Research Article

On the Stream Function-Vorticity Finite Element Formulation for Incompressible Flow in Porous Media

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Stream function-vorticity finite element formulation for incompressible flow in porous media is presented. The model consists of the heat equation, the equation for the concentration, and the equations of motion under the Darcy law. The existence of solution for the discrete problem is established. Optimal a priori error estimates are given.

1. Introduction

The stream function-vorticity formulation is widely used to perform the numerical simulation of an incompressible fluid flow in porous media [1, 2]. The main advantage of the formulation is a reduction of the numerical problem unknowns with the fact that the continuity equation remains satisfied. In this paper, we are interested in studying the finite element of stream function-vorticity formulation for propagating reaction front in porous media. The model considered is a system of reaction-diffusion equations coupled with the hydrodynamic under the Darcy-Boussinesq approximation in the open bounded convex domain $\Omega \subset \mathbb{R}^2$ [3–5]:

$$(P) \begin{cases} \partial_t T - \lambda \Delta T + u \cdot \nabla T = \alpha C \exp\left(-\frac{E}{RT}\right), \\ \partial_t C - \eta \Delta C + u \cdot \nabla C = -\alpha C \exp\left(-\frac{E}{RT}\right), \\ \partial_t u + \frac{\mu}{K} u + \nabla p = \beta (T - T_0) g \gamma, \\ \operatorname{div} u = 0, \end{cases} \quad (1)$$

where T is the temperature, C is the concentration, u is the velocity, p is the pressure, λ is the thermal diffusivity, η is the diffusion, μ is the viscosity, K is the permeability, E is

the activation energy, R is the universal gas constant, α is the Arrhenius preexponential factor, T_0 is a mean value of temperature, g is the gravity, γ is the upward unit vector, and β is the coefficient of the thermal expansion of the fluid.

The boundary conditions are of Dirichlet-Neumann type for the temperature and the concentration and of impermeability type for the normal component of velocity:

$$\begin{aligned} T|_{\Gamma_1} = C|_{\Gamma_1} = 0, \quad \frac{\partial T}{\partial n}|_{\Gamma_2} = \frac{\partial C}{\partial n}|_{\Gamma_2} = 0, \\ u \cdot n|_{\partial\Omega} = 0, \end{aligned} \quad (2)$$

$$u|_{t=0} = u_0, \quad T|_{t=0} = T_0, \quad C|_{t=0} = C_0,$$

where Γ_1 and Γ_2 are disjoint open parts of $\partial\Omega$ such that $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$.

Due to the incompressibility of the fluid, we introduce the stream function ψ :

$$u = \begin{pmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \end{pmatrix}, \quad (3)$$

and by introducing also the vorticity $\omega = \text{curl}(u)$, the problem (P) becomes

$$(\mathcal{P}) \begin{cases} \partial_t T - \lambda \Delta T + \text{rot}(\psi) \cdot \nabla T - Cg(T) = 0, \\ \partial_t C - \eta \Delta C + \text{rot}(\psi) \cdot \nabla C + Cg(T) = 0, \\ \partial_t \omega + \mu_p \omega = f(T), \\ -\Delta \psi = \omega, \end{cases} \quad (4)$$

where $g(T) = \alpha \exp(-E/RT)$, $f(T) = \beta g(\partial T / \partial x)$, and $\mu_p = \mu/K$.

Because there is no flow of fluid through the boundary (impermeability boundary), we will have the following zero-flux condition:

$$\psi|_{\partial\Omega} = 0. \quad (5)$$

The paper is organized as follows. We present the variational formulation of the problem in the next section. We establish the existence result in Section 3. A priori error estimates are given in Section 4. We conclude in the last section.

2. The Variational Formulation

In order to state the problem in the variational form, we first describe the functional framework needed for our study. We set

$$\begin{aligned} X &= L^2(0, t, H_0^1(\Omega)), & W &= L^2(0, t, H_{0,\Gamma_1}^1(\Omega)), \\ M &= L^2(0, t, L^2(\Omega)). \end{aligned} \quad (6)$$

We introduce now the constant of Friedrichs-Poincaré which depends on the geometry of the domain Ω :

$$\rho = \sup_{u \in H_0^1(\Omega)} \frac{\|u\|_{L^2(\Omega)}}{\|\nabla u\|_{L^2(\Omega)}}. \quad (7)$$

The variational form of the continued problem ($\mathcal{P}$) can be written as the following form:

$$(\mathcal{P}_v) \begin{cases} \text{Find } a(\psi, \omega, T, C) \in X \times \overline{M} \times \overline{W}^2, \text{ such as,} \\ \forall (\zeta, v, \phi, \xi) \in X \times M \times W^2, \text{ we have:} \\ (\partial_t T, \phi) + \lambda (\nabla T, \nabla \phi) + \frac{1}{2} \int_{\Omega} (\text{rot}(\psi) \nabla T) \phi \\ - \frac{1}{2} \int_{\Omega} (\text{rot}(\psi) \nabla \phi) T = \int_{\Omega} Cg(T) \phi, \\ (\partial_t C, \xi) + \eta (\nabla C, \nabla \xi) + \frac{1}{2} \int_{\Omega} (\text{rot}(\psi) \nabla C) \xi \\ - \frac{1}{2} \int_{\Omega} (\text{rot}(\psi) \nabla \xi) C = - \int_{\Omega} Cg(T) \xi, \\ (\partial_t \omega, v) + \mu_p (\omega, v) = \int_{\Omega} f(T) v, \\ (\nabla \psi, \nabla \zeta) = (\omega, \zeta), \end{cases} \quad (8)$$

where

$$\begin{aligned} \overline{M} &= \left\{ \omega \in M, \frac{\partial \omega}{\partial t} \in L^2(0, t, L^2(\Omega)) \right\}, \\ \overline{W} &= \left\{ T \in W, \frac{\partial T}{\partial t} \in L^2(0, t, (H_{0,\Gamma_1}^1(\Omega))^*) \right\}. \end{aligned} \quad (9)$$

Here, $(H_{0,\Gamma_1}^1(\Omega))^*$ is the dual of the space $H_{0,\Gamma_1}^1(\Omega)$. We can see clearly that the parameters and the functions f and g of the problem verify the following conditions:

The reals μ_p , η , and λ are strictly positives,

$$g \in W^{1,\infty}(\mathbb{R}), \quad \|g'\|_{L^\infty(\Omega)} = \frac{E\alpha}{RT_i^2}, \quad g \geq 0,$$

$$\|g\|_{L^\infty(\Omega)} = \alpha, \quad (10)$$

$$f \in W^{1,\infty}(\mathbb{R}), \quad f(T_0) = 0,$$

$$\forall (T_1, T_2) \in (H_{0,\Gamma_1}^1(\Omega))^2,$$

$$\|f(T_1) - f(T_2)\|_{L^2(\Omega)} \leq \beta g \|\nabla(T_1 - T_2)\|_{L^2(\Omega)},$$

where T_i is the temperature of the unburned mixture. For convenience, for all $(u, v) \in X^2$ and $(T, \phi, C, \psi) \in W^4$, we introduce the forms defined by

$$\begin{aligned} a(u, v) &= \left(\int_{\Omega} \nabla u \nabla v \, dx \right), \\ a_1(u, T, \phi) &= \frac{1}{2} \left(\int_{\Omega} (\text{rot}(u) \nabla) T \phi \, dx - \int_{\Omega} (\text{rot}(u) \nabla) \phi T \, dx \right), \\ j(T, \psi) &= \int_{\Omega} \nabla T \nabla \psi \, dx, \\ k(C, T, \psi) &= \int_{\Omega} Cg(T) \psi \, dx. \end{aligned} \quad (11)$$

The variational formulation ($\mathcal{P}_v$) is rewritten as follows:

$$(\mathcal{P}_v) \begin{cases} \text{Find } a(\psi, \omega, T, C) \in X \times \overline{M} \times \overline{W}^2, \text{ such as,} \\ \forall (v, u, \phi, \xi) \in M \times X \times W^2, \text{ we have:} \\ (\partial_t T, \phi) + \lambda j(T, \phi) + a_1(\psi, T, \phi) \\ - k(C, T, \phi) = 0, \quad \forall \phi \in W, \\ (\partial_t C, \xi) + \eta j(C, \xi) + a_1(\psi, C, \xi) \\ + k(C, T, \xi) = 0, \quad \forall \xi \in W, \\ (\partial_t \omega, v) + \mu_p (\omega, v) = (f(T), v), \quad \forall v \in M, \\ a(\psi, u) = (\omega, u), \quad \forall u \in X. \end{cases} \quad (12)$$

3. The Existence Result

3.1. The Semidiscrete Problem. In order to give the semidiscrete problem, we will need the following spaces:

$$X_h \subset H_0^1(\Omega), \quad M_h \subset L^2(\Omega), \quad W_h \subset H_{0,\Gamma_1}^1(\Omega), \quad (13)$$

where h is a strictly positive constant. We assume that the spaces X_h , M_h , and W_h satisfy the following assumptions.

- (1) For all $0 < \sigma \leq 1$, there exists a linear continuous operator R_h from $W^{1+\sigma,4}(\Omega) \cap \dot{W}^{1,4}(\Omega)$ onto X_h such that, for all $\psi \in W^{1+\sigma,4}(\Omega) \cap \dot{W}^{1,4}(\Omega)$,

$$\begin{aligned} \|\psi - R_h \psi\|_{1,4} &\leq h^\sigma |\psi|_{1+\sigma,4}, \\ \int_{\Omega} \nabla(\psi - R_h \psi) \nabla v_h &= 0 \quad \forall v_h \in X_h. \end{aligned} \quad (14)$$

- (2) For all $0 < \sigma \leq 1$, there exists a linear continuous operator i_h from $H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ onto W_h such that, for all $T \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$,

$$\begin{aligned} \|T - i_h T\|_{1,\Omega} &\leq h^\sigma |T|_{1+\sigma,\Omega}, \\ \int_{\Omega} \nabla(T - i_h T) \nabla v_h &= 0 \quad \forall v_h \in W_h. \end{aligned} \quad (15)$$

Examples of such spaces verifying these conditions are given in [6, 7].

The discretized form of the problem $(\mathcal{P}_v)$ is given as follows:

$$(\mathcal{P}_h) \left\{ \begin{array}{l} \text{Find } \omega_h \in C^1(0, t, M_h), \psi_h \in C^0(0, t, X_h), \\ \text{and } (C_h, T_h) \in (C^1(0, t, W_h))^2 \text{ such that:} \\ (\partial_t T_h, \phi_h) + \lambda j(T_h, \phi_h) + a_1(\psi_h, T_h, \phi_h) \\ - k(C_h, T_h, \phi_h) = 0, \quad \forall \phi_h \in W_h, \\ (\partial_t C_h, \xi_h) + \eta j(C_h, \xi_h) + a_1(\psi_h, C_h, \xi_h) \\ + k(C_h, T_h, \xi_h) = 0, \quad \forall \xi_h \in W_h, \\ (\partial_t \omega_h, v_h) + \mu_p(\omega_h, v_h) = (f(T_h), v_h), \\ \forall v_h \in M_h, \\ a(\psi_h, u_h) = (\omega_h, u_h), \quad \forall u_h \in X_h. \end{array} \right. \quad (16)$$

With the initial conditions,

$$\begin{aligned} \omega_h(t=0) &= \omega_h^0 \in M_h, \quad T_h(t=0) = T_h^0 \in W_h, \\ C_h(t=0) &= C_h^0 \in W_h. \end{aligned} \quad (17)$$

In the sequel, we assume for simplicity that

$$\omega_0 = \omega_h^0, \quad T_0 = T_h^0, \quad C_0 = C_h^0. \quad (18)$$

The main result of the paper is written as follows.

Theorem 1. *The problem $(\mathcal{P}_h)$ admits a unique solution. Moreover, for ψ , ω , T , and C solutions of the problem $(\mathcal{P}_v)$ and for ψ_h , ω_h , T_h , and C_h solutions of the problem $(\mathcal{P}_h)$, one has*

$$\begin{aligned} &\|\psi - \psi_h\|_{L^2(0,t,(W^{1,4}(\Omega)))} \\ &+ \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\ &+ \|C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\ &+ \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))} \\ &\leq h^\sigma \left(\|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|T\|_{H^1(0,t,H^\sigma(\Omega))} \right. \\ &\quad + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|C\|_{H^1(0,t,H^\sigma(\Omega))} \\ &\quad + \|\omega\|_{L^2(0,t,H^\sigma(\Omega))} + \|\omega\|_{H^1(0,t,H^\sigma(\Omega))} \\ &\quad \left. + \|\psi\|_{L^2(0,t,W^{1+\sigma,4}(\Omega))} \right), \end{aligned} \quad (19)$$

under the following conditions:

$$\delta_1 = 1 - \frac{\sqrt{5}N_C \alpha E \rho^2}{RT_b^2 \lambda} > 0, \quad (H_1)$$

$$\delta_2 = 1 - \frac{2\sqrt{5}\rho^4 \alpha^2 E N_C}{\delta_1 \lambda \eta RT_b^2} > 0, \quad (H_2)$$

$$\delta_2 \delta_1 > \frac{2\sqrt{3}\beta g N}{\lambda \mu_p} \left(\frac{2\sqrt{10}\alpha \rho^2 N_C}{\eta} + \sqrt{5}N_T \right), \quad (H_3)$$

where

$$\begin{aligned} N_T &= \sup \left(\|T_h\|_{L^\infty(0,t,H_{0,\Gamma_1}^1(\Omega))}, \|T\|_{L^\infty(0,t,H_{0,\Gamma_1}^1(\Omega))} \right), \\ N_C &= \sup \left(\|C_h\|_{L^\infty(0,t,H_{0,\Gamma_1}^1(\Omega))}, \|C\|_{L^\infty(0,t,H_{0,\Gamma_1}^1(\Omega))} \right), \\ M_1 &= \sup \left(\|\psi_h\|_{L^\infty(0,t,(W^{1,4}(\Omega)))}, \|\psi\|_{L^\infty(0,t,(W^{1,4}(\Omega)))} \right), \\ N &= \sup_{\psi \in \dot{W}(\Omega), v \in (H_0^1(\Omega))(\Omega), w \in (H_0^1(\Omega))(\Omega)} \frac{a_1(\psi, v, w)}{\|v\|_{H_0^1(\Omega)} \|\psi\|_{\dot{W}^{1,4}(\Omega)} \|w\|_{H_0^1(\Omega)}}, \\ &0 < \sigma \leq 1. \end{aligned} \quad (20)$$

Throughout the paper, we often use the following notation.

For each $\zeta, \eta > 0$: $\zeta \leq \eta \Leftrightarrow \exists C^* > 0$: $\zeta \leq C^* \eta$; without further specification, the constant C^* is independent of the mesh size and the solutions.

3.2. Existence of Semidiscrete Solutions. In order to prove the existence result of the problem $(\mathcal{P}_h)$, we need the following lemmas; first for the concentration, we have the following.

Lemma 2. *For any local solution C_h of the problem $(\mathcal{P}_h)$, one has the a priori estimate*

$$\|C_h\|_{L^\infty(0,t,L^2(\Omega))}^2 + 2\eta\|C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \leq 2\|C_h^0\|_{L^2(\Omega)}^2. \quad (21)$$

Proof. By choosing $\xi_h = C_h$, as test function in the third equation of the problem $(\mathcal{P}_h)$, we have

$$\frac{1}{2} \frac{d}{dt} \|C_h\|_{L^2(\Omega)}^2 + \eta \|\nabla C_h\|_{L^2(\Omega)}^2 + a_1(C_h, T_h, C_h) = 0. \quad (22)$$

Integrating the last equality and noticing that $a_1(C_h, T_h, C_h)$ is positive, we obtain

$$2\eta\|C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \leq \|C_h^0\|_{L^2(\Omega)}^2, \quad (23)$$

$$\|C_h(t=s)\|_{L^2(\Omega)}^2 \leq \|C_h^0\|_{L^2(\Omega)}^2. \quad (24)$$

By using the two last inequalities, we obtain the result of the lemma. $\square$

Also, for the temperature, we have the following result.

Lemma 3. *For any local solution T_h of the problem $(\mathcal{P}_h)$, one has the a priori estimate*

$$\begin{aligned} & \|T_h\|_{L^\infty(0,t,L^2(\Omega))}^2 + \lambda\|T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\ & \leq \frac{(\alpha\rho)^2}{\lambda\eta} \|C_h^0\|_{L^2(\Omega)}^2 + 2\|T_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (25)$$

Proof. By choosing $\phi_h = T_h$, as test function in the second equation of the problem $(\mathcal{P}_h)$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|T_h\|_{L^2(\Omega)}^2 + \lambda\|\nabla T_h\|_{L^2(\Omega)}^2 \\ & = k(C_h, T_h, T_h) \\ & \leq \rho\alpha\|C_h\|_{L^2(\Omega)}\|\nabla T_h\|_{L^2(\Omega)} \\ & \leq \frac{(\alpha\rho)^2}{2\lambda} \|C_h\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|\nabla T_h\|_{L^2(\Omega)}^2. \end{aligned} \quad (26)$$

Via integration of the last inequality, it follows that

$$\begin{aligned} & \|T_h(t=s)\|_{L^2(\Omega)}^2 + \lambda\|T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\ & \leq \frac{(\alpha\rho)^2}{\lambda} \|C_h\|_{L^2(0,t,L^2(\Omega))}^2 + \|T_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (27)$$

Thus, using inequality (23), we get

$$\begin{aligned} & \|T_h(t=s)\|_{L^2(\Omega)}^2 + \lambda\|T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\ & \leq \frac{(\alpha\rho)^2}{2\lambda\eta} \|C_h^0\|_{L^2(\Omega)}^2 + \|T_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (28)$$

It leads to

$$\|T_h\|_{L^\infty(0,t,L^2(\Omega))}^2 \leq \frac{(\alpha\rho)^2}{2\lambda\eta} \|C_h^0\|_{L^2(\Omega)}^2 + \|T_h^0\|_{L^2(\Omega)}^2. \quad (29)$$

From (28) and (29), it follows that

$$\begin{aligned} & \|T_h\|_{L^\infty(0,t,L^2(\Omega))}^2 + \lambda\|T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\ & \leq \frac{(\alpha\rho)^2}{\lambda\eta} \|C_h^0\|_{L^2(\Omega)}^2 + 2\|T_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (30)$$

$\square$

For the vorticity, we have the following.

Lemma 4. *For any local solution ω_h of the problem $(\mathcal{P}_h)$, one has the estimate*

$$\begin{aligned} & \|\omega_h\|_{L^\infty(0,t,L^2(\Omega))}^2 + \mu_p\|\omega_h\|_{L^2(0,t,L^2(\Omega))}^2 \\ & \leq \frac{2\alpha\rho^2}{\lambda^2\eta\mu_p} \|C_h^0\|_{L^2(\Omega)}^2 + \frac{4}{\lambda\mu_p} \|T_h^0\|_{L^2(\Omega)}^2 + 2\|\omega_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (31)$$

Proof. By choosing $v_h = \omega_h$, as test function in the first equation of the problem $(\mathcal{P}_h)$, we have

$$\frac{1}{2} \frac{d}{dt} \|\omega_h\|_{L^2(\Omega)}^2 + \mu_p\|\omega_h\|_{L^2(\Omega)}^2 \leq \|\nabla T_h\|_{L^2(\Omega)}\|\omega_h\|_{L^2(\Omega)}. \quad (32)$$

Then, by using triangular inequality, we get

$$\frac{1}{2} \frac{d}{dt} \|\omega_h\|_{L^2(\Omega)}^2 + \frac{\mu_p}{2} \|\omega_h\|_{L^2(\Omega)}^2 \leq \frac{1}{2\mu_p} \|\nabla T_h\|_{L^2(\Omega)}^2. \quad (33)$$

While integrating the last inequality, we obtain

$$\begin{aligned} & \|\omega_h(t=s)\|_{L^2(\Omega)}^2 + \mu_p\|\omega_h\|_{L^2(0,t,L^2(\Omega))}^2 \\ & \leq \frac{1}{\mu_p} \|T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 + \|\omega_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (34)$$

However, using Lemma 3, we have

$$\begin{aligned} & \|\omega_h(t=s)\|_{L^2(\Omega)}^2 + \mu_p\|\omega_h\|_{L^2(0,t,L^2(\Omega))}^2 \\ & \leq \frac{\alpha\rho^2}{\lambda^2\eta\mu_p} \|C_h^0\|_{L^2(\Omega)}^2 + \frac{2}{\lambda\mu_p} \|T_h^0\|_{L^2(\Omega)}^2 + \|\omega_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (35)$$

It leads to

$$\begin{aligned} & \|\omega_h\|_{L^\infty(0,t,L^2(\Omega))}^2 \leq \frac{\alpha\rho^2}{\lambda^2\eta\mu_p} \|C_h^0\|_{L^2(\Omega)}^2 \\ & + \frac{2}{\lambda\mu_p} \|T_h^0\|_{L^2(\Omega)}^2 + \|\omega_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (36)$$

From (35) and (36), we obtain

$$\begin{aligned} & \|\omega_h\|_{L^\infty(0,t,L^2(\Omega))}^2 + \mu_p\|\omega_h\|_{L^2(0,t,L^2(\Omega))}^2 \\ & \leq \frac{2\alpha\rho^2}{\lambda^2\eta\mu_p} \|C_h^0\|_{L^2(\Omega)}^2 + \frac{4}{\lambda\mu_p} \|T_h^0\|_{L^2(\Omega)}^2 + 2\|\omega_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (37)$$

$\square$

Finally, for the stream function, we have the following.

Lemma 5. *For any local solution ψ_h of the problem $(\mathcal{P}_h)$, one has the estimate*

$$\begin{aligned} \|\psi_h\|_{L^2(0,t,W^{1,4}(\Omega))}^2 &\lesssim \frac{2\alpha\rho^2}{\lambda^2\eta\mu_p^2} \|C_h^0\|_{L^2(\Omega)}^2 \\ &+ \frac{4}{\lambda\mu_p^2} \|T_h^0\|_{L^2(\Omega)}^2 + \frac{2}{\mu_p} \|\omega_h^0\|_{L^2(\Omega)}^2. \end{aligned} \quad (38)$$

Proof. Let ψ_h be the solution of the problem $(\mathcal{P}_h)$. From [8, 9], we have

$$|\psi_h|_{W^{1,4}(\Omega)} \leq \sup_{v \in W^{1,4/3}(\Omega)} \frac{\langle \nabla(\psi_h), \nabla v \rangle}{|v|_{W^{1,4/3}(\Omega)}}. \quad (39)$$

However,

$$\langle \nabla(\psi_h), \nabla v \rangle = \langle \nabla(\psi_h), \nabla(\Pi_h v) \rangle, \quad (40)$$

where Π_h is the projection operator defined from $W^{1,4/3}(\Omega)$ onto X_h , such as

$$\forall v \in W^{1,4/3}(\Omega), \quad \langle \nabla \psi_h, \nabla(\Pi_h v) - \nabla v \rangle = 0, \quad \forall \psi_h \in X_h. \quad (41)$$

By choosing $v_h = \psi_h$, as test function in the last equation of the problem $(\mathcal{P}_h)$, we get

$$\langle \nabla \psi_h, \nabla \Pi_h v \rangle = \langle \omega_h, \Pi_h v \rangle. \quad (42)$$

Therefore,

$$\langle \nabla \psi_h, \nabla v \rangle \leq \|\omega_h\|_{L^2(\Omega)} \|\Pi_h v\|_{L^2(\Omega)}. \quad (43)$$

Due to the embedding of $W^{1,4/3}(\Omega)$ onto $L^2(\Omega)$, we have

$$\|\Pi_h v\|_{L^2(\Omega)} \leq |\Pi_h v|_{W^{1,4/3}(\Omega)}, \quad (44)$$

and also, due to the propriety of the operator Π_h , we have

$$\|\Pi_h v\|_{W^{1,4/3}(\Omega)} \leq |v|_{W^{1,4/3}(\Omega)}, \quad (45)$$

so

$$\sup_{v \in W^{1,4/3}(\Omega)} \frac{\langle \nabla \psi_h, \nabla v \rangle}{|v|_{W^{1,4/3}(\Omega)}} \leq \|\omega_h\|_{L^2(\Omega)}. \quad (46)$$

We conclude that

$$|\psi_h|_{W^{1,4}(\Omega)} \leq \|\omega_h\|_{L^2(\Omega)}. \quad (47)$$

Then, we get

$$\|\psi_h\|_{L^2(0,t,(W^{1,4}(\Omega)))} \leq \|\omega_h\|_{L^2(0,t,L^2(\Omega))}. \quad (48)$$

Using Lemma 4, we conclude that

$$\begin{aligned} \|\psi_h\|_{L^2(0,t,W^{1,4}(\Omega))}^2 &\leq \left(\frac{2\alpha\rho^2}{\lambda^2\eta\mu_p^2} \|C_h^0\|_{L^2(\Omega)}^2 + \frac{4}{\lambda\mu_p^2} \|T_h^0\|_{L^2(\Omega)}^2 + \frac{2}{\mu_p} \|\omega_h^0\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (49)$$

□

Now, we are able to prove the main theorem of this section more precisely.

Theorem 6. *The problem $(\mathcal{P}_h)$ admits at least a solution*

$$\begin{aligned} &(\omega_h, \psi_h, C_h, T_h) \\ &\in H^1(0, t, M_h) \times L^2(0, t, X_h) \times (\mathcal{C}^1(0, t, W_h))^2. \end{aligned} \quad (50)$$

Proof. Indeed, it is obvious that the problem $(\mathcal{P}_h)$ admits a local solution in the interval $(0, t_h)$. For t_h which is rather small, Lemmas 2, 3, 4, and 5 show that this solution can be defined on the interval $(0, t)$ for $t > 0$. □

4. A Priori Error Estimations

In this section, we prove some error estimates on the stream function, on the vorticity, on the temperature, and on the concentration. Subsequently, we assume that there exists $0 < \sigma \leq 1$, such as

$$\omega \in L^2(0, t, H^\sigma(\Omega)) \cap H^1(0, t, H^\sigma(\Omega)),$$

$$T, C \in L^2(0, t, H^{1+\sigma}(\Omega)) \cap H^1(0, t, H^\sigma(\Omega)), \quad (51)$$

$$\psi \in \dot{W}^{1,4}(\Omega) \cap W^{1+\sigma,4}(\Omega).$$

Lemma 7. *For any ω and ω_h solutions of the problems $(\mathcal{P})$ and $(\mathcal{P}_h)$, respectively, one has*

$$\begin{aligned} &\|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))} \\ &\leq h^\sigma (\|\omega\|_{L^2(0,t,H^\sigma(\Omega))} + \|\omega\|_{H^1(0,t,H^\sigma(\Omega))} + \|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))}) \\ &+ \frac{\sqrt{3}\beta g}{\mu_p} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}. \end{aligned} \quad (52)$$

Proof. For ω and ω_h , respectively, solutions of the continuous and the semidiscrete problem, we have

$$\begin{aligned} &(\partial_t \omega - \partial_t \omega_h, v_h)_\kappa + \mu_p (\omega - \omega_h, v_h)_\kappa \\ &= (f(T) - f(T_h), v_h)_\kappa, \quad \forall v_h \in X_h, \end{aligned} \quad (53)$$

where κ is an element of the uniform meshes family τ_h . Let $\bar{\omega}_h$ be the approximation of ω in X_h defined by

$$\bar{\omega}_h|_\kappa = \frac{1}{\text{mes}(\kappa)} \int_\kappa \omega \, dx. \quad (54)$$

Hence, for all $0 < \sigma < 1$ and $\omega \in H^\sigma(\Omega)$, we have the following result:

$$\|\omega - \bar{\omega}_h\|_{L^2(\kappa)} \leq h_\kappa^\sigma \|\omega\|_{H^\sigma(\kappa)}, \quad (55)$$

where h_κ is the diameter of the element κ .

By using equality (53) and setting $v_h = \bar{\omega}_h - \omega_h$, we obtain

$$\begin{aligned} &(\partial_t \bar{\omega}_h - \partial_t \omega_h, \bar{\omega}_h - \omega_h)_\kappa + \mu_p (\bar{\omega}_h - \omega_h, \bar{\omega}_h - \omega_h)_\kappa \\ &= -(\partial_t \omega - \partial_t \bar{\omega}_h, \bar{\omega}_h - \omega_h)_\kappa + \mu_p (\bar{\omega}_h - \omega, \bar{\omega}_h - \omega_h)_\kappa \\ &+ (f(T) - f(T_h), \bar{\omega}_h - \omega_h)_\kappa. \end{aligned} \quad (56)$$

Therefore, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 + \mu_p \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 \\ & \leq \left(\|\partial_t \omega - \partial_t \bar{\omega}_h\|_{L^2(\kappa)} + \beta g \|\nabla(T - T_h)\|_{L^2(\kappa)} \right) \\ & \quad \times \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)} \\ & \quad + \mu_p \|\omega - \bar{\omega}_h\|_{L^2(\kappa)} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}. \end{aligned} \quad (57)$$

Then,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 + \mu_p \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 \\ & \leq \left(\|\partial_t \omega - \partial_t \bar{\omega}_h\|_{L^2(\kappa)} + \beta g \|\nabla(T - T_h)\|_{L^2(\kappa)} \right) \\ & \quad \times \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)} \\ & \quad + \mu_p \|\omega - \bar{\omega}_h\|_{L^2(\kappa)} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}. \end{aligned} \quad (58)$$

However, by using the triangular inequality, we can write

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 + \mu_p \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 \\ & \leq \frac{3}{2\mu_p} \|\partial_t \omega - \partial_t \bar{\omega}_h\|_{L^2(\kappa)}^2 + \frac{3(\beta g)^2}{2\mu_p} \|\nabla(T - T_h)\|_{L^2(\kappa)}^2 \\ & \quad + \frac{3\mu_p}{2} \|\bar{\omega}_h - \omega\|_{L^2(\kappa)}^2 + \frac{\mu_p}{2} \|\bar{\omega}_h - \omega\|_{L^2(\kappa)}^2. \end{aligned} \quad (59)$$

It follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 + \frac{\mu_p}{2} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 \\ & \leq \frac{3}{2\mu_p} \|\partial_t \omega - \partial_t \bar{\omega}_h\|_{L^2(\kappa)}^2 \\ & \quad + \frac{3\beta^2 g^2}{2\mu_p} \|T - T_h\|_{H_{0,\Gamma_1}^1(\kappa)}^2 + \frac{3\mu_p}{2} \|\bar{\omega}_h - \omega\|_{L^2(\kappa)}^2. \end{aligned} \quad (60)$$

However, for all $0 < \sigma < 1$ and $\omega \in H^1(0, t, H^\sigma(\kappa))$, we have

$$\|\partial_t \omega - \partial_t \bar{\omega}_h\|_{L^2(\kappa)}^2 \leq h_\kappa^{2\sigma} \|\partial_t \omega\|_{H^\sigma(\kappa)}^2. \quad (61)$$

While multiplying (60) by $2/\mu$ and using (55) and (61), we obtain

$$\begin{aligned} \|\bar{\omega}_h - \omega_h\|_{L^2(\kappa)}^2 & \leq h_\kappa^{2\sigma} \left(\|\omega\|_{H^\sigma(\kappa)} + \|\partial_t \omega\|_{H^\sigma(\kappa)} \right)^2 \\ & \quad + \frac{3(\beta g)^2}{\mu_p^2} \|T - T_h\|_{H_{0,\Gamma_1}^1(\kappa)}^2. \end{aligned} \quad (62)$$

Finally, by integrating and using Jensen's inequality, we get

$$\begin{aligned} & \|\bar{\omega}_h - \omega_h\|_{L^2(0,t,L^2(\Omega))} \\ & \leq h^\sigma \left(\|\omega\|_{L^2(0,t,H^\sigma(\Omega))} + \|\omega\|_{H^1(0,t,H^\sigma(\Omega))} \right) \\ & \quad + \frac{\sqrt{3}\beta g}{\mu_p} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}. \end{aligned} \quad (63)$$

However,

$$\begin{aligned} \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))} & \leq \|\bar{\omega}_h - \omega_h\|_{L^2(0,t,L^2(\Omega))} \\ & \quad + \|\omega - \bar{\omega}_h\|_{L^2(0,t,L^2(\Omega))}. \end{aligned} \quad (64)$$

We conclude that

$$\begin{aligned} \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))} & \leq h^\sigma \left(\|\omega\|_{L^2(0,t,H^\sigma(\Omega))} + \|\omega\|_{H^1(0,t,H^\sigma(\Omega))} \right) \\ & \quad + \frac{\sqrt{3}\beta g}{\mu_p} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}. \end{aligned} \quad (65)$$

□

Lemma 8. For any ψ and ψ_h solution of the problems $(\mathcal{P})$ and $(\mathcal{P}_h)$, respectively, one has

$$\begin{aligned} \|\psi - \psi_h\|_{L^2(0,t,W^{\circ 1,4}(\Omega))} & \leq h^\sigma \|\psi\|_{L^2(0,t,(W^{1+\sigma,4}(\Omega)))} \\ & \quad + \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))}. \end{aligned} \quad (66)$$

Proof. Let ψ and ψ_h be the solution of the problems $(\mathcal{P}_v)$ and $(\mathcal{P}_h)$, respectively. From [8, 9], we have

$$|\psi - \psi_h|_{W^{1,4}(\Omega)} \lesssim \sup_{v \in W^{\circ 1,4/3}(\Omega)} \frac{\langle \nabla(\psi - \psi_h), \nabla v \rangle}{|v|_{W^{1,4/3}(\Omega)}}. \quad (67)$$

However,

$$\begin{aligned} \langle \nabla(\psi - \psi_h), \nabla v \rangle & = \langle \nabla(\psi - \psi_h), \nabla(v - \Pi_h v) \rangle \\ & \quad + \langle \nabla(\psi - \psi_h), \nabla(\Pi_h v) \rangle, \end{aligned} \quad (68)$$

where Π_h is the projection operator verifying (41). Moreover, we have

$$\begin{aligned} \langle \nabla(R_h \psi - \psi_h), \nabla(\Pi_h v - v) \rangle & = 0, \\ \langle \nabla(R_h \psi - \psi), \nabla(\Pi_h v) \rangle & = 0. \end{aligned} \quad (69)$$

Then, we get

$$\begin{aligned} \langle \nabla(\psi - \psi_h), \nabla v \rangle & = \langle \nabla(\psi - R_h \psi), \nabla(v - \Pi_h v) \rangle \\ & \quad + \langle \nabla(\psi - \psi_h), \nabla(\Pi_h v) \rangle. \end{aligned} \quad (70)$$

By choosing $v_h = \Pi_h v$ as test function in the last equation of the problem (P_h) , we obtain

$$\langle \nabla(\psi - \psi_h), \nabla v \rangle = \langle \nabla(\psi - R_h \psi), \nabla v \rangle + \langle \omega - \omega_h, \Pi_h v \rangle. \quad (71)$$

It leads to

$$\begin{aligned} \langle \nabla(\psi - R_h \psi), \nabla v \rangle & \leq |\psi - R_h \psi|_{W^{1,4}(\Omega)} |v|_{W^{1,4/3}(\Omega)}, \\ \langle \omega - \omega_h, \Pi_h v \rangle & \leq \|\omega - \omega_h\|_{L^2(\Omega)} \|\Pi_h v\|_{L^2(\Omega)}. \end{aligned} \quad (72)$$

On the other hand, due to the embedding of $W^{1,4/3}(\Omega)$ onto $L^2(\Omega)$, we have

$$\|\Pi_h v\|_{L^2(\Omega)} \lesssim |\Pi_h v|_{W^{1,4/3}(\Omega)}. \quad (73)$$

In addition,

$$|\Pi_h v|_{W^{1,4/3}(\Omega)} \lesssim |v|_{W^{1,4/3}(\Omega)}, \quad (74)$$

so

$$\sup_{v \in W^{1,4/3}(\Omega)} \frac{\langle \omega - \omega_h, \Pi_h v \rangle}{|v|_{W^{1,4/3}(\Omega)}} \leq \|\omega - \omega_h\|_{L^2(\Omega)}. \quad (75)$$

From inequalities (67) and (72), we have

$$|\psi - \psi_h|_{W^{1,4}(\Omega)} \lesssim |\psi - R_h \psi|_{W^{1,4}(\Omega)} + \|\omega - \omega_h\|_{L^2(\Omega)}. \quad (76)$$

We conclude that

$$\begin{aligned} & \|\psi - \psi_h\|_{L^2(0,t; \dot{W}^{1,4}(\Omega))} \\ & \leq h^\sigma \|\psi\|_{L^2(0,t; (W^{1+\sigma,4}(\Omega)))} + \|\omega - \omega_h\|_{L^2(0,t; L^2(\Omega))}. \end{aligned} \quad (77)$$

□

Also, we have the following error estimate for temperature.

Lemma 9. *If we assume that the hypothesis (H_1) is verified, then one has*

$$\begin{aligned} & \|T - T_h\|_{L^2(0,t; H_{0,\Gamma_1}^1(\Omega))} \\ & \leq h^\sigma (\|T\|_{H^1(0,t; H^\sigma(\Omega))} + \|T\|_{L^2(0,t; H^{1+\sigma}(\Omega))}) \\ & \quad + \frac{\sqrt{5}\alpha\rho^2}{\delta_1\lambda} \|C - C_h\|_{L^2(0,t; H_{0,\Gamma_1}^1(\Omega))} \\ & \quad + \frac{\sqrt{5}NN_T}{\delta_1\lambda} \|\psi - \psi_h\|_{L^2(0,t; \dot{W}^{1,4}(\Omega))}. \end{aligned} \quad (78)$$

Proof. For T solution of the problem $(\mathcal{P})$ and T_h solution of the problem $(\mathcal{P}_h)$, we have

$$\begin{aligned} & (\partial_t T - \partial_t T_h, \phi_h) + \lambda a(T - T_h, \phi_h) \\ & \quad + a_1(\psi - \psi_h, T, \phi_h) + a_1(\psi_h, T - T_h, \phi_h) \\ & = k(C, T, \phi_h) - k(C_h, T_h, \phi_h). \end{aligned} \quad (79)$$

Therefore, using the propriety of the operator i_h , we have

$$a(i_h T - T, \phi_h) = 0. \quad (80)$$

So,

$$\begin{aligned} & (i_h \partial_t T - \partial_t T_h, \phi_h) + \lambda a(i_h T - T_h, \phi_h) \\ & = (i_h \partial_t T - \partial_t T, \phi_h) - a_1(\psi - \psi_h, T, \phi_h) \\ & \quad - a_1(\psi_h, T - T_h, \phi_h) + k(C, T, \phi_h) - k(C_h, T_h, \phi_h). \end{aligned} \quad (81)$$

By setting $\phi_h = i_h T - T_h$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|i_h T - T_h\|_{L^2(\Omega)}^2 + \lambda \|\nabla(i_h T - T_h)\|_{L^2(\Omega)}^2 \\ & = -(\partial_t T - i_h \partial_t T, \phi_h) - a_1(\psi - \psi_h, T, \phi_h) \\ & \quad - a_1(\psi_h, T - i_h T, i_h T - T_h) + k(C, T, \phi_h) \\ & \quad - k(C_h, T_h, \phi_h). \end{aligned} \quad (82)$$

Then, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|i_h T - T_h\|_{L^2(\Omega)}^2 + \lambda \|\nabla(i_h T - T_h)\|_{L^2(\Omega)}^2 \\ & \leq \left(\rho \|\partial_t T - i_h \partial_t T\|_{L^2(\Omega)} + NN_T \|\psi - \psi_h\|_{\dot{W}^{1,4}(\Omega)} \right. \\ & \quad \left. + NM_1 \|\nabla(T - i_h T)\|_{L^2(\Omega)} \right) \|\nabla(i_h T - T_h)\|_{L^2(\Omega)} \\ & \quad + |k(C, T, \phi_h) + k(C_h, T_h, \phi_h)|. \end{aligned} \quad (83)$$

We have also

$$\begin{aligned} & |k(C, T, \phi_h) - k(C_h, T_h, \phi_h)| \\ & \leq \alpha \rho^2 \|C - C_h\|_{H_{0,\Gamma_1}^1(\Omega)} \|\phi_h\|_{H_{0,\Gamma_1}^1(\Omega)} \\ & \quad + N_C \alpha \frac{E}{RT_i^2} \rho^2 \|T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)} \|\phi_h\|_{H_{0,\Gamma_1}^1(\Omega)}. \end{aligned} \quad (84)$$

From (83) and (84), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|i_h T - T_h\|_{L^2(\Omega)}^2 + \lambda \|\nabla(i_h T - T_h)\|_{L^2(\Omega)}^2 \\ & \leq \left(\rho \|\partial_t T - i_h \partial_t T\|_{L^2(\Omega)} + NN_T \|\psi - \psi_h\|_{\dot{W}^{1,4}(\Omega)} \right. \\ & \quad \left. + NM_1 \|\nabla(T - i_h T)\|_{L^2(\Omega)} \right) \|\nabla(i_h T - T_h)\|_{L^2(\Omega)} \\ & \quad + \left(\alpha \rho^2 \|C - C_h\|_{H_{0,\Gamma_1}^1(\Omega)} + N_C \alpha \frac{E}{RT_i^2} \rho^2 \|T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)} \right) \\ & \quad \times \|\nabla(i_h T - T_h)\|_{L^2(\Omega)}. \end{aligned} \quad (85)$$

Then, we deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|i_h T - T_h\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|\nabla(i_h T - T_h)\|_{L^2(\Omega)}^2 \\ & \leq \frac{5\rho^2}{2\lambda} \|\partial_t T - i_h \partial_t T\|_{L^2(\Omega)}^2 \\ & \quad + \frac{5N^2 M_1^2}{2\lambda} \|\nabla(T - i_h T)\|_{L^2(\Omega)}^2 \\ & \quad + \frac{5\alpha^2 \rho^4}{2\lambda} \|\nabla(C - C_h)\|_{L^2(\Omega)}^2 \\ & \quad + \frac{5N^2 N_T^2}{2\lambda} \|\psi - \psi_h\|_{\dot{W}^{1,4}(\Omega)}^2 \\ & \quad + \frac{5N_C^2 \alpha^2 E^2 \rho^4}{2\lambda R^2 T_i^4} \|\nabla(T - T_h)\|_{L^2(\Omega)}^2. \end{aligned} \quad (86)$$

By multiplying the last inequality by $2/\lambda$, we obtain

$$\begin{aligned} \|i_h T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)}^2 &\leq \frac{5\rho^2}{\lambda^2} \|\partial_t T - i_h \partial_t T\|_{L^2(\Omega)}^2 \\ &\quad + \frac{5N^2 M_1^2}{\lambda^2} \|\nabla (T - i_h T)\|_{L^2(\Omega)}^2 \\ &\quad + \frac{5\alpha^2 \rho^4}{\lambda^2} \|\nabla (C - C_h)\|_{L^2(\Omega)}^2 \\ &\quad + \frac{5N^2 N_T^2}{\lambda^2} \|\psi - \psi_h\|_{W^{1,4}(\Omega)}^2 \\ &\quad + 5 \frac{N_C^2 (\alpha E)^2 \rho^4}{R^2 T_i^4 \lambda^2} \|\nabla (T - T_h)\|_{L^2(\Omega)}^2. \end{aligned} \quad (87)$$

Via integration of the last inequality, we obtain

$$\begin{aligned} \|i_h T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 &\leq (h^\sigma \|T\|_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))})^2 \\ &\quad + \frac{5\alpha^2 \rho^4}{\lambda^2} \|C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\ &\quad + \frac{5N^2 N_T^2}{\lambda^2} \|\psi - \psi_h\|_{L^2(0,t,W^{1,4}(\Omega))}^2 \\ &\quad + 5 \frac{N_C^2 (\alpha E)^2 \rho^4}{R^2 T_i^4 \lambda^2} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2. \end{aligned} \quad (88)$$

Therefore,

$$\begin{aligned} \|i_h T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} &\leq (h^\sigma \|T\|_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))}) \\ &\quad + \frac{\sqrt{5}\alpha\rho^2}{\lambda} \|C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\ &\quad + \frac{\sqrt{5}NN_T}{\lambda} \|\psi - \psi_h\|_{L^2(0,t,W^{1,4}(\Omega))} \\ &\quad + \frac{\sqrt{5}N_C\alpha E\rho^2}{RT_i^2\lambda} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}. \end{aligned} \quad (89)$$

However,

$$\|T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)} \leq \|T - i_h T\|_{H_{0,\Gamma_1}^1(\Omega)} + \|i_h T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)}. \quad (90)$$

Then, we have

$$\|T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)} \leq h^\sigma \|T\|_{H^{1+\sigma}(\Omega)} + \|i_h T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)}. \quad (91)$$

Inequalities (89) and (91) allow us to obtain the following estimate:

$$\begin{aligned} \left(1 - \frac{\sqrt{5}N_C\alpha E\rho^2}{RT_i^2\lambda}\right) \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} &\leq h^\sigma (\|T\|_{H^1(0,t,H^\sigma(\Omega))} + \|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))}) \\ &\quad + \frac{\sqrt{5}\alpha\rho^2}{\lambda} \|C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\ &\quad + \frac{\sqrt{5}NN_T}{\lambda} \|\psi - \psi_h\|_{L^2(0,t,W^{1,4}(\Omega))}. \end{aligned} \quad (92)$$

Finally, if we set

$$\delta_1 = 1 - \frac{\sqrt{5}N_C\alpha E\rho^2}{RT_i^2\lambda} > 0, \quad (93)$$

we obtain the following estimate:

$$\begin{aligned} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} &\leq h^\sigma (\|T\|_{H^1(0,t,H^\sigma(\Omega))} + \|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))}) \\ &\quad + \frac{\sqrt{5}\alpha\rho^2}{\delta_1\lambda} \|C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\ &\quad + \frac{\sqrt{5}NN_T}{\delta_1\lambda} \|\psi - \psi_h\|_{L^2(0,t,W^{1,4}(\Omega))}. \end{aligned} \quad (94)$$

□

In addition, we have the following error estimate on concentration.

Lemma 10. For C and C_h , respectively, solution of the problems $(\mathcal{P})$ and $(\mathcal{P}_h)$, one has

$$\begin{aligned} \|C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} &\leq h^\sigma (\|C\|_{H^1(0,t,H^\sigma(\Omega))} + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))}) \\ &\quad + \frac{2\sqrt{2}NN_C}{\eta} \|\psi - \psi_h\|_{L^2(0,t,W^{1,4}(\Omega))} \\ &\quad + 2 \frac{N_C\alpha E\rho^2}{\eta RT_i^2} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}. \end{aligned} \quad (95)$$

Proof. By choosing $\chi_h = i_h C - C_h$ as test function, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|i_h C - C_h\|_{0,t,L^2(\Omega)}^2 + \eta \|\nabla (i_h C - C_h)\|_{0,t,L^2(\Omega)}^2 &= -(\partial_t C - i_h \partial_t C, \chi_h) - a_1 (\psi - \psi_h, C, \chi_h) \\ &\quad - a_1 (\psi_h, C - i_h C, i_h C - C_h) \\ &\quad - k(C, T, \chi_h) + k(C_h, T_h, \chi_h). \end{aligned} \quad (96)$$

However,

$$\begin{aligned}
& k(C, T, \chi_h) - k(C_h, T_h, \chi_h) \\
&= k(C, T, \chi_h) - k(i_h C, T_h, \chi_h) + k(\chi_h, T_h, \chi_h), \\
& k(C, T, \chi_h) - k(i_h C, T_h, \chi_h) \\
&= k(C - i_h C, T, \chi_h) - k(i_h C, T - T_h, \chi_h) \\
&\leq \alpha \rho^2 \|\nabla(C - i_h C)\|_{L^2(\Omega)} \|\nabla \chi_h\|_{L^2(\Omega)} \\
&\quad + \int i_h C (g(T) - g(T_h)) \chi_h.
\end{aligned} \tag{97}$$

Then, we have

$$\begin{aligned}
& k(C, T, \chi_h) - k(i_h C, T_h, \chi_h) \\
&\leq \alpha \rho^2 \|C - i_h C\|_{H_{0,\Gamma_1}^1(\Omega)} \|\chi_h\|_{H_{0,\Gamma_1}^1(\Omega)} \\
&\quad + N_C \alpha \frac{E}{RT_i^2} \rho^2 \|T - T_h\|_{H_{0,\Gamma_1}^1(\Omega)} \|\chi_h\|_{H_{0,\Gamma_1}^1(\Omega)}.
\end{aligned} \tag{98}$$

Using (96) and (98), we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|i_h C - C_h\|_{L^2(\Omega)}^2 + \eta \|\nabla(i_h C - C_h)\|_{L^2(\Omega)}^2 \\
&\leq \left(\rho \|\partial_t C - i_h \partial_t C\|_{L^2(\Omega)} + NN_C \|\psi - \psi_h\|_{\dot{W}^{1,4}(\Omega)} \right. \\
&\quad \left. + NM_1 \|\nabla(C - i_h C)\|_{L^2(\Omega)} \right) \|\nabla(i_h C - C_h)\|_{L^2(\Omega)} \\
&\quad + \left(\alpha \rho^2 \|C - i_h C\|_{H_{0,\Gamma_1}^1(\Omega)} + N_C \alpha \frac{E}{RT_i^2} \rho^2 \|\nabla(T - T_h)\|_{L^2(\Omega)} \right) \\
&\quad \times \|\nabla \chi_h\|_{L^2(\Omega)}.
\end{aligned} \tag{99}$$

By using Young's inequality, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|i_h C - C_h\|_{L^2(\Omega)}^2 + \frac{\eta}{2} \|\nabla(i_h C - C_h)\|_{L^2(\Omega)}^2 \\
&\leq \frac{2\rho^2}{\eta} \|\partial_t C - i_h \partial_t C\|_{L^2(\Omega)}^2 \\
&\quad + \frac{2N^2 M_1^2}{\eta} \|\nabla(C - i_h C)\|_{L^2(\Omega)}^2 \\
&\quad + \frac{2\alpha^2 \rho^4}{\eta} \|\nabla(C - i_h C)\|_{L^2(\Omega)}^2 \\
&\quad + \frac{2N^2 N_C^2}{\eta} \|\psi - \psi_h\|_{\dot{W}^{1,4}(\Omega)}^2 \\
&\quad + N_C \alpha \frac{E}{RT_i^2} \rho^2 \|\nabla(T - T_h)\|_{L^2(\Omega)} \|\nabla \chi_h\|_{L^2(\Omega)}.
\end{aligned} \tag{100}$$

While multiplying the last inequality by $2/\eta$ and via integration, we obtain

$$\begin{aligned}
& \|i_h C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\
&\leq (h^\sigma \|C\|_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))})^2 \\
&\quad + \frac{4N^2 N_C^2}{\eta^2} \|\psi - \psi_h\|_{L^2(0,t,\dot{W}^{1,4}(\Omega))}^2 \\
&\quad + 2 \frac{N_C \alpha E \rho^2}{\eta RT_i^2} \|\nabla(T - T_h)\|_{L^2(0,t,L^2(\Omega))} \|\nabla \chi_h\|_{L^2(0,t,L^2(\Omega))}.
\end{aligned} \tag{101}$$

So, we get

$$\begin{aligned}
& \|i_h C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\
&\leq (h^\sigma \|C\|_{H^1(0,t,H^\sigma(\Omega))} + h^\sigma \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))})^2 \\
&\quad + \frac{4N^2 N_C^2}{\eta^2} \|\psi - \psi_h\|_{L^2(0,t,\dot{W}^{1,4}(\Omega))}^2 \\
&\quad + 2 \frac{N_C^2 \alpha^2 E^2 \rho^4}{R^2 T_i^4 \eta^2} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2 \\
&\quad + \frac{1}{2} \|i_h C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}^2.
\end{aligned} \tag{102}$$

Finally, by the triangular inequality, we obtain

$$\begin{aligned}
& \|C - C_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\
&\leq h^\sigma (\|C\|_{H^1(0,t,H^\sigma(\Omega))} + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))}) \\
&\quad + \frac{2\sqrt{2}NN_C}{\eta} \|\psi - \psi_h\|_{L^2(0,t,\dot{W}^{1,4}(\Omega))} \\
&\quad + 2 \frac{N_C \alpha E \rho^2}{\eta RT_i^2} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}.
\end{aligned} \tag{103}$$

□

Lemma 11. For T and T_h solution of the problems $(\mathcal{P})$ and $(\mathcal{P}_h)$, respectively, one has

$$\begin{aligned}
& \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\
&\leq h^\sigma (\|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|T\|_{H^1(0,t,H^\sigma(\Omega))}) \\
&\quad + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|C\|_{H^1(0,t,H^\sigma(\Omega))} \\
&\quad + \left(\frac{2\sqrt{10}\alpha\rho^2 NN_C}{\delta_2 \delta_1 \lambda \eta} + \frac{\sqrt{5}NN_T}{\delta_2 \delta_1 \lambda} \right) \\
&\quad \times \|\psi - \psi_h\|_{L^2(0,t,\dot{W}^{1,4}(\Omega))}.
\end{aligned} \tag{104}$$

Proof. By using Lemmas 9 and 10, we have

$$\begin{aligned}
& \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\
& \leq h^\sigma (\|C\|_{H^1(0,t,H^\sigma(\Omega))} + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))} \\
& \quad + \|T\|_{H^1(0,t,H^\sigma(\Omega))} + \|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))}) \\
& \quad + \left(\frac{2\sqrt{10}\alpha\rho^2 NN_C}{\delta_1 \lambda \eta} + \frac{\sqrt{5} NN_T}{\delta_1 \lambda} \right) \|\psi - \psi_h\|_{L^2(0,t,W^{\circ,1,4}(\Omega))} \\
& \quad + \frac{2\sqrt{5}\rho^4 \alpha^2 EN_C}{\delta_1 \lambda \eta RT_i^2} \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))}.
\end{aligned} \tag{105}$$

If we set

$$\delta_2 = 1 - \frac{2\sqrt{5}\rho^4 \alpha^2 EN_C}{\delta_1 \lambda \eta RT_i^2} > 0, \tag{106}$$

then we get

$$\begin{aligned}
& \delta_2 \|T - T_h\|_{L^2(0,t,H_{0,\Gamma_1}^1(\Omega))} \\
& \leq h^\sigma (\|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|T\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|C\|_{H^1(0,t,H^\sigma(\Omega))}) \\
& \quad + \left(\frac{2\sqrt{10}\alpha\rho^2 NN_C}{\delta_1 \lambda \eta} + \frac{\sqrt{5} NN_T}{\delta_1 \lambda} \right) \times \|\psi - \psi_h\|_{L^2(0,t,W^{\circ,1,4}(\Omega))}.
\end{aligned} \tag{107}$$

□

Lemma 12. One assumes that the hypotheses (H_1) , (H_2) , and (H_3) are verified, so

$$\begin{aligned}
& \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))} \\
& \leq h^\sigma (\|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|T\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|C\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|\psi\|_{L^2(0,t,W^{1+\sigma,4}(\Omega))} + \|\omega\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|\omega\|_{L^2(0,t,H^\sigma(\Omega))}).
\end{aligned} \tag{108}$$

Proof. First of all, by using Lemmas 7, 8, and 11, we obtain

$$\begin{aligned}
& \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))} \\
& \leq h^\sigma (\|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|T\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|C\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|\omega\|_{L^2(0,t,H^\sigma(\Omega))} + \|\omega\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|\psi\|_{L^2(0,t,W^{1+\sigma,4}(\Omega))}) \\
& \quad + \sqrt{3} \frac{\beta g}{\mu_p} \left(\frac{2\sqrt{10}\alpha\rho^2 NN_C}{\delta_2 \delta_1 \lambda \eta} + \frac{\sqrt{5} NN_T}{\delta_2 \delta_1 \lambda} \right) \\
& \quad \times \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))}.
\end{aligned} \tag{109}$$

If we set

$$A = \left(\frac{2\sqrt{10}\alpha\rho^2 NN_C}{\eta} + \sqrt{5} NN_T \right), \tag{110}$$

assuming that

$$\frac{\sqrt{3}\beta g N}{\delta_2 \delta_1 \lambda \mu_p} A \leq 1 - \delta_7 \quad \text{with } 0 < \delta_7 < 1, \tag{111}$$

for example, if $\delta_7 = 1/2$, we obtain

$$\delta_2 \delta_1 > \frac{2\sqrt{3}\beta g N}{\lambda \mu_p} A, \tag{112}$$

and then we obtain

$$\begin{aligned}
& \|\omega - \omega_h\|_{L^2(0,t,L^2(\Omega))} \\
& \leq h^\sigma (\|T\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|T\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|C\|_{L^2(0,t,H^{1+\sigma}(\Omega))} + \|C\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|\omega\|_{L^2(0,t,H^\sigma(\Omega))} + \|\omega\|_{H^1(0,t,H^\sigma(\Omega))} \\
& \quad + \|\psi\|_{L^2(0,t,W^{1+\sigma,4}(\Omega))}).
\end{aligned} \tag{113}$$

□

Finally, using Lemmas 8, 10, 11, and 12, we obtain the a priori error estimate at the same time on the temperature, the concentration, the stream function, and the vorticity, so the main Theorem 1 is proved.

5. Conclusion

The propagation of reaction front in porous media is modelled by a system of equations, coupling hydrodynamic equations and the reaction-diffusion equations. We have taken into account Darcy-Boussinesq approximation. We have adopted the stream function-vorticity formulation of the Darcy equation and we have chosen the appropriate functional framework for our variational problem. We have proved the existence result for the semidiscrete solution. Furthermore, we have established an optimal a priori estimate on the temperature, the concentration, the stream function, and the vorticity.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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