results in this paper.

Let $\Omega \subset R^N$ ($N \geq 1$) be an open (possibly unbounded) set, $\Gamma \subset \partial\Omega$ a fixed closed set (Γ may be an empty set), $\bar{\Omega} = \Omega \times T$, and $T = (-\pi, \pi)$. We assume throughout this paper that $\rho(x)$, $p_i(x)$ ($i = 1, 2, \dots, N$) are positive $C^0(\Omega)$ functions, function $q(x)$ is a nonnegative $C^0(\Omega)$ function ($q(x)$ may be zero), and

$$\begin{aligned} \int_{\Omega} \rho(x) dx < \infty, \quad \int_{\Omega} p_i(x) dx < \infty, \quad i = 1, 2, \dots, N, \\ \exists K > 0, \text{ s.t. } 0 \leq q(x) \leq K\rho(x), \quad \forall x \in \Omega. \end{aligned} \quad (6)$$

We define $p = (p_1, \dots, p_N)$, $D_i u = \partial u / \partial x_i$, $i = 1, 2, \dots, N$, and $\mathcal{A}$ is a set of real-valued functions defined as

$$\begin{aligned} \mathcal{A} = \{u : u \in C^0(\bar{\Omega} \times R), u(x, t + 2\pi) = u(x, t), \\ \forall (x, t) \in \bar{\Omega} \times R\}. \end{aligned} \quad (7)$$

We consider the following pre-Hilbert spaces (see [1]):

$$\bar{C}_{\rho}^0(\bar{\Omega}) = \left\{ u \in C^0(\bar{\Omega}) \mid \int_{\bar{\Omega}} |u(x, t)|^2 \rho(x) dx dt < \infty \right\}, \quad (8)$$

with inner product $\langle u, v \rangle_{\bar{C}_{\rho}^0} = \int_{\bar{\Omega}} u(x, t)v(x, t)\rho(x)dx dt$, and

$$\begin{aligned} \bar{C}_{p, \rho}^1(\bar{\Omega}, \Gamma) \\ = \left\{ u \in \mathcal{A} \cap C^1(\Omega \times R) \mid u(x, t) = 0, \right. \\ \left. \forall (x, t) \in \Gamma \times R; \right. \\ \left. \int_{\bar{\Omega}} \left(\sum_{i=1}^N |D_i u|^2 p_i + (u^2 + |D_t u|^2) \rho \right) < \infty \right\}, \end{aligned} \quad (9)$$

with inner product

$$\langle u, v \rangle_{\bar{H}} = \int_{\bar{\Omega}} \left(\sum_{i=1}^N p_i D_i u D_i v + (uv + D_t u D_t v) \rho \right) dx dt. \quad (10)$$

We denote $\tilde{L}_{\rho}^2 \triangleq L_{\rho}^2(\bar{\Omega})$ which is the Hilbert space obtained from the completion of $\bar{C}_{\rho}^0(\bar{\Omega})$ with the norm $\|u\|_{\rho} = (\langle u, u \rangle_{\rho})^{1/2}$, and denote $\tilde{H} \triangleq \tilde{H}(\bar{\Omega}, \Gamma)$ which is the Hilbert space obtained from the completion of the space $\bar{C}_{p, \rho}^1(\bar{\Omega}, \Gamma)$ with the norm $\|u\|_{\tilde{H}} = \langle u, u \rangle_{\tilde{H}}^{1/2}$. Similarly, we have $\tilde{L}_{p_i}^2$ ($i = 1, 2, \dots, N$) and $\tilde{L}_q^2$.

It is assumed throughout the paper that $\sigma_i(u)$ ($i = 0, 1, \dots, N$), $b_{ij}(x)$ ($i, j = 1, 2, \dots, N$), and $b_0(x)$ meet the following assumptions:

- (σ-1) $\sigma_i(u): \tilde{H} \rightarrow R$ is weakly sequentially continuous;
- (σ-2) there are $\eta_0, \eta_1 > 0$, s.t. $\eta_0 \leq \sigma_i(u) \leq \eta_1$, and $\sigma_i(u)$ is measurable, for $u \in \tilde{H}$;
- (b-1) $b_0(x), b_{ij}(x) \in C^0(\Omega) \cap L^{\infty}(\Omega)$, where $b_{ij}(x) = b_{ji}(x)$, for $x \in \Omega$;
- (b-2) $b_0(x) \geq \varepsilon_0 > 0$, for $x \in \Omega$;
- (b-3) there is a constant $c_0 > 0$, for $x \in \Omega$ and $\xi \in R^N$, such that

$$\sum_{i,j=1}^N b_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2, \quad |\xi|^2 = \sum_{i=1}^N \xi_i^2. \quad (11)$$

Definition 1. For the quasilinear differential operator $\mathcal{M}$, the two-form is

$$\begin{aligned} \mathcal{M}(u, v) = \sum_{i,j=1}^N \int_{\bar{\Omega}} (p_i^{1/2} p_j^{1/2} \sigma_i^{1/2}(u) \sigma_j^{1/2}(u) b_{ij} D_j u D_i v) \\ + \int_{\bar{\Omega}} q \sigma_0(u) b_0 uv, \quad \text{for } u, v \in \tilde{H}. \end{aligned} \quad (12)$$

Defining

$$\mathcal{L}u = - \sum_{i=1}^N D_i (p_i D_i u) + qu, \quad (13)$$

for $u \in H_{p, q, \rho}(\Omega, \Gamma)$ (as described in [5]), and

$$\tilde{\mathcal{L}}u = - \sum_{i=1}^N D_i (p_i D_i u) + qu, \quad u \in \tilde{H}, \quad (14)$$

the two-form of $\mathcal{L}$ is

$$\mathcal{L}(u, v) = \sum_{i=1}^N \int_{\Omega} p_i D_i u D_i v + \int_{\Omega} uvq, \quad u, v \in H_{p, q, \rho}(\Omega, \Gamma), \quad (15)$$

and the two-form of $\mathcal{L}$ is

$$\mathcal{L}(u, v) = \sum_{i=1}^N \int_{\Omega} p_i D_i u D_i v + \int_{\Omega} uvq, \quad u, v \in \tilde{H}. \quad (16)$$

Definition 2. One says that $\mathcal{M}$ is $\# \tilde{H}$ -related to $\mathcal{L}$ if the following condition holds:

$$\lim_{\|u\|_{\tilde{H}} \rightarrow \infty} \frac{\mathcal{M}(u, v) - \mathcal{L}(u, v)}{\|u\|_{\tilde{H}}} = 0, \quad \text{uniformly for } \|v\|_{\tilde{H}} \leq 1. \quad (17)$$

Definition 3. $(\tilde{\Omega}, \Gamma)$ is a simple $V_L(\tilde{\Omega}, \Gamma)$ if

(V_L -1) there is a complete orthonormal system $\{\varphi_n\}_{n=1}^{\infty}$ in L^2_{ρ} . Also $\varphi_n \in H_{p,q,\rho} \cap C^2$ for all n ;

(V_L -2) there is a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \rightarrow \infty$ such that $\mathcal{L}(\varphi_n, v) = \lambda_n \langle \varphi_n, v \rangle_{\rho}$ for all $v \in H_{p,q,\rho}(\Omega, \Gamma)$. Also $\varphi_1 > 0$ in Ω ;

(V_L -3) $\Omega = \Omega_1 \times \dots \times \Omega_N$, where $\Omega_i \subset R$ is an open set for $i = 1, \dots, N$;

(V_L -4) for each Ω_i , there are positive functions $p_i^*(x)$, $\rho_i^*(x) \in C^0(\Omega_i)$ satisfying $\int_{\Omega_i} (p_i^*(x) + \rho_i^*(x)) dx < \infty$, and $\rho(x) = \rho_1^*(x_1) \dots \rho_N^*(x_N)$, $p_i(x) = \rho_1^*(x_1) \dots \rho_{i-1}^*(x_{i-1}) p_i^*(x_i) \rho_{i+1}^*(x_{i+1}) \dots \rho_N^*(x_N)$;

(V_L -5) for each Ω_i , $p_i^*(x)$, and $\rho_i^*(x)$ ($i = 1, \dots, N$), there exist $h_i \in C^0(\Omega_i) \cap L^{\theta}_{\rho_i^*}(\Omega_i)$ and $y_i(t) \in C^0(T) \cap L^{\theta}(T)$ for $2 < \theta < \infty$, such that

$$|v(s, t)| \leq h_i(s) y_i(t) \|v\|_{p_i^*(s), \rho_i^*(s)}, \quad (18)$$

$$\forall v \in C^1(\Omega_i \times T), \quad \forall s \in \Omega_i, \quad t \in T,$$

where

$$\|v\|_{p_i^*(s), \rho_i^*(s)}^2 = \int_{\Omega_i} (p_i^*(s) |D_s v|^2 + \rho_i^*(s) (v^2 + |D_t v|^2)) ds dt. \quad (19)$$

There are many examples to illustrate the simple V_L region. One can refer to [1].

Remark 4. We give an example to establish existence results for $\mathcal{M}$ and $\mathcal{L}$ which meet Definition 2. Taking $N > 1$, $\Omega \subset R^N$ (a bounded open connected set), and $\Gamma = \partial\Omega$, we set $\rho = q = p_i = b_0 = b_{ij} = 1$, for $i = 1, 2, \dots, N$, $\sigma_i(u) = \sigma_i(u)$, and $\mathcal{L} = -\sum_{i=1}^N D_{ii} u + u$. Then, it is easy to see from [9] that (Ω, Γ) is a simple V_L region. We set

$$\sigma_i(u) = D_i u + \frac{D_i u}{(1 + u^2 + |Du|^2)^{1/2}}, \quad (20)$$

$$\sigma_0(u) = 1 + \frac{1}{(1 + u^2 + |Du|^2)^{1/2}}.$$

Also, we define

$$\mathcal{M} = -\sum_{i=1}^N D_i (\sigma_i(u) D_i u) + \sigma_0(u) u. \quad (21)$$

We observe with $|Du|^2 = \sum_{i=1}^N |D_i u|^2$ that

$$\begin{aligned} & |\mathcal{M}(u, v) - \mathcal{L}(u, v)| \\ &= \left| \int_{\Omega} \frac{1}{1 + u^2 + |Du|^2} \left(uv + \sum_{i=1}^N D_i u D_i v \right) \right| \\ &\leq (2\pi |\Omega|)^{1/2} |\mathcal{L}(v, v)|^{1/2}. \end{aligned} \quad (22)$$

Clearly, $\mathcal{M}$ and $\mathcal{L}$ meet Definition 2.

Remark 5. If $\mathcal{M}$ (as (12)) meets $(\sigma-1)$ -($\sigma-2$), $(b-1)$ -($b-3$), then

$$\tilde{M}(v, D_t v) = 0, \quad \forall v \in \tilde{C}_{p,\rho}^{1b} = \{v : v \in \tilde{C}_{p,\rho}^1; D_t v \in \tilde{C}_{p,\rho}^1\}. \quad (23)$$

$f(x, s)$ meets the following conditions:

($f-1$) $f(x, s)$ satisfies the Carathéodory conditions;

($f-2$) (*Superlinear condition*) there exists θ with $2 < \theta < 2N/(N-1)$ such that $|f(x, s)| \leq h_0(x) + K|s|^{\theta-1}$, $\forall s \in R$, a.e. $x \in \Omega$, where $h_0(x) \in L^{\theta^*}_{\rho}$. K is a nonnegative constant and $\theta^* = \theta/(\theta-1)$.

Now we state our main results in this paper.

Theorem 6. Assume that $(\tilde{\Omega}, \Gamma)$ is a simple $V_L(\tilde{\Omega}, \Gamma)$. Let $\mathcal{L}$ given by (13) satisfy the conditions of simple $V_L(\tilde{\Omega}, \Gamma)$, and let $\mathcal{M}$ given by (12) satisfy $(b-1)$ -($b-3$), $(\sigma-1)$ -($\sigma-2$) and be $\# \tilde{H}$ -related to $\mathcal{L}$. Suppose that $G \in (\tilde{H})'$ (the dual of $\tilde{H}$) and that ($f-1$)-($f-2$) hold. Suppose furthermore there is a nonnegative function $\tilde{h}_0(x) \in L^{\theta^*}_{\rho}$ and a constant $\epsilon_0^* > 0$ such that

$$sf(x, s) \leq -\epsilon_0^* |s|^2 + \tilde{h}_0(x) |s|, \quad \forall s \in R, \quad \text{a.e. } x \in \Omega. \quad (24)$$

Then problem (1) has at least one nontrivial weak solution, that is, $\exists u^* \in \tilde{H}$, such that

$$\begin{aligned} & \langle D_t u^*, v \rangle_{\rho}^{\sim} + \mathcal{M}(u^*, v) \\ &= \lambda_1 \langle u^*, v \rangle_{\rho}^{\sim} + \int_{\Omega} f(x, u^*) v \rho - G(v), \quad \forall v \in \tilde{H}. \end{aligned} \quad (25)$$

Remark 7. Observing that, for $N = 2$, $f(x, s) = -g(x)s|s|^{5/3} - \epsilon_0^* s$, where $g(x) \in C^0(\Omega) \cap L^{\infty}(\Omega)$ is a positive function and $g(x)$ meets ($f-2$) and (24).

3. Preliminary Lemmas

In this section, we introduce some lemmas. First, we introduce some notions.

If the conditions of simple $V_L(\tilde{\Omega}, \Gamma)$ hold, we get

$$\{\tilde{\varphi}_{jk}^c\}_{j=1,k=0}^{\infty,\infty} \cup \{\tilde{\varphi}_{jk}^s\}_{j=1,k=1}^{\infty,\infty} \text{ is a CONS for } \tilde{L}_\rho^2, \quad (26)$$

where

$$\begin{aligned} \text{(i)} \quad \tilde{\varphi}_{jk}^c(x, t) &= \begin{cases} \frac{1}{\sqrt{2\pi}} \varphi_j(x), & k=0, \quad j=1, 2, \dots, \\ \frac{1}{\sqrt{\pi}} \varphi_j(x) \cos kt, & j, k=1, 2, \dots, \end{cases} \\ \text{(ii)} \quad \tilde{\varphi}_{jk}^s(x, t) &= \frac{1}{\sqrt{\pi}} \varphi_j(x) \sin kt, \quad j, k=1, 2, \dots \end{aligned} \quad (27)$$

Obviously, both $\tilde{\varphi}_{jk}^c$ and $\tilde{\varphi}_{jk}^s$ are in $\tilde{H}$. Defining

$$\mathcal{L}_1(u, v) = \tilde{\mathcal{L}}(u, v) + \langle u, v \rangle_\rho, \quad \forall u, v \in \tilde{H}, \quad (28)$$

it is clear that $\mathcal{L}_1(u, v)$ is an inner product on $\tilde{H}$. From (10), (16), and (28), there are $K_1, K_2 > 0$ such that

$$K_1 \|v\|_{\tilde{H}}^2 \leq \mathcal{L}_1(v, v) + \|D_t v\|_\rho^2 \leq K_2 \|v\|_{\tilde{H}}^2, \quad \forall v \in \tilde{H}. \quad (29)$$

For $v \in \tilde{L}_\rho^2$, setting

$$\hat{v}^c(j, k) = \langle v, \tilde{\varphi}_{jk}^c \rangle_\rho, \quad \hat{v}^s(j, k) = \langle v, \tilde{\varphi}_{jk}^s \rangle_\rho, \quad (30)$$

and from (V_L-2) , (16), and (28), we see that, for $v \in \tilde{H}$,

$$\begin{aligned} \mathcal{L}_1(v, \tilde{\varphi}_{jk}^s) &= (\lambda_j + 1) \hat{v}^s(j, k), \\ \mathcal{L}_1(v, \tilde{\varphi}_{jk}^c) &= (\lambda_j + 1) \hat{v}^c(j, k). \end{aligned} \quad (31)$$

Lemma 8. If $\{\tilde{\varphi}_{jk}^c\}_{j=1,k=0}^{\infty,\infty} \cup \{\tilde{\varphi}_{jk}^s\}_{j=1,k=1}^{\infty,\infty}$ is a CONS for $L_\rho^2(\tilde{\Omega})$ defined by (27), setting

$$\tau_n(v) = \sum_{j=1}^n \hat{v}^c(j, 0) \tilde{\varphi}_{j0}^c + \sum_{j,k=1}^n (\hat{v}^c(j, k) \tilde{\varphi}_{jk}^c + \hat{v}^s(j, k) \tilde{\varphi}_{jk}^s), \quad (32)$$

then

$$\lim_{n \rightarrow \infty} \|\tau_n(v) - v\|_{\tilde{H}} = 0, \quad \forall v \in \tilde{H}. \quad (33)$$

Lemma 9. (i) If $v \in \tilde{H}$, then

$$\begin{aligned} &\mathcal{L}_1(v, v) + \|D_t v\|_\rho^2 \\ &= \sum_{j=1}^{\infty} |\hat{v}^c(j, 0)|^2 (\lambda_j + 1) \\ &\quad + \sum_{j,k=1}^{\infty} (|\hat{v}^c(j, k)|^2 + |\hat{v}^s(j, k)|^2) (\lambda_j + 1 + k^2). \end{aligned} \quad (34)$$

(ii) If $v \in L_\rho^2(\tilde{\Omega})$ and $\mathcal{L}_1(v, v) + \|D_t v\|_\rho^2 < \infty$, then $v \in \tilde{H}$.

Lemma 10. Let $\mathcal{L}$ be as Theorem 6 and assume that $(\tilde{\Omega}, \Gamma)$ is a simple $V_L(\tilde{\Omega}, \Gamma)$. Then $\tilde{H}$ is compactly imbedded in $L_\rho^2(\tilde{\Omega})$.

For the proofs of Lemmas 8–10, one can refer to [5].

We define

$$S_n = \left\{ v \in \tilde{H} : v = \sum_{j=1}^n \eta_{j0}^c \tilde{\varphi}_{j0}^c + \sum_{j,k=1}^n \eta_{jk}^c \tilde{\varphi}_{jk}^c + \eta_{jk}^s \tilde{\varphi}_{jk}^s \right\}. \quad (35)$$

Remark 11. If $u_n \in S_n$, then $\mathcal{M}(u_n, D_t u_n) = 0$.

Lemma 12. Let $\mathcal{L}$ be given as (13) and suppose that $(\tilde{\Omega}, \Gamma)$ is a simple $V_L(\tilde{\Omega}, \Gamma)$. Then $\tilde{H}$ is continuously imbedded in $L_\rho^\theta(\tilde{\Omega})$ for every θ satisfying $2 < \theta < 2N/(N-1)$, that is, $\exists K_\theta > 0$, such that

$$\left(\int_{\tilde{\Omega}} |u|^\theta \rho dx dt \right)^{1/\theta} \leq K_\theta \|u\|_{\tilde{H}}, \quad \forall u \in \tilde{H}. \quad (36)$$

Proof. To establish the lemma, we need to only prove the case that

$$\left(\int_{\tilde{\Omega}} |v|^\theta \rho dx dt \right)^{1/\theta} \leq K_\theta \|v\|_{\tilde{H}}, \quad (37)$$

where $v \in \tilde{C}_{p,\rho}^{1b}(\tilde{\Omega}, \Gamma)$, for $N \geq 2$.

For fixed $(x_2, \dots, x_N)$, $v(x_1, x_2, \dots, x_N, t)$ is a function only depending on (x_1, t) in $C^1(\tilde{\Omega}_1)$. Hence, with (18) and (19), one gets

$$|v(x_1, x_2, \dots, x_N, t)| \leq h_1(x_1) y_1(t) g_1^{1/2}(x_2, \dots, x_N), \quad (38)$$

where $g_1(x_2, \dots, x_N) = \int_{\tilde{\Omega}_1} [|D_1 v(x, t)|^2 p_1^*(x_1) + (|v(x, t)|^2 + |D_t v(x, t)|^2) \rho_1^*(x_1)] dx_1 dt$. Applying the same kind of reasoning to (x_i, t) for fixed $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$, we have

$$|v(x, t)|^N \leq \prod_{i=1}^N h_i(x_i) y_i(t) g_i^{1/2}(\tilde{x}_i), \quad (39)$$

where

$$\tilde{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N), \quad (40)$$

$$g_i(\tilde{x}_i)$$

$$= \int_{\tilde{\Omega}_i} [|D_i v(x, t)|^2 p_i^*(x_i) \quad (41)$$

$$+ (|D_t v(x, t)|^2 + |v(x, t)|^2) \rho_i^*(x_i)] dx_i dt.$$

By (39), it follows that

$$\begin{aligned} &|v(x, t)|^\theta \\ &\leq \prod_{i=1}^N |y_i(t)|^{\theta/N} \prod_{i=2}^N |h_i(x_i)|^{\theta/N} |g_1(\tilde{x}_1)|^{\theta/2N} |h_1(x_1)|^{\theta/N} \\ &\quad \times \prod_{i=2}^N |g_i(\tilde{x}_i)|^{\theta/2N}, \end{aligned} \quad (42)$$

with $2 < \theta < 2N/(N-1)$ ($N \geq 2$). Taking $r_1 = \beta = 2N/(2N - (N-1)\theta)$ and $r_2 = \dots = r_N = \gamma = 2N/\theta$ and applying the generalized version of Hölder inequality on Ω_1 , we deduce

$$\begin{aligned} & \int_{\Omega_1} |v(x, t)|^\theta \rho_1^*(x_1) dx_1 \\ & \leq \prod_{i=1}^N |y_i(t)|^{\theta/N} \prod_{i=2}^N |h_i(x_i)|^{\theta/N} |g_1(\check{x}_1)|^{\theta/2N} \\ & \quad \cdot \left[\int_{\Omega_1} |h_1(x_1)|^{\beta\theta/N} \rho_1^*(x_1) dx_1 \right]^{1/\beta} \\ & \quad \times \prod_{i=2}^N \left[\int_{\Omega_1} g_i(\check{x}_i) \rho_1^*(x_1) dx_1 \right]^{1/\gamma}. \end{aligned} \quad (43)$$

Integrating the last inequality on both sides with respect to x_2 , we get

$$\begin{aligned} & \int_{\Omega_2} \int_{\Omega_1} |v(x, t)|^\theta \rho_1^*(x_1) \rho_2^*(x_2) dx_1 dx_2 \\ & \leq \prod_{i=1}^N |y_i(t)|^{\theta/N} \prod_{i=3}^N |h_i(x_i)|^{\theta/N} \left[\int_{\Omega_1} |g_2(\check{x}_2)| \rho_1^*(x_1) dx_1 \right]^{1/\gamma} \\ & \quad \cdot \prod_{i=1}^2 \left[\int_{\Omega_i} |h_i(x_i)|^{\beta\theta/N} \rho_i^*(x_i) dx_i \right]^{1/\beta} \\ & \quad \times \left[\int_{\Omega_2} |g_1(\check{x}_1)| \rho_2^*(x_2) dx_2 \right]^{1/\gamma} \\ & \quad \cdot \prod_{i=3}^N \left[\int_{\Omega_2} \int_{\Omega_1} g_i(\check{x}_i) \rho_1^*(x_1) \rho_2^*(x_2) dx_1 dx_2 \right]^{1/\gamma}. \end{aligned} \quad (44)$$

We continue integrating with respect to $x_3, \dots, x_{N-1}$ and eventually find

$$\begin{aligned} & \int_{\Omega_{N-1}} \int_{\Omega_{N-2}} \dots \int_{\Omega_1} |v(x, t)|^\theta \rho_1^*(x_1) \dots \rho_{N-2}^*(x_{N-2}) \\ & \quad \times \rho_{N-1}^*(x_{N-1}) dx_1 \dots dx_{N-2} dx_{N-1} \\ & \leq \prod_{i=1}^N |y_i(t)|^{\theta/N} \\ & \quad \times \prod_{i=1}^{N-1} \left[\int_{\Omega_i} |h_i(x_i)|^{\beta\theta/N} \rho_i^*(x_i) dx_i \right]^{1/\beta} |h_N(x_N)|^{\theta/N} \\ & \quad \cdot \left[\int_{\Omega_{N-2}} \dots \int_{\Omega_1} g_{N-1}(\check{x}_{N-1}) \rho_1^*(x_1) \right. \\ & \quad \left. \dots \rho_{N-2}^*(x_{N-2}) dx_1 \dots dx_{N-2} \right]^{1/\gamma} \end{aligned}$$

$$\begin{aligned} & \cdot \left[\int_{\Omega_{N-1}} \int_{\Omega_{N-2}} \dots \int_{\Omega_1} g_N(\check{x}_N) \rho_1^*(x_1) \dots \rho_{N-2}^*(x_{N-2}) \right. \\ & \quad \left. \times \rho_{N-1}^*(x_{N-1}) dx_1 \dots dx_{N-2} dx_{N-1} \right]^{1/\gamma} \\ & \cdot \left[\int_{\Omega_{N-1}} \int_{\Omega_{N-3}} \dots \int_{\Omega_1} g_{N-2}(\check{x}_{N-2}) \rho_1^*(x_1) \dots \rho_{N-3}^*(x_{N-3}) \right. \\ & \quad \left. \times \rho_{N-1}^*(x_{N-1}) dx_1 \dots dx_{N-3} dx_{N-1} \right]^{1/\gamma} \\ & \dots \\ & \cdot \left[\int_{\Omega_{N-1}} \dots \int_{\Omega_3} \int_{\Omega_1} g_2(\check{x}_2) \rho_1^*(x_1) \rho_3^*(x_3) \dots \rho_{N-1}^* \right. \\ & \quad \left. \times (x_{N-1}) dx_1 dx_3 \dots dx_{N-1} \right]^{1/\gamma} \\ & \cdot \left[\int_{\Omega_{N-1}} \int_{\Omega_{N-2}} \dots \int_{\Omega_3} \int_{\Omega_2} g_1(\check{x}_1) \right. \\ & \quad \times \rho_2^*(x_2) \rho_3^*(x_3) \\ & \quad \dots \rho_{N-2}^*(x_{N-2}) \\ & \quad \times \rho_{N-1}^*(x_{N-1}) dx_2 dx_3 \\ & \quad \left. \dots dx_{N-2} dx_{N-1} \right]^{1/\gamma}. \end{aligned} \quad (45)$$

Furthermore, we have

$$\begin{aligned} & \int_{\bar{\Omega}_N} \int_{\Omega_{N-1}} \int_{\Omega_{N-2}} \dots \int_{\Omega_1} |v(x, t)|^\theta \rho_1^*(x_1) \dots \rho_{N-2}^*(x_{N-2}) \\ & \quad \times \rho_{N-1}^*(x_{N-1}) \rho_N^*(x_N) dx_1 \\ & \quad \dots dx_{N-2} dx_{N-1} dx_N dt \\ & \leq \int_T \prod_{i=1}^N |y_i(t)|^{\theta/N} dt \\ & \quad \times \prod_{i=1}^N \left[\int_{\Omega_i} |h_i(x_i)|^{\beta\theta/N} \rho_i^*(x_i) dx_i \right]^{1/\beta} \left[\prod_{i=1}^N J_i \right]^{1/\gamma} \end{aligned} \quad (46)$$

with

$$\begin{aligned} J_i &= \int_{\Omega_N} \dots \int_{\Omega_{i+1}} \int_{\Omega_{i-1}} \dots \int_{\Omega_1} g_i(\check{x}_i) \rho_1^*(x_1) \\ & \quad \dots \rho_{i-1}^*(x_{i-1}) \rho_{i+1}^*(x_{i+1}) \\ & \quad \dots \rho_N^*(x_N) dx_1 \\ & \quad \dots dx_{i-1} dx_{i+1} \dots dx_N. \end{aligned} \quad (47)$$

From (18) we see that $h_i \in C^0(\Omega_i) \cap L_{\rho_i^*}^\theta(\Omega_i)$ and $y_i(t) \in C^0(T) \cap L^\theta(T)$. Hence, we obtain from (46) that there is a constant K_θ such that

$$\left(\int_{\bar{\Omega}} |v(x, t)|^\theta \rho(x) dx dt \right)^{1/\theta} \leq K_\theta \left(\prod_{i=1}^N J_i \right)^{1/2N}. \quad (48)$$

By (41), (47), and $p_i(x) = \rho_1^*(x_1) \cdots \rho_{i-1}^*(x_{i-1}) p_i^*(x_i) \rho_{i+1}^*(x_{i+1}) \cdots \rho_N^*(x_N)$, we see that

$$J_i = \int_{\bar{\Omega}} \left[|D_t v(x, t)|^2 p_i(x) + (|D_t v(x, t)|^2 + |v(x, t)|^2) \rho(x) \right] dx dt. \quad (49)$$

Therefore $J_i \leq \|v\|_{\bar{H}}^2$ for $i = 1, \dots, N$, and $\prod_{i=1}^N J_i \leq \|v\|_{\bar{H}}^{2N}$.

By using (48), we get (36). The proof of Lemma 12 is complete. $\square$

Lemma 13. Let $\mathcal{L}u$ be given by (13) and suppose that $(\bar{\Omega}, \Gamma)$ is a simple $V_L(\bar{\Omega}, \Gamma)$. Then, for $N \geq 2$, $\bar{H}$ is compactly imbedded in $L_\rho^\theta(\bar{\Omega})$ for $2 < \theta < 2N/(N-1)$. For $N = 1$, $\bar{H}$ is compactly imbedded in $L_\rho^\theta(\bar{\Omega})$ for $2 < \theta < \infty$.

Lemma 13 is an immediate byproduct of [1, Theorem 9].

4. Proof of Theorem 6

In this section, we will give the proof of Theorem 6. In order to do this, we divide the proof into three parts.

Lemma 14. Let all the assumptions in Theorem 6 hold. Then, for $n \geq 2$, $\exists u_n \in S_n$ such that

$$\begin{aligned} & \langle D_t u_n, v \rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, v) \\ &= \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, v \rangle_\rho^\sim + \langle f(x, u_n), v \rangle_\rho^\sim - G(v), \quad \forall v \in S_n. \end{aligned} \quad (50)$$

Proof. To prove the lemma, we first observe from (35) that

$$\begin{aligned} & \text{(i) } v \in S_n \implies D_t v \in S_n, \\ & \text{(ii) } \langle D_t (\alpha \tilde{\varphi}_{jk}^c + \beta \tilde{\varphi}_{jk}^s), \alpha \tilde{\varphi}_{jk}^c + \beta \tilde{\varphi}_{jk}^s \rangle_\rho^\sim = 0, \end{aligned} \quad (51)$$

for $j, k \geq 1$, $\alpha, \beta \in \mathbb{R}$.

Let $\{\psi_i\}_{i=1}^{2n^2+n}$ be an enumeration of $\{\tilde{\varphi}_{jk}^c\}_{j=1, k=0}^{n, n} \cup \{\tilde{\varphi}_{jk}^s\}_{j=1, k=1}^{n, n}$, and set

$$n^* = (j_0 + j_1 - 1)(2n + 1). \quad (52)$$

So $\{\psi_i\}_{i=1}^{n^*}$ is an enumeration of $\{\tilde{\varphi}_{jk}^c\}_{j=1, k=0}^{j_0+j_1-1, n} \cup \{\tilde{\varphi}_{jk}^s\}_{j=1, k=1}^{j_0+j_1-1, n}$, where $n \geq n_0$.

For $\alpha = (\alpha_1, \dots, \alpha_{2n^2+n})$, setting

$$u = \sum_{i=1}^{2n^2+n} \alpha_i \psi_i, \quad \tilde{u} = \sum_{i=1}^{2n^2+n} \delta_i \alpha_i \psi_i, \quad (53)$$

where

$$\delta_i = \begin{cases} -1, & 1 \leq i \leq n^*, \\ 1, & n^* + 1 \leq i \leq 2n^2 + n, \end{cases} \quad (54)$$

from (52)–(54), we have

$$\tilde{u} = \sum_{j=1}^n \delta_j \tilde{u}^c(j, 0) \tilde{\varphi}_{j0}^c + \sum_{j,k=1}^n \delta_j [\tilde{u}^c(j, k) \tilde{\varphi}_{jk}^c + \tilde{u}^s(j, k) \tilde{\varphi}_{jk}^s],$$

$$u = \sum_{j=1}^n \tilde{u}^c(j, 0) \tilde{\varphi}_{j0}^c + \sum_{j,k=1}^n [\tilde{u}^c(j, k) \tilde{\varphi}_{jk}^c + \tilde{u}^s(j, k) \tilde{\varphi}_{jk}^s],$$

$$\delta_j = \begin{cases} -1, & 1 \leq j \leq j_0 + j_1 - 1, \\ 1, & j_0 + j_1 \leq j \leq n, \end{cases}$$

$$\begin{aligned} & \mathcal{L}_1(u, u) + \|D_t u\|_\rho^2 \\ &= \sum_{j=1}^n |\tilde{u}^c(j, 0)|^2 (\lambda_j + 1) \\ &+ \sum_{j=1}^n \sum_{k=1}^n [|\tilde{u}^c(j, k)|^2 + |\tilde{u}^s(j, k)|^2] (\lambda_j + 1 + k^2). \end{aligned} \quad (55)$$

Further, (29) yields

$$\begin{aligned} & K_1 \|u\|_{\bar{H}}^2 \leq \mathcal{L}_1(u, u) + \|D_t u\|_\rho^2 \\ &\leq \sum_{j=1}^n |\tilde{u}^c(j, 0)|^2 (\lambda_n + 1) \\ &+ \sum_{j,k=1}^n [|\tilde{u}^c(j, k)|^2 + |\tilde{u}^s(j, k)|^2] (\lambda_n + 1 + n^2) \\ &\leq (\lambda_n + 1 + n^2) \|u\|_\rho^2, \quad \forall u \in S_n. \end{aligned} \quad (56)$$

For $m \geq 2$, we set

$$f_m(x, s) = \begin{cases} f(x, m), & \text{for } m \leq s, \\ f(x, s), & \text{for } -m \leq s \leq m, \\ f(x, m), & \text{for } s \leq -m, \end{cases} \quad (57)$$

and we claim that $\exists u_m^* \in S_n$ such that

$$\begin{aligned} & \langle D_t u_m^*, v \rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_m^*, v) \\ &= \left(\lambda_1 - \frac{1}{n} \right) \langle u_m^*, v \rangle_\rho^\sim + \int_{\bar{\Omega}} f_m(x, u_m^*) v \rho - G(v), \quad \forall v \in S_n. \end{aligned} \quad (58)$$

By (f-2) and the above,

$$\begin{aligned} & |f_m(x, s)| \leq h_0(x) + K |m|^{\theta-1}, \\ & \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \end{aligned} \quad (59)$$

where $h_0(x) \in L_{\rho}^{\theta^*}$ and $\theta^* = \theta/(\theta - 1)$. Therefore, $f_m(x, u) \in L_{\rho}^{\theta^*}(\bar{\Omega})$ and $\int_{\bar{\Omega}} f_m(x, u) v \rho$ is well defined for $v \in S_n$ by Lemma 12.

To establish (58), we set

$$F_i(\alpha) = \langle D_t u, \psi_i \rangle_{\rho}^{\sim} + \mathcal{M}(u, \psi_i) - \left(\lambda_1 - \frac{1}{n} \right) \langle u, \psi_i \rangle_{\rho}^{\sim} - \langle f_m(x, u), \psi_i \rangle_{\rho}^{\sim} + G(\psi_i) \quad (60)$$

for $i = 1, \dots, n$, and we observe from (51) that

$$\sum_{i=1}^{2n^2+n} F_i(\alpha) \alpha_i = I(\alpha) + II(\alpha), \quad (61)$$

where

$$I(\alpha) = \mathcal{L}(u, u) - \left(\lambda_1 - \frac{1}{n} \right) \langle u, u \rangle_{\rho}^{\sim}, \quad (62)$$

$$II(\alpha) = \mathcal{M}(u, u) - \mathcal{L}(u, u) - \int_{\bar{\Omega}} f_m(x, u) u \rho + G(u). \quad (63)$$

Now, from (62),

$$I(\alpha) = \sum_{i=1}^{2n^2+n} (\lambda_i - \lambda_1) \alpha_i^2 + \frac{|\alpha|^2}{n}, \quad (64)$$

and according to Lemma 12, (56), (59), (63), and the fact that $\mathcal{M}$ is $\# \tilde{H}$ -related to $\mathcal{L}$,

$$\lim_{|\alpha| \rightarrow \infty} \frac{|II(\alpha)|}{|\alpha|^2} = 0. \quad (65)$$

We see from (61) that $\exists s_0 > 0$ such that

$$\sum_{i=1}^{2n^2+n} F_i(\alpha) \alpha_i \geq \frac{|\alpha|^2}{2n}, \quad \text{for } |\alpha| > s_0. \quad (66)$$

Therefore, by virtue of generalized Brouwer's theorem [10], there exists $\alpha^* = (\alpha_1^*, \dots, \alpha_{2n^2+n}^*)$ such that $F_i(\alpha^*) = 0$, for $i = 1, \dots, n$. We set $u_m^* = \sum_{i=1}^{2n^2+n} \alpha_i^* \psi_i$ and see from (60) that (58) does indeed hold.

By the definition of $f_m(x, s)$ and (24), for $m \geq 2$,

$$sf_m(x, s) \leq \tilde{h}_0(x) |s|, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \quad (67)$$

where $\tilde{h}_0(x) \in L_{\rho}^{\theta^*}(\Omega)$.

We claim

$$\exists K_1 > 0 \quad \text{s.t.} \quad \|u_m^*\|_{\rho} \leq K_1, \quad \forall m > 2. \quad (68)$$

Suppose that (68) is false. Then there is a subsequence (which for ease of notation, we take to be the full sequence) such that

$$\lim_{|m| \rightarrow \infty} \|u_m^*\|_{\rho} = \infty. \quad (69)$$

Inserting u_m^* in place of v in (58), we find

$$\begin{aligned} & \mathcal{L}(u_m^*, u_m^*) - \left(\lambda_1 - \frac{1}{n} \right) \langle u_m^*, u_m^* \rangle_{\rho}^{\sim} \\ &= \int_{\bar{\Omega}} f_m(x, u_m^*) u_m^* \rho - G(u_m^*) + \mathcal{L}(u_m^*, u_m^*) - \mathcal{M}(u_m^*, u_m^*), \end{aligned} \quad (70)$$

and the left-hand side of (70) is

$$\begin{aligned} & \sum_{j=1}^n |\hat{u}_m^{*c}(j, 0)|^2 (\lambda_j - \lambda_1) \\ &+ \sum_{j,k=1}^n (|\hat{u}_m^{*c}(j, k)|^2 + |\hat{u}_m^{*s}(j, k)|^2) (\lambda_j - \lambda_1) + \frac{\|u_m^*\|_{\rho}^2}{n}. \end{aligned} \quad (71)$$

So we see from (67) and Hölder inequality that

$$\begin{aligned} \frac{\|u_m^*\|_{\rho}^2}{n} &\leq \|\tilde{h}_0\|_{L_{\rho}^{\theta^*}} \|u_m^*\|_{L_{\rho}^{\theta}} + |G(u_m^*)| \\ &+ |\mathcal{L}(u_m^*, u_m^*) - \mathcal{M}(u_m^*, u_m^*)|. \end{aligned} \quad (72)$$

From (36), (56), (69), and the fact that $\mathcal{M}$ is $\# \tilde{H}$ -related to $\mathcal{L}$, dividing both sides of (72) by $\|u_m^*\|_{\rho}^2$ and taking the limit as $m \rightarrow \infty$, we obtain that $1/n \leq 0$. But n is a positive integer. So we get a contradiction. Therefore, (68) does indeed hold.

Hence, from (56), (68), and $u_m^* \in S_n$, for $m \geq 2$, there is a subsequence $\{u_{m_j}^*\}_{j=1}^{\infty}$ and a $u_n \in S_n$ [11] such that

$$\begin{aligned} & \lim_{j \rightarrow \infty} \|u_{m_j}^* - u_n\|_{\tilde{H}} = 0; \\ & \lim_{j \rightarrow \infty} u_{m_j}^*(x, t) = u_n(x, t), \quad \text{a.e. } (x, t) \in \bar{\Omega}; \\ & \lim_{j \rightarrow \infty} D_i u_{m_j}^*(x, t) = D_i u_n(x, t), \\ & \text{a.e. } (x, t) \in \bar{\Omega}, \quad i = 1, \dots, N. \end{aligned} \quad (73)$$

It is easy to check that $\tilde{M}(u_{m_j}^*, v) \rightarrow \tilde{M}(u_n, v)$ as $j \rightarrow \infty$, $\forall v \in S_n$, and from Lemma 12 we obtain that $\lim_{j \rightarrow \infty} \int_{\bar{\Omega}} |u_{m_j}^* - u_n|^{\theta} \rho = 0$. Consequently, there exist a function $W \in L_{\rho}^{\theta}(\bar{\Omega})$ and a subsequence $\{u_{m_{j_k}}^*\}_{k=1}^{\infty}$ such that

$$\begin{aligned} & |u_{m_{j_k}}^*(x, t)| \leq W(x, t) \quad \text{a.e. } (x, t) \in \bar{\Omega}, \quad \forall k, \\ & \lim_{k \rightarrow \infty} f_{m_{j_k}}(x, u_{m_{j_k}}^*) = f(x, u_n) \quad \text{a.e. } (x, t) \in \bar{\Omega}. \end{aligned} \quad (74)$$

By using (f-2), Hölder inequality, and the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{k \rightarrow \infty} \int_{\bar{\Omega}} f_{m_{j_k}}(x, u_{m_{j_k}}^*) v \rho = \int_{\bar{\Omega}} f(x, u_n) v \rho, \quad \forall v \in S_n. \quad (75)$$

Hence replacing u_m^* by $u_{m_{j_k}}^*$ in (58) and passing to the limit as $k \rightarrow \infty$, we consequently obtain that

$$\begin{aligned} & \langle D_t u_n, v \rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, v) \\ &= \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, v \rangle_\rho^\sim + \langle f(x, u_n), v \rangle_\rho^\sim - G(v), \quad \forall v \in S_n, \end{aligned} \quad (76)$$

and the proof of the lemma is complete. $\square$

Lemma 15. *Let the conditions in Lemma 14 hold. Then the sequence $\{u_n\}$ obtained in Lemma 14 is uniformly bounded in $\widetilde{H}$ with respect to the norm $\|u_n\|_{\widetilde{H}} = \langle u_n, u_n \rangle_{\widetilde{H}}^{1/2}$.*

Proof. According to Lemma 14, we obtain a sequence $\{u_n\}_{n=2}^\infty$, where

$$u_n = \sum_{i=1}^{2n^2+n} \alpha_{(i,n)} \psi_i, \quad (77)$$

and u_n satisfies

$$\begin{aligned} & \langle D_t u_n, v \rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, v) \\ &= \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, v \rangle_\rho^\sim + \langle f(x, u_n), v \rangle_\rho^\sim - G(v), \quad \forall v \in S_n. \end{aligned} \quad (78)$$

We claim there is a constant K such that

$$\|u_n\|_{\widetilde{H}} \leq K, \quad \forall n. \quad (79)$$

Suppose that the assertion is false. Then, without loss of generality, we assume

$$\lim_{n \rightarrow \infty} \|u_n\|_{\widetilde{H}} = \infty. \quad (80)$$

Putting u_n in place of v in (78), we have

$$\begin{aligned} & \langle D_t u_n, u_n \rangle_\rho^\sim + \widetilde{\mathcal{M}}(u_n, u_n) \\ &= \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, u_n \rangle_\rho^\sim + \langle f(x, u_n), u_n \rangle_\rho^\sim - G(u_n). \end{aligned} \quad (81)$$

Next, we observe from (77) and Lemma 8 that

$$\begin{aligned} \widetilde{\mathcal{L}}(u_n, u_n) &= \sum_{j=1}^n |\widehat{u}_n^c(j, 0)|^2 \lambda_j \\ &+ \sum_{j,k=1}^n (|\widehat{u}_n^c(j, k)|^2 + |\widehat{u}_n^s(j, k)|^2) \lambda_j \\ &\geq \lambda_1 \|u_n\|_\rho^2 \end{aligned} \quad (82)$$

and from (78), (24), and Hölder inequality that

$$\begin{aligned} \varepsilon_0^* \|u_n\|_\rho^2 &\leq \|\widetilde{h}_0\|_{\widetilde{L}^{\theta^*}} \|u_n\|_{\widetilde{L}^\theta} + |G(u_n)| \\ &+ |\widetilde{\mathcal{L}}(u_n, u_n) - \widetilde{\mathcal{M}}(u_n, u_n)|. \end{aligned} \quad (83)$$

Hence it follows from Lemma 12, $G \in (\widetilde{H})'$, $\widetilde{\mathcal{M}}$ being $\# \widetilde{H}$ -related to $\widetilde{\mathcal{L}}$, ε_0^* being strictly positive, and (83) that

$$\lim_{n \rightarrow \infty} \frac{\|u_n\|_\rho}{\|u_n\|_{\widetilde{H}}} = 0. \quad (84)$$

Next, we observe from (12) and (16) that there is a positive constant c such that

$$c \widetilde{\mathcal{L}}(u_n, u_n) \leq \widetilde{\mathcal{M}}(u_n, u_n), \quad \forall n. \quad (85)$$

Putting $D_t u_n$ for v in (78), we see that there exists $K > 0$ such that

$$\|D_t u_n\|_\rho \leq K. \quad (86)$$

But then it follows from (81), (24), and $\|D_t u_n\|_\rho \leq K$ that

$$\begin{aligned} & c \widetilde{\mathcal{L}}(u_n, u_n) + \varepsilon_0^* \|u_n\|_\rho^2 + \|D_t u_n\|_\rho^2 \\ & \leq \|\widetilde{h}_0\|_{\widetilde{L}^{\theta^*}} \|u_n\|_{\widetilde{L}^\theta} + \lambda_1 \|u_n\|_\rho^2 + |G(u_n)| + K. \end{aligned} \quad (87)$$

So, we see from (28), (29), and (87) that there is a positive constant K_1 such that

$$\|u_n\|_{\widetilde{H}}^2 \leq K_1 \left(\|\widetilde{h}_0\|_{\widetilde{L}^{\theta^*}} \|u_n\|_{\widetilde{L}^\theta} + \lambda_1 \|u_n\|_\rho^2 + |G(u_n)| + K \right). \quad (88)$$

Consequently, dividing both sides of (88) by $\|u_n\|_{\widetilde{H}}^2$ and passing to the limit as $n \rightarrow \infty$, from Lemma 12, the fact that $G \in (\widetilde{H})'$, (80), and (84), we obtain that $1 \leq 0$. This is a contradiction. Hence (80) does not hold and (79) is true. $\square$

Proof of Theorem 6. Since $\widetilde{H}$ is a separable Hilbert space, we see from (79) and Lemmas 9 and 13 that there exist a subsequence (for the sake of simplicity, we take to be a full sequence) and a function $u^* \in \widetilde{H}$ with the following properties [11]:

- (1) $\lim_{n \rightarrow \infty} \left(\|u_n - u^*\|_\rho + \int_{\Omega} |u_n - u^*|^\theta \rho \right) = 0$;
- (2) $\exists w(x, t) \in \widetilde{L}_\rho^2 \cap \widetilde{L}_\rho^\theta$, s.t. $|u_n(x, t)| \leq w(x, t)$,
a.e. $(x, t) \in \widetilde{\Omega}$, $\forall n$;
- (3) $\lim_{n \rightarrow \infty} u_n(x, t) = u^*(x, t)$, a.e. $(x, t) \in \widetilde{\Omega}$;
- (4) $\lim_{n \rightarrow \infty} \langle D_t u_n, v \rangle_{p_i}^\sim = \langle D_t u^*, v \rangle_{p_i}^\sim$,
 $\forall v \in \widetilde{L}_{p_i}^2$, $i = 1, \dots, N$;
- (5) $\lim_{n \rightarrow \infty} \langle u_n, v \rangle_q^\sim = \langle u^*, v \rangle_q^\sim$, $\forall v \in \widetilde{L}_q^2$;
- (6) $\lim_{n \rightarrow \infty} G(u_n) = G(u^*)$;
- (7) $\lim_{n \rightarrow \infty} \langle D_t u_n, v \rangle_\rho^\sim = \langle D_t u^*, v \rangle_\rho^\sim$, $\forall v \in \widetilde{L}_\rho^2$.

We let $v_J \in S_J$. Then it follows from (78) that, for $n \geq J$,

$$\begin{aligned} & \langle D_t u_n, v_J \rangle_\rho^\sim + \mathcal{M}(u_n, v_J) \\ &= \left(\lambda_1 - \frac{1}{n} \right) \langle u_n, v_J \rangle_\rho^\sim + \langle f(x, u_n), v_J \rangle_\rho^\sim - G(v_J). \end{aligned} \quad (90)$$

Now, from $(\sigma-1)$ -($\sigma-2$) and (89)(1)(4)(7), we obtain that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \langle D_t u_n, v_J \rangle_\rho^\sim + \mathcal{M}(u_n, v_J) \\ &= \langle D_t u_n, v_J \rangle_\rho^\sim + \mathcal{M}(u_n, v_J). \end{aligned} \quad (91)$$

Next, we observe from (f-2) and (89)(2) that

$$\begin{aligned} & |f(x, u_n)| \leq K |w(x, t)|^{\theta-1} + h_0(x) \\ & \text{a.e. } (x, t) \in \tilde{\Omega}, \quad n = 1, 2, \dots, \end{aligned} \quad (92)$$

where $w \in \tilde{L}_\rho^\theta$ and $h_0 \in L_\rho^{\theta^*}$. From (f-1) and (89)(3), we further obtain that

$$\lim_{n \rightarrow \infty} f(x, u_n(x, t)) = f(x, u^*(x, t)) \quad \text{a.e. } (x, t) \in \tilde{\Omega}. \quad (93)$$

Note that $|w|^{\theta-1} \in \tilde{L}_\rho^{\theta^*}$. Hence we conclude from Hölder inequality, the Lebesgue dominated convergence theorem, Lemma 12, and (92) that

$$\lim_{n \rightarrow \infty} \int_{\tilde{\Omega}} f(x, u_n(x, t)) v_J \rho = \int_{\tilde{\Omega}} f(x, u^*(x, t)) v_J \rho. \quad (94)$$

Then passing to the limit on both sides of (90), we obtain from (91), (94), and (89)(1) that

$$\begin{aligned} & \langle D_t u^*, v_J \rangle_\rho + \mathcal{M}(u^*, v_J) \\ &= \lambda_1 \langle u^*, v_J \rangle_\rho + \int_{\tilde{\Omega}} f(x, u^*(x, t)) v_J \rho - G(v_J). \end{aligned} \quad (95)$$

Next, for given $v \in \tilde{H}$, we replace v_J with $\tau_J(v)$ (defined as (32)) in (95). From (29) and Lemma 8, $\|\tau_J(v) - v\|_{\tilde{H}} \rightarrow 0$ as $J \rightarrow \infty$. From (95) and Lemma 12, it is then an easy matter to obtain that

$$\begin{aligned} & \langle D_t u^*, v \rangle_\rho^\sim + \mathcal{M}(u^*, v) \\ &= \lambda_1 \langle u^*, v \rangle_\rho^\sim + \langle f(x, u^*), v \rangle_\rho^\sim \\ & \quad - G(v), \quad \forall v \in \tilde{H}, \end{aligned} \quad (96)$$

and the proof of Theorem 6 is established. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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