Research Article

Powers of Complex Persymmetric Antitridiagonal Matrices with Constant Antidiagonals

Haibo Wang

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

Correspondence should be addressed to Haibo Wang; nunwanghaibo@126.com

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We derive a general expression for the p th power ($p \in \mathbb{N}$) of any complex persymmetric antitridiagonal Hankel (constant antidiagonals) matrices. Numerical examples are presented, which show that our results generalize the results in the related literature (Rimas 2008, Wu 2010, and Rimas 2009).

1. Introduction

Solving some difference, differential, and delay differential equations, we meet the necessity to compute the arbitrary positive integer powers of square matrix. Recently, computing the integer powers of antitridiagonal matrices has been a very popular problem. There have been several papers on computing the positive integer powers of various kinds of square matrices by Rimas et al., and others [1–5]. In 2011, the general expression for the entries of the power of complex persymmetric or skew-persymmetric antitridiagonal matrices with constant antidiagonals is presented by Gutiérrez-Gutiérrez [1]. Rimas [2] gave the general expression of the p th power for this type of symmetric odd order antitridiagonal matrices ($\text{antitridiag}_n(1, 0, 1)$) in 2008. In [3, 4] a similar problem is solved for antitridiagonal matrices having zeros in main skew diagonal and units in the neighbouring diagonals. In 2010, the general expression for the entries of the power of odd order antitridiagonal matrices with zeros in main skew diagonal and elements $1, 1, \dots, 1; -1, -1, \dots, -1$ in neighbouring diagonals is derived by Rimas [5]. In 2013, Rimas [6] gave the eigenvalue decomposition for real odd order skew-persymmetric antitridiagonal matrices with constant antidiagonals ($\text{antitridiag}_n(a, b, a)$) and derived the general expression for integer powers of such matrices.

In the present paper, we derive a general expression for the p th power ($p \in \mathbb{N}$) of any complex

persymmetric antitridiagonal matrices with constant antidiagonals ($\text{antitridiag}_n(b, a, b)$). This novel expression is both an extension of the one obtained by Rimas for the powers of the matrix $\text{antitridiag}_n(1, 0, 1)$ with $n \in \mathbb{N}$ (see [2] for the odd case and [5] for the even case) and an extension of the one obtained by Honglin Wu for the powers of the matrix $\text{antitridiag}_n(1, 1, 1)$ with $n \in \mathbb{N}$ (see [3] for the even case).

2. Derivation of General Expression

In this present paper, we study the entries of positive integer power of an $n \times n$ complex persymmetric antitridiagonal matrix with constant antidiagonals as follows:

$$B_n = \text{antitridiag}_n(b, a, b) = \begin{pmatrix} & & & & b & a \\ & & & & b & a & b \\ & & & b & a & b \\ & & \ddots & \ddots & \ddots & \\ b & a & b \\ a & b \end{pmatrix}, \quad (1)$$

where $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \{0\}$.

Consider the following $n \times n$ complex Toeplitz tridiagonal matrix:

$$A_n = \text{tridiag}_n(b, a, b) = \begin{pmatrix} a & b & & & \\ b & a & b & & \\ & b & a & b & \\ & & \ddots & \ddots & \ddots \\ & & & b & a & b \\ & & & & b & a \end{pmatrix}. \quad (2)$$

The next trivial result relates the matrix B_n with A_n and with the $n \times n$ backward identity [1]:

$$J_n = (\delta_{j,n-k+1})_{j,k=1}^n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}, \quad (3)$$

where δ is the Kronecker delta.

Lemma 1. Let $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus (0)$, and $n \in \mathbb{N}$. Then

$$B_n = J_n A_n, \quad (4)$$

where $B_n = \text{antitridiag}_n(b, a, b)$ and $A_n = \text{tridiag}_n(b, a, b)$.

Proof. We have

$$\begin{aligned} [J_n A_n]_{j,k} &= \sum_{h=1}^n [J_n]_{j,h} [A_n]_{h,k} \\ &= [A_n]_{n+1-j,k} \\ &= \begin{cases} b, & \text{if } n+1-(j+k) = -1, \\ a, & \text{if } n+1-(j+k) = 0, \\ b, & \text{if } n+1-(j+k) = 1, \\ 0, & \text{other.} \end{cases} \quad (5) \\ &= [B_n]_{j,k}. \end{aligned}$$

This completes the proof. $\square$

We will find the q th power ($q \in \mathbb{N}$) of the matrix (1). Theorem 2 relates all positive integer powers of the matrix B_n to A_n and J_n .

Theorem 2. If $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus (0)$, and $n \in \mathbb{N}$ and if $B_n = \text{antitridiag}_n(b, a, b)$, then

$$B_n^q = \begin{cases} A_n^q, & \text{if } q \text{ is even,} \\ J_n A_n^q, & \text{if } q \text{ is odd,} \end{cases} \quad (6)$$

where $A_n = \text{tridiag}_n(b, a, b)$.

Proof. We will proceed by induction on q . The case $q = 1$ is obvious.

Suppose that the result is true for $q \geq 1$ and consider case $q + 1$.

By the induction hypothesis we have

$$B_n^{q+1} = B_n B_n^q = \begin{cases} B_n J_n A_n^q, & \text{if } q + 1 \text{ is even,} \\ B_n A_n^q, & \text{if } q + 1 \text{ is odd.} \end{cases} \quad (7)$$

Since $B_n = J_n A_n$ we obtain that

$$B_n^{q+1} = \begin{cases} J_n A_n J_n A_n^q, & \text{if } q + 1 \text{ is even,} \\ J_n A_n (A_n^q), & \text{if } q + 1 \text{ is odd.} \end{cases} \quad (8)$$

Since A_n is symmetric and $J_n^{-1} = J_n$, we have

$$B_n^q = \begin{cases} A_n^q, & \text{if } q \text{ is even,} \\ J_n A_n^q, & \text{if } q \text{ is odd.} \end{cases} \quad (9)$$

This completes the proof. $\square$

Next, we have to solve A_n^q .

We begin this work by reviewing a theorem regarding the Hermitian Toeplitz tridiagonal matrix A_n .

Theorem 3. Let $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus (0)$, and $n \in \mathbb{N}$. Then A_n has eigenvalues

$$\lambda_i = a + 2|b| \cos \frac{i\pi}{n+1}, \quad 1 \leq i \leq n. \quad (10)$$

Proof. See [7]. $\square$

With the tridiagonal matrix $A_n = \text{tridiag}_n(b, a, b)$, we associate the polynomial sequence $\{P_i\}_{-1 \leq i \leq n}$ characterized by a three-term recurrence relation:

$$\begin{aligned} xP_i(x) &= bP_{i+1}(x) + aP_i(x) + bP_{i-1}(x), \\ i &= 0, 1, \dots, n-1. \end{aligned} \quad (11)$$

With initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, we can write the relations (11) in matrix form:

$$xQ_{n-1}(x) = A_n Q_{n-1}(x) + P_n(x) E_n, \quad (12)$$

where $Q_{n-1}(x) = [P_0(x), P_1(x), \dots, P_{n-1}(x)]^T$ and $E_n = [0, 0, \dots, 0, 1]^T \in \mathbb{R}^n$.

Lemma 4. For $i \geq 0$, the degree of the polynomial P_i is i and P_i and P_{i+1} has no common root.

Proof. See [7]. $\square$

One can show that the characteristic polynomial of A_n is precisely $(\prod_{k=1}^n c_k) P_n(x)$. Hence the eigenvalues of A_n are exactly the roots of P_n .

If $(\lambda_j)_{0 \leq j \leq n-1}$ are the roots of the polynomial P_n , then it follows from (12) that each λ_j is an eigenvalue of the matrix A_n and $Q_{n-1}(x) = [P_0(x), P_1(x), \dots, P_{n-1}(x)]^T$ is a corresponding eigenvector [5, 7, 8]. This observation should be taken into account elsewhere in the paper.

The polynomials $\{P_i\}_{0 \leq i \leq n}$ verify the well-known Christoffel-Darboux Identity.

Lemma 5. We have

$$\sum_{i=0}^{n-1} P_i(x) P_i(y) = \frac{P_n(y) P_{n-1}(x) - P_n(x) P_{n-1}(y)}{y - x}, \quad (13)$$

for $x \neq y$.

Proof. See [7]. $\square$

Tending y to x in formula (13), we get

$$\sum_{i=0}^{n-1} P_i(x) P_i(x) = P'_n(x) P_{n-1}(x) - P_n(x) P'_{n-1}(x). \quad (14)$$

Since the matrix A_n has distinct eigenvalues $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$, thus, the eigendecomposition of the matrix A_n is

$$A_n = TDT^{-1}, \quad (15)$$

where $D = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$ and T is the transforming matrix formed by the eigenvectors of A_n . Namely, $T = (t_{i,j} = P_{i-1}(\lambda_{j-1}))_{1 \leq i, j \leq n}$, where $\{P_i\}_{-1 \leq i \leq n}$ are defined as above.

Lemma 6. If $T^{-1} = (s_{i,j})_{1 \leq i, j \leq n}$, then

$$s_{i,j} = \frac{P_{j-1}(\lambda_{i-1})}{P'_n(\lambda_{i-1}) P_{n-1}(\lambda_{i-1})}. \quad (16)$$

Proof. By using the relations (13) and (14), we obtain

$$\sum_{k=1}^n s_{i,k} t_{k,j} = \sum_{k=1}^n \frac{P_{k-1}(\lambda_{i-1}) P_{k-1}(\lambda_{j-1})}{P'_n(\lambda_{i-1}) P_{n-1}(\lambda_{i-1})} = \delta_{i,j}, \quad (17)$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

This completes the proof. $\square$

For $q \in N$, we have $A_n^q = TD^qT^{-1}$.

We get immediately the following.

Theorem 7. Assume that $q \in N$ and $A_n^q = (\alpha_{i,j}^{(q)})_{1 \leq i, j \leq n}$. Then

$$\alpha_{i,j}^{(q)} = \sum_{k=1}^n \lambda_{k-1}^q \frac{P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_n(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}. \quad (18)$$

By using the Cauchy Integral Formula, we can give another expressions of the coefficients $\alpha_{i,j}^{(q)}$ as follows:

$$\alpha_{i,j}^{(q)} = \int_C z^q \frac{P_{i-1}(z) P_{j-1}(z)}{P_n(z) P_{n-1}(z)} dz, \quad (19)$$

where C is a closed curve containing the roots of P_n and no roots of P_{n-1} .

Corollary 8. If the matrix A_n is nonsingular with $A^{-1} = (\alpha_{i,j}^{(-1)})_{1 \leq i, j \leq n}$, then

$$A_n^{-1} = TD^{-1}T^{-1}, \quad (20)$$

$$\alpha_{i,j}^{(-1)} = \sum_{k=1}^n \frac{1}{\lambda_{k-1}} \frac{P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_n(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}.$$

By using the Cauchy Integral Formula, we can give other expressions of the coefficients $\alpha_{i,j}^{(-1)}$:

$$\alpha_{i,j}^{(-1)} = \int_C z^{-1} \frac{P_{i-1}(z) P_{j-1}(z)}{P_n(z) P_{n-1}(z)} dz, \quad (21)$$

where C is a closed curve containing the roots of P_n and no roots of P_{n-1} .

Theorem 9. Assume that $q \in N$ and $B_n^q = (\beta_{i,j}^{(q)})_{1 \leq i, j \leq n}$. Then

$$\beta_{i,j}^{(q)} = \begin{cases} \sum_{k=1}^n \lambda_{k-1}^q \frac{P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_n(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}, & \text{if } q \text{ is even,} \\ \sum_{k=1}^n \lambda_{k-1}^q \frac{P_{n-i}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_n(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}, & \text{if } q \text{ is odd.} \end{cases} \quad (22)$$

By using the Cauchy Integral Formula, we can give other expressions of the coefficients $\beta_{i,j}^{(q)}$:

$$\beta_{i,j}^{(q)} = \begin{cases} \int_C z^q \frac{P_{i-1}(z) P_{j-1}(z)}{P_n(z) P_{n-1}(z)} dz, & \text{if } q \text{ is even,} \\ \int_C z^q \frac{P_{n-i}(z) P_{j-1}(z)}{P_n(z) P_{n-1}(z)} dz, & \text{if } q \text{ is odd,} \end{cases} \quad (23)$$

where C is a closed curve containing the roots of P_n and no roots of P_{n-1} .

Proof. From Theorem 2 we get

$$[B_n^q]_{i,j} = \begin{cases} [A_n^q]_{i,j}, & \text{if } q \text{ is even,} \\ [J_n A_n^q]_{i,j} = \sum_{h=1}^n [J_n]_{i,h} [A_n^q]_{h,j}, & \text{if } q \text{ is odd.} \end{cases} \quad (24)$$

$$= \begin{cases} [A_n^q]_{i,j}, & \text{if } q \text{ is even,} \\ [A_n^q]_{n+1-i,j}, & \text{if } q \text{ is odd.} \end{cases}$$

Namely,

$$\beta_{i,j}^{(q)} = \begin{cases} \alpha_{i,j}^{(q)}, & \text{if } q \text{ is even,} \\ \alpha_{n+1-i,j}^{(q)}, & \text{if } q \text{ is odd.} \end{cases} \quad (25)$$

From Theorem 7 it follows that

$$\beta_{i,j}^{(q)} = \begin{cases} \sum_{k=1}^n \lambda_{k-1}^q \frac{P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_n(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}, & \text{if } q \text{ is even,} \\ \sum_{k=1}^n \lambda_{k-1}^q \frac{P_{n-i}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_n(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}, & \text{if } q \text{ is odd.} \end{cases} \quad (26)$$

By using the Cauchy Integral Formula, we can give other expressions of the coefficients $\beta_{i,j}^{(q)}$:

$$\beta_{i,j}^{(q)} = \begin{cases} \int_C z^q \frac{P_{i-1}(z) P_{j-1}(z)}{P_n(z) P_{n-1}(z)} dz, & \text{if } q \text{ is even,} \\ \int_C z^q \frac{P_{n-i}(z) P_{j-1}(z)}{P_n(z) P_{n-1}(z)} dz, & \text{if } q \text{ is odd.} \end{cases} \quad (27)$$

This completes the proof. $\square$

Corollary 10. Assume that $q \in \mathbb{N}$ and $B_n^{-1} = (\beta_{i,j}^{(-1)})_{1 \leq i,j \leq n}$. Then

$$\beta_{i,j}^{(-1)} = \sum_{k=1}^n \frac{1}{\lambda_{k-1}} \frac{P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_n(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}. \quad (28)$$

By using the Cauchy Integral Formula, we can give other expressions of the coefficients $\beta_{i,j}^{(-1)}$:

$$\beta_{i,j}^{(-1)} = \int_C z^{(-1)} \frac{P_{i-1}(z) P_{j-1}(z)}{P_n(z) P_{n-1}(z)} dz, \quad (29)$$

where C is a closed curve containing the roots of P_n and no roots of P_{n-1} .

3. Numerical Examples

Consider the n order antitridiagonal matrix B_n of the following type:

$$B_n = \begin{pmatrix} & & & & 1 & a \\ & & & & a & 1 \\ & & & 1 & a & 1 \\ & & \ddots & \ddots & \ddots & \\ & 1 & a & 1 & \ddots & \\ a & 1 & & & & \end{pmatrix}. \quad (30)$$

Assume that

$$A_n = \begin{pmatrix} a & 1 & & & \\ 1 & a & 1 & & \\ & 1 & a & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & a & 1 \\ & & & & 1 & a \end{pmatrix}, \quad (31)$$

$$J_n = \begin{pmatrix} & & & & 1 \\ & & & & \\ & & & 1 & \\ & & \ddots & \ddots & \\ & 1 & & & \\ 1 & & & & \end{pmatrix},$$

where A_n and J_n are $n \times n$ matrix. The polynomial sequence $\{P_i\}_{-1 \leq i \leq n}$ verifies

$$xP_i(x) = P_{i+1}(x) + aP_i(x) + P_{i-1}(x), \quad (32)$$

$$i = 0, 1, \dots, n-1,$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$.

By simple calculation we can show that

$$P_i(x) = U_i\left(\frac{x-a}{2}\right), \quad i = 0, \dots, n, \quad (33)$$

where U_i are the Chebyshev polynomials [8] of the second kind which satisfies the three-term recurrence relations:

$$2xU_i(x) = U_{i+1}(x) + U_{i-1}(x), \quad (34)$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$.

Each U_n satisfies

$$U_n(x) = \frac{\sin((n+1) \arccos x)}{\sin(\arccos x)}, \quad (35)$$

and thus the roots of $U_n(x)$ are $z_k = \cos(k\pi/(n+1))$, $k = 1, \dots, n$. Then, the eigenvalues of A are

$$\lambda_k = a + 2 \cos \frac{k\pi}{n+1}, \quad k = 1, \dots, n. \quad (36)$$

We get by Theorem 7 the following.

Assume that $q \in \mathbb{N}$ and $A_n^q = (\alpha_{i,j}^{(q)})_{1 \leq i,j \leq n}$. Then

$$\begin{aligned} & \alpha_{i,j}^{(q)} \\ &= 2 \sum_{k=1}^n \left(a + 2 \cos \frac{k\pi}{n+1} \right)^q \\ & \quad \times \frac{U_{i-1}(\cos(k\pi/(n+1))) U_{j-1}(\cos(k\pi/(n+1)))}{U'_n(\cos(k\pi/(n+1))) U_{n-1}(\cos(k\pi/(n+1)))}. \end{aligned} \quad (37)$$

If the matrix A_n is nonsingular and $A^{-1} = (\alpha_{i,j}^{(-1)})_{1 \leq i,j \leq n}$, then

$$\begin{aligned} & \alpha_{i,j}^{(-1)} \\ &= 2 \sum_{k=1}^n \frac{1}{a + 2 \cos(k\pi/(n+1))} \\ & \quad \times \frac{U_{i-1}(\cos(k\pi/(n+1))) U_{j-1}(\cos(k\pi/(n+1)))}{U'_n(\cos(k\pi/(n+1))) U_{n-1}(\cos(k\pi/(n+1)))}. \end{aligned} \quad (38)$$

We can obtain the following:

$$\begin{aligned} & \beta_{i,j}^{(q)} \\ &= \begin{cases} 2 \sum_{k=1}^n \left(a + 2 \cos \frac{k\pi}{n+1} \right)^q \\ \quad \times \frac{U_{i-1}(\cos(k\pi/(n+1))) U_{j-1}(\cos(k\pi/(n+1)))}{U'_n(\cos(k\pi/(n+1))) U_{n-1}(\cos(k\pi/(n+1)))}, & \text{if } q \text{ is even,} \\ 2 \sum_{k=1}^n \left(a + 2 \cos \frac{k\pi}{n+1} \right)^q \\ \quad \times \frac{U_{n-i}(\cos(k\pi/(n+1))) U_{j-1}(\cos(k\pi/(n+1)))}{U'_n(\cos(k\pi/(n+1))) U_{n-1}(\cos(k\pi/(n+1)))}, & \text{if } q \text{ is odd,} \end{cases} \\ & \beta_{i,j}^{(-1)} \\ &= 2 \sum_{k=1}^n \left(a + 2 \cos \frac{k\pi}{n+1} \right)^{-1} \\ & \quad \times \frac{U_{i-1}(\cos(k\pi/(n+1))) U_{j-1}(\cos(k\pi/(n+1)))}{U'_n(\cos(k\pi/(n+1))) U_{n-1}(\cos(k\pi/(n+1)))}. \end{aligned} \quad (39)$$

Theorem 11. Consider an odd natural number $n = 2m+1$, $m \in N$. Let $B_n = \text{antitridiag}_n(1, 0, 1)$ and $\lambda_k = -2 \cos(k\pi/(n+1))$ for every $1 \leq k \leq n$. Then

$$[B_n^q]_{i,j} = \frac{1 + (-1)^{q+i+j}}{2n+2} \times \sum_{k=1}^{(n-1)/2} \beta_{i,j}(k) \lambda_k^q (4 - \lambda_k^2) U_{n-i}\left(\frac{\lambda_k}{2}\right) U_{n-j}\left(\frac{\lambda_k}{2}\right), \quad (40)$$

for all $q \in N$ and $1 \leq i, j \leq n$, where

$$\beta_{i,j}(k) = \begin{cases} 1, & \text{if } i+j \text{ is even,} \\ (-1)^{k-1}, & \text{if } i+j \text{ is odd.} \end{cases} \quad (41)$$

λ_k ($k = 1, 2, \dots, n$) are the eigenvalues of the matrix B_n and $U_k(x)$ is the k th degree Chebyshev polynomial of the second kind.

Theorem 12. Consider an even natural number $n = 2m$, $m \in N$. Let $B_n = \text{antitridiag}_n(1, 0, 1)$ and $\lambda_k = -2 \cos(k\pi/(n+1))$ for every $1 \leq k \leq n$. Then

$$[B_n^q]_{i,j} = \frac{1}{n+1} \gamma_{i,j} \sum_{k=1}^{n/2} \lambda_{2k}^q (4 - \lambda_{2k}^2) U_{n-i}\left(\frac{\lambda_{2k}}{2}\right) U_{n-j}\left(\frac{\lambda_{2k}}{2}\right) \quad (42)$$

for all $q \in N$ and $1 \leq i, j \leq n$, where

$$\gamma_{i,j} = \begin{cases} 1, & \text{if } i+j \text{ is even,} \\ 0, & \text{if } i+j \text{ is odd.} \end{cases} \quad (43)$$

For even order matrix B the following condition is fulfilled: $\lambda_k \neq 0$ ($k = 2, 4, \dots, n$). This means that even order matrix $B_n = \text{antitridiag}_n(1, 0, 1)$ is nonsingular (its determinant is not equal to zero) and derived expression of B_n^q can be applied for computing negative integer powers, as well. Taking $q = -1$, we get the following expression for elements of the inverse matrix B_n^{-1} :

$$[B_n^{-1}]_{i,j} = \frac{1}{n+1} \gamma_{i,j} \sum_{k=1}^{n/2} \lambda_{2k}^{-1} (4 - \lambda_{2k}^2) \times U_{n-i}\left(\frac{\lambda_{2k}}{2}\right) U_{n-j}\left(\frac{\lambda_{2k}}{2}\right), \quad i, j = 0, 1, \dots, n. \quad (44)$$

Theorem 13. Consider an even natural number $n = 2m$, $m \in N$. Let $B_n = \text{antitridiag}_n(1, 1, 1)$ and $\lambda_k = (-1)^{k-1}(1 + 2 \cos(k\pi/(n+1)))$, ($k = 1, 2, \dots, n$). Then

$$\begin{aligned} [B_n^q]_{i,j} &= \frac{2}{n+1} \sum_{k=1}^n \lambda_k^q \sin \frac{k\pi}{n+1} \sin \frac{kj\pi}{n+1} \\ &= \frac{2}{n+1} \sum_{k=1}^n \left[\lambda_k^q + (-1)^{i+j} (\lambda_k + 2(-1)^k)^q \right] \\ &\quad \times \sin \frac{k\pi}{n+1} \sin \frac{kj\pi}{n+1}. \end{aligned} \quad (45)$$

4. Conclusion and Discussion

In this paper, we derive a general expression for the p th power ($p \in N$) of any complex persymmetric antitridiagonal Hankel (constant antidiagonals) matrices with constant antidiagonals ($\text{antitridiag}_n(b, a, b)$). This novel expression is both an extension of the one obtained by Rimas for the powers of the matrix $\text{antitridiag}_n(1, 0, 1)$ with $n \in N$ (see [2] for the odd case and [5] for the even case) and an extension of the one obtained by Honglin Wu for the powers of the matrix $\text{antitridiag}_n(1, 1, 1)$ with $n \in N$ (see [3] for the even case). We may safely draw the conclusion that our results generalize the results in the related literature [2, 3, 5].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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