Research Article

Existence of Nontrivial Solutions of p -Laplacian Equation with Sign-Changing Weight Functions

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This paper shows the existence and multiplicity of nontrivial solutions of the p -Laplacian problem $-\Delta_p u = (1/\sigma)(\partial F(x, u)/\partial u) + \lambda a(x)|u|^{q-2}u$ for $x \in \Omega$ with zero Dirichlet boundary conditions, where Ω is a bounded open set in $\mathbb{R}^n$, $1 < q < p < \sigma < p^*$ ($p^* = np/(n-p)$ if $p < n$, $p^* = \infty$ if $p \geq n$), $\lambda \in \mathbb{R} \setminus \{0\}$, a is a smooth function which may change sign in $\bar{\Omega}$, and $F \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$. The method is based on Nehari results on three submanifolds of the space $W_0^{1,p}(\Omega)$.

1. Introduction

In this paper, we are concerned with the multiplicity of nontrivial nonnegative solutions of the following elliptic equation:

$$\begin{aligned} -\Delta_p u &= \frac{1}{\sigma} \frac{\partial F(x, u)}{\partial u} + \lambda a(x) |u|^{q-2} u \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 3$), $1 < q < p < \sigma < p^*$ ($p^* = np/(n-p)$ if $p < n$, $p^* = \infty$ if $p \geq n$), $\lambda \in \mathbb{R} \setminus \{0\}$, $F \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree σ ; that is, $F(x, tu) = t^\sigma F(x, u)$ holds for all $(x, u) \in \bar{\Omega} \times \mathbb{R}$ and the sign-changing weight function a satisfies the following condition:

(A) $a \in C(\Omega)$ with $\|a\|_\infty = 1$, $a^+ := \max(+a, 0) \not\equiv 0$, and $a^- := \max(-a, 0) \not\equiv 0$.

In recent years, several authors have used the Nehari manifold and fibering maps (i.e., maps of the form $t \mapsto J_\lambda(tu)$, where J_λ is the Euler function associated with the equation) to solve semilinear and quasilinear problems. For instance, we cite papers [1–9] and references therein. More precisely, Brown and Zhang [10] studied the following

subcritical semilinear elliptic equation with sign-changing weight function:

$$\begin{aligned} -\Delta u &= \lambda a(x) u + b(x) |u|^{\gamma-2} u(x), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (2)$$

where $\gamma > 2$. Also, the authors in [10] by the same arguments considered the following semilinear elliptic problem:

$$\begin{aligned} -\Delta u &= \lambda f(x) |u|^{q-2} u(x) + g(x) |u|^{p-2} u(x), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (3)$$

where $1 < q < 2 < p$. Exploiting the relationship between the Nehari manifold and fibering maps, they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value.

Inspired by the work of Brown and Zhang [10], Nyamouadi [11] treated the following problem:

$$\Delta(|\Delta u|^{p-2} \Delta u) = \frac{1}{p^*} f(x, u) + \lambda |u|^{q-2} u(x); \quad \text{in } \Omega, \quad (4)$$

$$u = \Delta u = 0, \quad \text{on } \partial\Omega,$$

where f is positively homogeneous of degree $p^* - 1$.

In this work, motivated by the above works, we give a very simple variational method to prove the existence of at least two nontrivial solutions of problem (1). In fact, we use the decomposition of the Nehari manifold as λ vary to prove our main result.

Before stating our main result, we need the following assumptions:

(H₁) $F : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function such that

$$F(x, tu) = t^\sigma F(x, u) \quad (t > 0) \quad \forall x \in \overline{\Omega}, u \in \mathbb{R}. \quad (5)$$

$$(H_2) \quad \begin{aligned} F(x, 0) &= (\partial F / \partial u)(x, 0) = 0, & F^+(x, u) &= \\ \max(+F(x, u), 0) &\neq 0, & \text{and } F^-(x, u) &= \\ \max(-F(x, u), 0) &\neq 0 \text{ for all } u \neq 0. \end{aligned}$$

We remark that using assumption (H₁), for all $x \in \overline{\Omega}, u \in \mathbb{R}$, we have the so-called Euler identity:

$$u \frac{\partial F(x, u)}{\partial u} = \sigma F(x, u), \quad (6)$$

$$|F(x, u)| \leq K|u|^\sigma \text{ for some constant } K > 0.$$

Our main result is the following.

Theorem 1. *Under the assumptions (A), (H₁), and (H₂), there exists $\lambda_0 > 0$ such that for all $0 < |\lambda| < \lambda_0$, problem (1) has at least two nontrivial nonnegative solutions.*

This paper is organized as follows. In Section 2, we give some notations and preliminaries and we present some technical lemmas which are crucial in the proof of Theorem 1. Theorem 1 is proved in Section 3.

2. Some Notations and Preliminaries

Throughout this paper, we denote by S_l the best Sobolev constant for the operators $W_0^{1,p}(\Omega) \hookrightarrow L^l(\Omega)$, given by

$$S_l = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^l dx\right)^{p/l}}, \quad (7)$$

where $1 < l \leq p^*$. In particular, we have

$$\int_\Omega |u|^l dx \leq S_l^{-l/p} \|u\|^l \quad \forall u \in W_0^{1,p}(\Omega), \quad (8)$$

with the standard norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx \right)^{1/p}. \quad (9)$$

Problem (1) is posed in the framework of the Sobolev space $E = W_0^{1,p}(\Omega)$. Moreover, a function u in E is said to be a weak solution of problem (1) if

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx - \frac{1}{\sigma} \int_\Omega \frac{\partial F(x, u)}{\partial u} \varphi dx \\ - \lambda \int_\Omega a |u|^{q-2} u \varphi dx = 0, \quad \forall \varphi \in E. \end{aligned} \quad (10)$$

Thus, by (6) the corresponding energy functional of problem (1) is defined in E by

$$J_\lambda(u) = \frac{1}{p} \|u\|^p - \frac{1}{\sigma} \int_\Omega F(x, u) dx - \frac{\lambda}{q} \int_\Omega a(x) |u|^q dx. \quad (11)$$

In order to verify $J_\lambda \in C^1(E, \mathbb{R})$, we need the following lemmas.

Lemma 2. *Assume that $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree σ ; then $\partial F / \partial u \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree $\sigma - 1$.*

Proof. The proof is the same as that in Chu and Tang [4].

In addition, by Lemma 2, we get the existence of positive constant M such that

$$\left| \frac{\partial F(x, u)}{\partial u} \right| \leq M |u|^{\sigma-1}, \quad \forall x \in \overline{\Omega}, u \in \mathbb{R}. \quad (12)$$

□

Lemma 3 (see [12], Theorem A.2). *Let $p, r \in [1, \infty)$ and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ such that*

$$|f(x, u)| \leq c(1 + |u|^{p/r}), \quad \forall x \in \overline{\Omega}, \forall u \in \mathbb{R}. \quad (13)$$

Then for every $u \in L^p(\Omega)$, one has $f(\cdot, u) \in L^r(\Omega)$; moreover the operator $A : L^p(\Omega) \rightarrow L^r(\Omega)$ defined by $A(u)(x) = f(x, u(x))$ is continuous.

Lemma 4 (See Proposition 1 in [13]). *Suppose that $\partial F(x, u) / \partial u \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ verifies condition (12). Then, the functional J_λ belongs to $C^1(E, \mathbb{R})$, and*

$$\langle J'_\lambda(u), u \rangle = \|u\|^p - \int_\Omega F(x, u) dx - \lambda \int_\Omega a(x) |u|^q dx, \quad (14)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality between E and $E^* := W^{-1,p'}(\Omega)$ (the dual space of the Sobolev space E).

As the energy functional J_λ is not bounded below in E , it is useful to consider the functional on the Nehari manifold:

$$N_\lambda = \{u \in E \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0\}. \quad (15)$$

Thus, $u \in N_\lambda$ if and only if

$$\|u\|^p - \int_\Omega F(x, u) dx - \lambda \int_\Omega a(x) |u|^q dx = 0. \quad (16)$$

Note that N_λ contains every nonzero solution of problem (1). Moreover, one has the following result.

Lemma 5. *The energy functional J_λ is coercive and bounded below on N_λ .*

Proof. If $u \in N_\lambda$, then by (16) and condition (A) we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{\sigma - p}{\sigma p} \|u\|^p - \lambda \frac{\sigma - q}{\sigma q} \int_\Omega a(x) |u|^q dx, \\ &\geq \frac{\sigma - p}{\sigma p} \|u\|^p - |\lambda| \frac{\sigma - q}{\sigma q} \int_\Omega |u|^q dx. \end{aligned} \quad (17)$$

So, it follows from (8) that

$$J_\lambda(u) \geq \frac{\sigma-p}{\sigma p} \|u\|^p - \frac{|\lambda|}{S_q^{q/p}} \frac{\sigma-q}{\sigma q} \|u\|^q. \quad (18)$$

Thus, J_λ is coercive and bounded below on N_λ .

Define

$$\phi_\lambda(u) = \langle J'_\lambda(u), u \rangle. \quad (19)$$

Then, by (16) it is easy to see that for $u \in N_\lambda$,

$$\begin{aligned} \langle \phi'_\lambda(u), u \rangle &= p\|u\|^p - \sigma \int_\Omega F(x, u) dx - \lambda q \int_\Omega a(x) |u|^q dx, \end{aligned} \quad (20)$$

$$= \lambda(p-q) \int_\Omega a(x) |u|^q dx - (\sigma-p) \int_\Omega F(x, u) dx, \quad (21)$$

$$= \lambda(\sigma-q) \int_\Omega a(x) |u|^q dx - (\sigma-p) \|u\|^p, \quad (22)$$

$$= (p-q) \|u\|^p - (\sigma-q) \int_\Omega F(x, u) dx. \quad (23)$$

Now, we split N_λ into three parts

$$\begin{aligned} N_\lambda^+ &= \{u \in N_\lambda : \langle \phi'_\lambda(u), u \rangle > 0\}, \\ N_\lambda^0 &= \{u \in N_\lambda : \langle \phi'_\lambda(u), u \rangle = 0\}, \\ N_\lambda^- &= \{u \in N_\lambda : \langle \phi'_\lambda(u), u \rangle < 0\}. \end{aligned} \quad (24)$$

□

Lemma 6. Assume that u_0 is a local minimizer for J_λ on N_λ and that $u_0 \notin N_\lambda^0$. Then, $J'_\lambda(u_0) = 0$ in E^{-1} (the dual space of the Sobolev space E).

Proof. Our proof is the same as that in Brown-Zhang [10, Theorem 2.3]. □

Lemma 7. One has the following:

- (i) if $u \in N_\lambda^+$, then $\lambda \int_\Omega a(x) |u|^q dx > 0$;
- (ii) if $u \in N_\lambda^0$, then $\lambda \int_\Omega a(x) |u|^q dx > 0$ and $\int_\Omega F(x, u) dx > 0$;
- (iii) if $u \in N_\lambda^-$, then $\int_\Omega F(x, u) dx > 0$.

Proof. The proof is immediate from (21), (22), and (23).

From now on, we denote by λ_0 the constant defined by

$$\lambda_0 = \frac{q(\sigma-p)}{p(\sigma-q)} S_q^{q/p} \left(\frac{p-q}{K(\sigma-q)} S_\sigma^{\sigma/p} \right)^{(p-q)(\sigma-p)}, \quad (25)$$

then we have the following. □

Lemma 8. If $0 < |\lambda| < \lambda_0$, then $N_\lambda^0 = \emptyset$.

Proof. Suppose otherwise, that $0 < |\lambda| < \lambda_0$ such that $N_\lambda^0 \neq \emptyset$. Then for $u \in N_\lambda^0$, we have

$$0 = \langle \phi'_\lambda(u), u \rangle = \lambda(\sigma-q) \int_\Omega a(x) |u|^q dx - (\sigma-p) \|u\|^p \quad (26)$$

$$= (p-q) \|u\|^p - (\sigma-q) \int_\Omega F(x, u) dx. \quad (27)$$

From the Hölder inequality, (6) and (8), it follows that

$$\begin{aligned} \int_\Omega F(x, u) dx &\leq \int_\Omega |F(x, u)| dx \\ &\leq K \int_\Omega |u|^\sigma dx \leq K S_\sigma^{-\sigma/p} \|u\|^\sigma. \end{aligned} \quad (28)$$

Hence, it follows from (27) that

$$\|u\|^p = \frac{\sigma-q}{p-q} \int_\Omega F(x, u) dx \leq \frac{\sigma-q}{p-q} K S_\sigma^{-\sigma/p} \|u\|^\sigma, \quad (29)$$

then,

$$\|u\| \geq \left(\frac{p-q}{K(\sigma-q)} S_\sigma^{\sigma/p} \right)^{1/(\sigma-p)}. \quad (30)$$

On the other hand, from condition (A), (8) and (26) we have

$$\|u\|^p = \lambda \frac{\sigma-q}{\sigma-p} \int_\Omega a(x) |u|^q dx \leq |\lambda| \frac{\sigma-q}{\sigma-p} S_q^{-q/p} \|u\|^q. \quad (31)$$

So,

$$\|u\| \leq \left(|\lambda| \frac{\sigma-q}{\sigma-p} S_q^{-q/p} \right)^{1/(p-q)}. \quad (32)$$

Combining (30) and (32), we obtain $\lambda_0 \leq |\lambda|$, which is a contradiction.

By Lemma 8, for $0 < |\lambda| < \lambda_0$, we write $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and define

$$\theta_\lambda = \inf_{u \in N_\lambda} J_\lambda(u), \quad \theta_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u), \quad \theta_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u). \quad (33)$$

Then, we have the following. □

Lemma 9. If $0 < |\lambda| < \lambda_0$, then

$$\theta_\lambda \leq \theta_\lambda^+ < 0, \quad \theta_\lambda^- > d_0 \quad (34)$$

for some $d_0 > 0$ depending on $p, q, \sigma, K, \lambda, S_q$, and S_σ .

Proof. Let $u \in N_\lambda^+$. Then, from (23) we have

$$\frac{p-q}{\sigma-q} \|u\|^p > \int_\Omega F(x, u) dx. \quad (35)$$

So

$$\begin{aligned} J_\lambda(u) &= \frac{q-p}{pq} \|u\|^p + \frac{\sigma-q}{\sigma q} \int_\Omega F(x, u) dx \\ &< \left(\frac{q-p}{pq} + \frac{\sigma-q}{\sigma q} \frac{p-q}{\sigma-q} \right) \|u\|^p \\ &= -\frac{(p-q)(\sigma-p)}{pq\sigma} \|u\|^p < 0. \end{aligned} \quad (36)$$

Thus, from the definition of θ_λ and θ_λ^+ , we can deduce that $\theta_\lambda \leq \theta_\lambda^+ < 0$.

Now, let $u \in N_\lambda^-$. Then, using (6) and (8) we obtain

$$\frac{p-q}{\sigma-q} \|u\|^p < \int_\Omega F(x, u) dx \leq K S_\sigma^{-\sigma/p} \|u\|^\sigma, \quad (37)$$

this implies that

$$\|u\| > \left(\frac{p-q}{\sigma-q} \frac{S_\sigma^{\sigma/p}}{K} \right)^{1/(\sigma-p)}, \quad \forall u \in N_\lambda^-. \quad (38)$$

In addition, by (18) and (38)

$$\begin{aligned} J_\lambda(u) &\geq \frac{\sigma-p}{p\sigma} \|u\|^p - |\lambda| S_q^{-q/p} \frac{\sigma-p}{\sigma q} \|u\|^q \\ &\geq \|u\|^q \left[\frac{\sigma-p}{p\sigma} \|u\|^{p-q} - |\lambda| S_q^{-q/p} \frac{\sigma-q}{\sigma q} \right] \\ &> \left(\frac{p-q}{\sigma-q} \frac{S_\sigma^{\sigma/p}}{K} \right)^{q/(\sigma-p)} \left(\frac{\sigma-p}{p\sigma} \left(\frac{p-q}{\sigma-q} \frac{S_\sigma^{\sigma/p}}{K} \right)^{(p-q)/(\sigma-p)} \right. \\ &\quad \left. - |\lambda| S_q^{-q/p} \frac{\sigma-q}{\sigma q} \right). \end{aligned} \quad (39)$$

Thus, since $0 < |\lambda| < \lambda_0$, we conclude that $J_\lambda > d_0$ for some $d_0 > 0$. This completes the proof. $\square$

For $u \in E$ with $\int_\Omega F(x, u) dx > 0$, set

$$T = \left(\frac{(p-q) \|u\|^p}{(\sigma-q) \int_\Omega F(x, u) dx} \right)^{1/(\sigma-p)} > 0. \quad (40)$$

Then, the following lemma holds.

Lemma 10. For each $u \in E$ with $\int_\Omega F(x, u) dx > 0$, one has the following:

(i) if $\lambda \int_\Omega a(x) |u|^q dx \leq 0$, then there exists unique $t^- > T$ such that $t^- u \in N_\lambda^-$ and

$$J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu); \quad (41)$$

(ii) if $\lambda \int_\Omega a(x) |u|^q dx > 0$, then there are unique $0 < t^+ < T < t^-$ such that $(t^- u, t^+ u) \in N_\lambda^- \times N_\lambda^+$ and

$$J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu), \quad J_\lambda(t^+ u) = \inf_{0 \leq t \leq T} J_\lambda(tu). \quad (42)$$

Proof. We fix $u \in E$ with $\int_\Omega F(x, u) dx > 0$ and we let

$$m(t) = t^{p-q} \|u\|^p - t^{\sigma-q} \int_\Omega F(x, u) dx \quad \text{for } t \geq 0. \quad (43)$$

Then, it is easy to check that $m(t)$ achieves its maximum at T . Moreover,

$$\begin{aligned} m(T) &= \|u\|^q \left[\left(\frac{p-q}{\sigma-q} \right)^{(p-q)(\sigma-p)} - \left(\frac{p-q}{\sigma-q} \right)^{(\sigma-q)/(\sigma-p)} \right] \\ &\quad \times \left(\frac{\|u\|^\sigma}{\int_\Omega F(x, u) dx} \right)^{(p-q)/(\sigma-p)} \\ &\geq \|u\|^q \left(\frac{\sigma-p}{\sigma-q} \right) \left(\frac{(\sigma-q) S_\sigma^{\sigma/p}}{K(p-q)} \right)^{(p-q)/(\sigma-p)}. \end{aligned} \quad (44)$$

(i) We suppose that $\lambda \int_\Omega a(x) |u|^q dx \leq 0$. Since $m(0) = 0$, $m(t) \rightarrow -\infty$ as $t \rightarrow \infty$, $m'(t) > 0$ for $t < T$ and $m'(t) < 0$ for $t > T$. There is a unique $t^- > T$ such that $m(t^-) = \lambda \int_\Omega a(x) |u|^q dx \leq 0$.

Now, it follows from (14) and (27) that

$$\begin{aligned} \phi'_\lambda(t^- u) t^- u &= (t^-)^{1+q} m'(t^-) < 0, \\ J'_\lambda(t^- u) t^- u &= (t^-)^q \left(m(t^-) - \lambda \int_\Omega a(x) |u|^q dx \right) = 0. \end{aligned} \quad (45)$$

Hence, $t^- u \in N_\lambda^-$. On the other hand, it is easy to see that for all $t > T$

$$\frac{d^2}{dt^2} J_\lambda(tu) < 0, \quad \frac{d}{dt} J_\lambda(tu) = 0 \quad \text{for } t = t^-. \quad (46)$$

Thus, $J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu)$.

(ii) We suppose that $\lambda \int_\Omega a(x) |u|^q dx > 0$. Then, by (A), (8) and the fact that $|\lambda| < \lambda_0$ we obtain

$$m(0) = 0 < \lambda \int_\Omega a(x) |u|^q dx \leq |\lambda| S_q^{-q/p} \|u\|^q < m(T). \quad (47)$$

Then, there are unique t^+ and t^- such that $0 < t^+ < T < t^-$, $m(t^+) = \lambda \int_\Omega a(x) |u|^q dx = m(t^-)$, and $m'(t^-) < 0 < m'(t^+)$. We have $(t^- u, t^+ u) \in N_\lambda^- \times N_\lambda^+$, and

$$\begin{aligned} J_\lambda(t^+ u) &\leq J_\lambda(tu) \leq J_\lambda(t^- u) \quad \forall t \in [t^+, t^-], \\ J_\lambda(t^+ u) &\leq J_\lambda(tu) \quad \forall 0 \leq t \leq t^+. \end{aligned} \quad (48)$$

Thus,

$$J_\lambda(t^- u) = \sup_{t \geq 0} J_\lambda(tu), \quad J_\lambda(t^+ u) = \inf_{0 \leq t \leq T} J_\lambda(tu). \quad (49)$$

This completes the proof. $\square$

For each $u \in E$ with $\lambda \int_{\Omega} a(x)|u|^q dx > 0$, set

$$\tilde{T} = \left(\frac{(\sigma - q) \lambda \int_{\Omega} a(x)|u|^q dx}{(\sigma - p) \|u\|^p} \right)^{1/(p-q)} > 0. \quad (50)$$

Then we have the following.

Lemma 11. For each $u \in E$ with $\lambda \int_{\Omega} a(x)|u|^q dx > 0$, one has the following:

(i) if $\int_{\Omega} F(x, u) dx \leq 0$, then there exists a unique $0 < t^+ < \tilde{T}$ such that $t^+ u \in N_{\lambda}^+$ and

$$J_{\lambda}(t^+ u) = \inf_{t \geq 0} J_{\lambda}(tu); \quad (51)$$

(ii) if $\int_{\Omega} F(x, u) dx > 0$, then there are unique $0 < t^+ < \tilde{T} < t^-$ such that $(t^- u, t^+ u) \in N_{\lambda}^- \times N_{\lambda}^+$ and

$$J_{\lambda}(t^- u) = \sup_{t \geq 0} J_{\lambda}(tu), \quad J_{\lambda}(t^+ u) = \inf_{0 \leq t \leq \tilde{T}} J_{\lambda}(tu). \quad (52)$$

Proof. For $u \in E$ with $\lambda \int_{\Omega} a(x)|u|^q dx > 0$, we can take

$$\tilde{m}(t) = t^{p-\sigma} \|u\|^p - \lambda t^{q-\sigma} \int_{\Omega} a(x)|u|^q dx \quad \text{for } t > 0, \quad (53)$$

and similar to the argument in Lemma 9, we obtain the results of Lemma 10. $\square$

Proposition 12. (i) There exist minimizing sequences $\{u_n^+\}$ in N_{λ}^+ such that

$$J_{\lambda}(u_n^+) = \theta_{\lambda}^+ + o(1), \quad J'_{\lambda}(u_n^+) = o(1) \quad \text{in } E^{-1}. \quad (54)$$

(ii) There exist minimizing sequences $\{u_n^-\}$ in N_{λ}^- such that

$$J_{\lambda}(u_n^-) = \theta_{\lambda}^- + o(1), \quad J'_{\lambda}(u_n^-) = o(1) \quad \text{in } E^{-1}. \quad (55)$$

Proof. The proof is almost the same as that in Wu [14, Proposition 9] and is omitted here. $\square$

3. Proof of Our Result

Throughout this section, the norm L^s is denoted by $\|\cdot\|_s$ for $1 \leq s \leq \infty$ and the parameter λ satisfies $0 < |\lambda| < \lambda_0$.

Theorem 13. If $0 < |\lambda| < \lambda_0$, then, problem (1) has a positive solution u_0^+ in N_{λ}^+ such that

$$J_{\lambda}(u_0^+) = \theta_{\lambda} = \theta_{\lambda}^+. \quad (56)$$

Proof. By Proposition 12(i), there exists a minimizing sequence $\{u_n^+\}$ for J_{λ} on N_{λ}^+ such that

$$J_{\lambda}(u_n^+) = \theta_{\lambda}^+ + o(1), \quad J'_{\lambda}(u_n^+) = o(1) \quad \text{in } E^{-1}. \quad (57)$$

Then by Lemma 5, there exists a subsequence $\{u_n\}$ and u_0^+ in E such that

$$u_n \rightharpoonup u_0^+ \quad \text{weakly in } E, \quad (58)$$

$$u_n \rightarrow u_0^+ \quad \text{strongly in } L^q(\Omega) \text{ and in } L^{\sigma}(\Omega).$$

This implies that $\int_{\Omega} a(x)|u_n|^q dx \rightarrow \int_{\Omega} a(x)|u_0^+|^q dx$ as $n \rightarrow \infty$.

Next, we will show that

$$\int_{\Omega} F(x, u_n) dx \rightarrow \int_{\Omega} F(x, u_0^+) dx \quad \text{as } n \rightarrow \infty. \quad (59)$$

By Lemma 3, we have

$$\frac{\partial F(x, u_n)}{\partial u} \in L^{\gamma}(\Omega), \quad \frac{\partial F(x, u_n)}{\partial u} \rightarrow \frac{\partial F(x, u_0^+)}{\partial u} \quad \text{in } L^{\gamma}(\Omega), \quad (60)$$

where $\gamma = \sigma/(\sigma - 1)$. On the other hand, it follows from the Hölder inequality that

$$\begin{aligned} & \int_{\Omega} \left| u_n \frac{\partial F(x, u_n)}{\partial u} - u_0^+ \frac{\partial F(x, u_0^+)}{\partial u} \right| dx \\ & \leq \int_{\Omega} \left| (u_n - u_0^+) \frac{\partial F(x, u_n)}{\partial u} \right| dx \\ & \quad + \int_{\Omega} |u_0^+| \left| \frac{\partial F(x, u_n)}{\partial u} - \frac{\partial F(x, u_0^+)}{\partial u} \right| dx \\ & \leq \|u_n - u_0^+\|_{\sigma} \left\| \frac{\partial F(x, u_n)}{\partial u} \right\|_{\gamma} \\ & \quad + \|u_0^+\|_{\sigma} \left\| \frac{\partial F(x, u_n)}{\partial u} - \frac{\partial F(x, u_0^+)}{\partial u} \right\|_{\gamma} \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (61)$$

Hence, $\int_{\Omega} F(x, u_n) dx \rightarrow \int_{\Omega} F(x, u_0^+) dx$ as $n \rightarrow \infty$.

By (57) and (58) it is easy to prove that u_0^+ is a weak solution of (1).

Since

$$\begin{aligned} J_{\lambda}(u_n) &= \frac{\sigma - p}{p\sigma} \|u_n\|^p - \lambda \frac{\sigma - q}{q\sigma} \int_{\Omega} a(x)|u_n|^q dx \\ &\geq -\lambda \frac{\sigma - q}{q\sigma} \int_{\Omega} a(x)|u_n|^q dx \end{aligned} \quad (62)$$

then by (57) and Lemma 9, we have $J_{\lambda}(u_n) \rightarrow \theta_{\lambda} < 0$ as $n \rightarrow \infty$. Letting $n \rightarrow \infty$, we obtain

$$\lambda \int_{\Omega} a(x)|u_0^+|^q dx > 0. \quad (63)$$

Now, we aim to prove that $u_n \rightarrow u_0^+$ strongly in E and $J_{\lambda}(u_0^+) = \theta_{\lambda}$.

Using the fact that $u_0^+ \in N_\lambda$ and by Fatou's lemma, we get

$$\begin{aligned} \theta_\lambda &\leq J_\lambda(u_0^+) = \frac{1}{p} \|u_0^+\|^p - \frac{1}{\sigma} \int_\Omega F(x, u_0^+) dx \\ &\quad - \frac{\lambda}{q} \int_\Omega a(x) |u_0^+|^q dx \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{1}{p} \|u_n\|^p - \frac{1}{\sigma} \int_\Omega F(x, u_n) dx \right. \\ &\quad \left. - \frac{\lambda}{q} \int_\Omega a(x) |u_n|^q dx \right) \\ &\leq \liminf_{n \rightarrow \infty} J_\lambda(u_n) = \theta_\lambda. \end{aligned} \quad (64)$$

This implies that

$$J_\lambda(u_0^+) = \theta_\lambda, \quad \lim_{n \rightarrow \infty} \|u_n\|^p = \|u_0^+\|^p. \quad (65)$$

Let $\tilde{u}_n = u_n - u_0^+$; then by Brézis-Lieb Lemma [3] we obtain

$$\lim_{n \rightarrow \infty} (\|u_n\|^p - \|\tilde{u}_n\|^p) = \|u_0^+\|^p. \quad (66)$$

Therefore, $u_n \rightarrow u_0^+$ strongly in E .

Moreover, we have $u_0^+ \in N_\lambda^+$. In fact, if $u_0^+ \in N_\lambda^-$ then, there exist t_0^+, t_0^- such that $t_0^- u_0^+ \in N_\lambda^-$ and $t_0^+ u_0^+ \in N_\lambda^+$. In particular we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d^2}{dt^2} J_\lambda(t_0^+ u_0^+) > 0, \quad \frac{d}{dt} J_\lambda(t_0^+ u_0^+) = 0, \quad (67)$$

there exists $t_0^+ < \tilde{t} < t_0^-$ such that $J_\lambda(t_0^+ u_0^+) < J_\lambda(\tilde{t} u_0^+)$. By Lemma 10, we have

$$J_\lambda(t_0^+ u_0^+) < J_\lambda(\tilde{t} u_0^+) \leq J_\lambda(t_0^- u_0^+) = J_\lambda(u_0^+), \quad (68)$$

which is a contradiction.

Finally, by (63) we may assume that u_0^+ is a nontrivial nonnegative solution of problem (1). $\square$

Theorem 14. *If $0 < |\lambda| < \lambda_0$, then, problem (1) has a positive solution u_0^- in N_λ^- such that*

$$J_\lambda(u_0^-) = \theta_\lambda^-. \quad (69)$$

Proof. By Proposition 12(ii), there exists a minimizing sequence $\{u_n\}$ for J_λ on N_λ^- such that

$$J_\lambda(u_n) = \theta_\lambda^- + o(1), \quad J'_\lambda(u_n) = o(1) \text{ in } E^{-1}, \quad (70)$$

$$u_n \rightharpoonup u_0^- \text{ weakly in } E, \quad (71)$$

$$u_n \rightarrow u_0^- \text{ strongly in } L^q(\Omega) \text{ and in } L^\sigma(\Omega).$$

Moreover, by (23) we obtain

$$\int_\Omega F(x, u_n) dx > \frac{p-q}{\sigma-q} \|u_n\|^p. \quad (72)$$

So, by (38) and (72) there exists a positive constant $\tilde{C}$ such that

$$\int_\Omega F(x, u_n) > \tilde{C}. \quad (73)$$

This implies that

$$\int_\Omega F(x, u_0^-) \geq \tilde{C}. \quad (74)$$

By (70) and (71), we obtain clearly that u_0^- is a weak solution of (1).

Now, we aim to prove that $u_n \rightarrow u_0^-$ strongly in E . Supposing otherwise, then

$$\|u_0^-\| < \liminf_{n \rightarrow \infty} \|u_n\|. \quad (75)$$

By Lemma 9, there is a unique t_0^- such that $t_0^- u_0^- \in N_\lambda^-$. Since $u_n \in N_\lambda^-$, $J_\lambda(u_n) \geq J_\lambda(t u_n)$ for all $t \geq 0$, we have

$$J_\lambda(t_0^- u_0^-) < \lim_{n \rightarrow \infty} J_\lambda(t_0^- u_n) \leq \lim_{n \rightarrow \infty} J_\lambda(u_n) = \theta_\lambda^-, \quad (76)$$

which is a contradiction. Hence $u_n \rightarrow u_0^-$ strongly in E .

This imply that

$$J_\lambda(u_n) \rightarrow J_\lambda(u_0^-) = \theta_\lambda^- \text{ as } n \rightarrow \infty. \quad (77)$$

By Lemma 5 and (74) we may assume that u_0^- is a nontrivial solution of problem (1).

Now, we begin to show the proof of Theorem 1: by Theorem 13, we obtain that for all $0 < \lambda < \lambda_0$, problem (1) has a nontrivial solution $u_0^+ \in N_\lambda^+$. On the other hand, from Theorem 14, we get the second solution $u_0^- \in N_\lambda^-$. Since $N_\lambda^- \cap N_\lambda^+ = \emptyset$, then u_0^- and u_0^+ are distinct. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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