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Research Article

On the Sumudu Transform and Its Extension to a Class of Boehmians

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Boehmians are used for all objects obtained by an algebraic construction similar to that of the field of quotients. In literature, several integral transforms have been extended to various Boehmian spaces but a few to the space of strong Boehmians. As shown in the work of Al-Omari (2013), this work describes certain spaces of Boehmians. The Sumudu transform is therefore established and it is one-one and continuous in the space of Boehmians. The inverse transform is given and some results are also discussed.

1. Introduction

Integral transforms are widely used in the literature, where some are often used for solving differential equations. The Sumudu transform was introduced by Watugala [1] and discussed by Weerakoon in [2]. Many properties are established in [3–5]. Having scale and unit-preserving properties, the Sumudu transform can be used to solve problems without resorting to new frequency domains. In [6] the classical Sumudu transform has been investigated over functions of two variables. In [7] the classical theorems in [6] are extended to distribution spaces and a space of Boehmians.

In this note we discuss the cited transform on certain space of strong Boehmians. Definitions and classical properties of Sumudu transforms are briefly given in Section 1. The general construction of usual and strong Boehmians is given in Sections 2 and 3, respectively. In Section 4, the Sumudu transform is extended to Boehmians and many properties are also obtained.

The Sumudu transform of $f(x)$ is given by [1]

$$(\mu f)(\zeta) = \tilde{f}(\zeta) =: \frac{1}{\zeta}\int_{\mathbb{R}_+} f(t)\exp\left(\frac{-t}{\zeta}\right)dt, \tag{1}$$

over a set of functions given as

$$\begin{aligned}\mathbb{A} = \{ & f(t) : \exists m, \tau_1, \tau_2 > 0, |f(t)| < me^{t/\tau j}, \\ & t \in (-1)^j \times (0, \infty]\},\end{aligned} \tag{2}$$

where $f(t)$ is of convergent series type and $\zeta \in (-\tau_1, \tau_2)$.

Denote by

$$(f \circledast g)(x) = \int_{\mathbb{R}_+} f(x-t)\, g(t)\, dt, \tag{3}$$

the usual convolution product of f and g.

Then the Sumudu transform of the convolution product is given as

$$\mu(f \circledast g)(\zeta) = \zeta \tilde{f}(\zeta)\, \tilde{g}(\zeta), \tag{4}$$

where $\tilde{f}$ and $\tilde{g}$ denote the Sumudu transform of f and g, respectively.

Following are general properties of Sumudu transforms.

(i) If $k_1, k_2 \in \mathbb{R}_+$ and $\tilde{f}, \tilde{g}$ are the Sumudu transforms of f and g, respectively, then

$$\mu(k_1 f(t) + k_2 g(t))(\zeta) = k_1 \tilde{f}(\zeta) + k_2 \tilde{g}(\zeta); \tag{5}$$

(ii)

$$\mu f(kt)(\zeta) = \mu(k\zeta), \quad k \in \mathbb{R}_+; \tag{6}$$

(iii)

$$\lim_{t\to 0} f(t) = \lim_{\zeta \to 0} \mu(\zeta) = f(0). \tag{7}$$

For details, see [8].

For strong Boehmians, see [9]. For usual Boehmians, see [10–19].

2. Strong Boehmians

Denote by $\mathbb{R}_+$ the set of positive real numbers. By $\mathfrak{d}(\mathbb{R}_+)$ denote the Schwartz space of test functions of compact support over $\mathbb{R}_+$ and by $\mathfrak{e}(\Omega)$ the space of smooth functions over Ω, where $\Omega = \mathbb{R}_{+1} \times \mathbb{R}_+$, ($\mathbb{R}_{+1} = [1, \infty)$). The dual space of $\mathfrak{e}(\Omega)$, namely $\mathfrak{e}'(\Omega)$, consists of distributions of compact support [14, 20, 21].

Between $f \in \mathfrak{e}(\Omega)$ and $v \in \mathfrak{d}(\mathbb{R}_+)$, the convolution product $\otimes$ of f and v is given by

$$(f \otimes v)(x) = \int_{\mathbb{R}_+} f(\alpha, t)\, v(x-t)\, dt, \tag{8}$$

where $\alpha \in \mathbb{R}_{+1}$.

Since notations $\otimes$ and $\circledast$ denote a same product, when f is either defined over Ω or $\mathbb{R}_+$, the role of $\otimes$ and $\circledast$ may be interchanged in the text of this paper.

Let $\mathfrak{u}(\mathbb{R}_+)$ be the subset of $\mathfrak{d}(\mathbb{R}_+)$ of test functions such that

$$\int_{\mathbb{R}_+} v(x)\, dx = 1. \tag{9}$$

The pair (f, v) of functions, $f \in \mathfrak{e}(\Omega)$, $v \in \mathfrak{u}(\mathbb{R}_+)$, is said to be a quotient of functions, denoted as f/v, if and only if

$$f(\alpha, x) \otimes \mathtt{d}_\beta v(x) = f(\beta, x) \otimes \mathtt{d}_\alpha v(x), \tag{10}$$

where

$$\mathtt{d}_r v(x) = r v(rx), \quad \alpha, \beta \in \mathbb{R}_{+1}. \tag{11}$$

Two quotients f/v and g/ψ are said to be equivalent, $f/v \sim g/\psi$, if and only if

$$f(\alpha, x) \otimes \mathtt{d}_\beta \psi(x) = g(\beta, x) \otimes \mathtt{d}_\alpha v(x), \quad \alpha, \beta \in \mathbb{R}_{+1}. \tag{12}$$

Let the set $\mathtt{A}$ be given as

$$\mathtt{A} = \left\{ \frac{f}{v} : \forall f \in \mathfrak{e}(\Omega), v \in \mathfrak{u}(\mathbb{R}_+) \right\}. \tag{13}$$

Then the equivalence class $[f/v]$ containing f/v is said to be a strong Boehmian.

The space of all such Boehmians is denoted by $\pounds_{\mathrm{st}}(\mathfrak{e}, \mathfrak{u}, \otimes)$ and is the so-called the space of strong Boehmians.

Following conclusions are useful in the sequel [9, p.p. 886].

($\mathfrak{c}_1$) Let $v, \psi \in \mathfrak{u}(\mathbb{R}_+)$; then $v \circledast \psi \in \mathfrak{u}(\mathbb{R}_+)$.

($\mathfrak{c}_2$) Let $f \in \mathfrak{e}(\Omega)$ and $v \in \mathfrak{u}(\mathbb{R}_+)$; then $f \otimes v \in \mathfrak{e}(\Omega)$.

($\mathfrak{c}_3$) Let $(f/v) \in \mathtt{A}$ and $\psi \in \mathfrak{u}(\mathbb{R}_+)$; then we have that $(f \otimes \psi / v \circledast \psi) \in \mathtt{A}$ and

$$\frac{f}{v} \text{ is equivalent to } \frac{f \otimes \psi}{v \circledast \psi}. \tag{14}$$

($\mathfrak{c}_4$) Let $v \in \mathfrak{u}(\mathbb{R}_+)$; then $\mathtt{d}_r v \in \mathfrak{u}(\mathbb{R}_+)$, $r \geq 1$.

($\mathfrak{c}_5$) Let $(f/v) \in \mathtt{A}$, $w > 0$ and $g(\alpha, x) = f(\alpha + w, x)$; then we have

$$\frac{g}{\psi} \in \mathtt{A}, \qquad \frac{g}{\psi} \sim \frac{f}{v}, \tag{15}$$

where $\psi = \mathtt{d}_w v$.

Addition and scalar multiplications in $\pounds_{\mathrm{st}}(\mathfrak{e}, \mathfrak{u}, \otimes)$ are defined in the usual way as

$$\left[\frac{f}{v}\right] + \left[\frac{g}{\psi}\right] = \left[\frac{f \otimes \psi + g \otimes v}{v \circledast \psi}\right], \qquad \lambda \left[\frac{f}{v}\right] = \left[\frac{\lambda f}{v}\right]. \tag{16}$$

Differentiation and the operation $\otimes$ in the space in $\pounds_{\mathrm{st}}(\mathfrak{e}, \mathfrak{u}, \otimes)$ are given as

$$\mathscr{D}^k \left[\frac{f}{v}\right] = \left[\frac{\mathscr{D}^k f}{v}\right], \qquad \left[\frac{f}{v}\right] \otimes \psi = \left[\frac{f \otimes \psi}{v}\right]. \tag{17}$$

Convergence, in $\pounds_{\mathrm{st}}(\mathfrak{e}, \mathfrak{u}, \otimes)$, is defined as follows. A sequence (x_n) of strong Boehmians is said to be $\mathfrak{u}$ convergent to a Boehmian x in $\pounds_{\mathrm{st}}(\mathfrak{e}, \mathfrak{u}, \otimes)$ if for some $f_n, f \in \mathfrak{e}(\Omega)$, $n \in \mathbb{N}$, and $v \in \mathfrak{u}(\mathbb{R}_+)$ such that $x_n = [f_n/v]$, $x = [f/v]$, we have $f_n \to f$ as $n \to \infty$ in $\pounds_{\mathrm{st}}(\mathfrak{e}, \mathfrak{u}, \otimes)$.

3. General Boehmians

Boehmians were first constructed as a generalization of regular Mikusinski operators [16]. The minimal structure necessary for the construction of Boehmians consists of the following elements: (i) A set $\mathfrak{I}$; (ii) a commutative semigroup $(\mathfrak{R}, *)$; (iii) an operation $\odot : \mathfrak{I} \times \mathfrak{R} \to \mathfrak{I}$ such that for each $x \in \mathfrak{I}$ and v_1, v_2, a collection $\Delta \subset \mathfrak{R}$, $x \odot (v_1 * v_2) = (x \odot v_1) \odot v_2$; ($vi$) a collection $\Delta \subset \mathfrak{R}^{\mathbb{N}}$ such that (a) if $x, y \in \mathfrak{I}$, $(v_n) \in \Delta$, $x \odot v_n = y \odot v_n$ $\forall n$, then $x = y$; (b) if $(v_n), (\sigma_n) \in \Delta$, then $(v_n * \sigma_n) \in \Delta$. Δ is the set of all delta sequences. Consider

$$\mathscr{A} = \left\{ (x_n, v_n) : x_n \in \mathfrak{I}, (v_n) \in \Delta, x_n \odot v_m = x_m \odot v_n, \forall m, n \in \mathbb{N} \right\}. \tag{18}$$

If $(x_n, v_n), (y_n, \sigma_n) \in \mathscr{A}$, $x_n \odot \sigma_m = y_m \odot v_n$, $\forall m, n \in \mathbb{N}$, then we say $(x_n, v_n) \sim (y_n, \sigma_n)$. The relation $\sim$ is an equivalence relation in $\mathscr{A}$. The space of equivalence classes in $\mathscr{A}$ is denoted by $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$. Elements of $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$ are general Boehmians. Between $\mathfrak{I}$ and $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$ there is a canonical embedding expressed as $x \to (x \odot s_n)/s_n$. The operation $\odot$ can be extended to $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta) \times \mathfrak{I}$ by $x_n/v_n \odot t = (x_n \odot t)/v_n$. In $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$, there are two types of convergence. (1) A sequence (h_n) in $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$ is said to be δ convergent to h in $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$, denoted by $h_n \xrightarrow{\delta} h$, if there exists a delta sequence (v_n) such that $(h_n \odot v_n)$, $(h \odot v_n) \in \mathfrak{I}$, $\forall k, n \in \mathbb{N}$, and $(h_n \odot v_k) \to (h \odot v_k)$ as $n \to \infty$, in $\mathfrak{I}$, for every $k \in \mathbb{N}$. (2) A sequence (h_n) in $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$ is said to be Δ convergent to h in $\pounds_{\mathrm{ge}}(\mathfrak{I}, \mathfrak{R}, \Delta)$,

denoted by $h_n \xrightarrow{\Delta} h$, if there exists a $(v_n) \in \Delta$ such that $(h_n - h) \odot v_n \in \mathfrak{I}$, $\forall n \in \mathbb{N}$, and $(h_n - h) \odot v_n \to 0$ as $n \to \infty$ in $\mathfrak{I}$.

The following is equivalent for the statement of δ convergence: $h_n \xrightarrow{\delta} h$ $(n \to \infty)$ in $\pounds_{ge}(\mathfrak{I}, \mathfrak{R}, \Delta)$ if and only if there is $f_{n,k}, f_k \in \mathfrak{I}$ and $v_k \in \Delta$ such that $h_n = [f_{n,k}/v_k]$, $h = [f_k/v_k]$ and for each $k \in \mathbb{N}$, $f_{n,k} \to f_k$ as $n \to \infty$ in $\mathfrak{I}$.

For further discussion see [10–19].

4. Sumudu Transform of Strong Boehmians

We start investigation by establishing the following theorem.

Theorem 1 (the convolution theorem). *Let $f \in \mathfrak{e}(\Omega)$ and $v \in \mathfrak{u}(\mathbb{R}_+)$. Then one has*

$$\mu\left(f \otimes \mathtt{d}_\beta v\right)(\zeta) = \zeta \tilde{f}(\alpha, \zeta)\, \mathtt{d}_\beta \tilde{v}(\zeta), \tag{19}$$

where $\mathtt{d}_\beta \tilde{v}(\zeta) = \beta \tilde{v}(\beta\zeta)$, $\zeta \in \mathbb{R}_+$.

Proof. By using the definition of Sumudu transform, Fubini's theorem then gives

$$\begin{aligned} &\mu\left(f(\alpha, x) \otimes \mathtt{d}_\beta v(x)\right)(\zeta) \\ &\quad = \int_{\mathbb{R}_+} f(\alpha, t)\, dt \int_{\mathbb{R}_+} \mathtt{d}_\beta v(x - t) \frac{e^{-x/\zeta}}{\zeta} dx. \end{aligned} \tag{20}$$

The substitution $\beta x - \beta t = y$ implies $x = (y + \beta t)/\beta$ and, hence

$$\begin{aligned} &\mu\left(f(\alpha, x) \otimes \mathtt{d}_\beta v(x)\right)(\zeta) \\ &\quad = \int_{\mathbb{R}_+} f(\alpha, t)\, dt \int_{\mathbb{R}_+} \mathtt{d}_\beta v(y) \frac{e^{(-y+\beta t)/\beta\zeta}}{\beta\zeta} dy. \end{aligned} \tag{21}$$

Hence

$$\mu\left(f(\alpha, x) \otimes \mathtt{d}_\beta v(x)\right)(\zeta) = \zeta \tilde{f}(\alpha, \zeta)\, \mathtt{d}_\beta \tilde{v}(\zeta). \tag{22}$$

This completes the proof of the theorem. □

Next, we are describing a usual space of Boehmians by images of Sumudu transforms of strong Boehmians.

Consider the following definition.

Definition 2. Let $\Delta_1(\mathbb{R}_+)$, or simply Δ_1, be a set of delta sequences such that $(v_n) \in \mathfrak{u}(\mathbb{R}_+)$ and $\operatorname{supp} v_n \subset (0, \varepsilon_n)$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$.

Let $\mathfrak{n}(\mathbb{R}_+)$ be the set of images of Sumudu transforms of all $\mathfrak{u}(\mathbb{R}_+)$ elements and, $\Delta_2(\mathbb{R}_+)$ be the set of Sumudu transforms of all delta sequences from Δ_1.

For $f \in \mathfrak{e}(\Omega)$ and $\tilde{v} \in \mathfrak{n}(\mathbb{R}_+)$, we introduce the operation $\bullet$ as

$$\begin{aligned} f(\alpha, \zeta) \bullet \tilde{v}(\zeta) &= \zeta f(\alpha, \zeta)\, \mathtt{d}_\beta \tilde{v}(\zeta), \\ &\qquad \beta \in \mathbb{R}_{+1}. \end{aligned} \tag{23}$$

Lemma 3. *Let $f \in \mathfrak{e}(\Omega)$ and $\tilde{v} \in \mathfrak{n}(\mathbb{R}_+)$; then $f \bullet \tilde{v} \in \mathfrak{e}(\Omega)$, $\forall n \in \mathbb{N}$.*

Proof is immediate since $f \in \mathfrak{e}(\Omega)$ and $\mathtt{d}_\beta \tilde{v} \in \mathfrak{n}(\mathbb{R}_+)$.

Lemma 4. *The following hold true.*

(i) *If $(\tilde{v}_n), (\tilde{\psi}_n) \in \Delta_2(\mathbb{R}_+)$, then $\tilde{v}_n \bullet \tilde{\psi}_n \in \Delta_2(\mathbb{R}_+)$, $\forall n \in \mathbb{N}$.*

(ii) *Let $f, g \in \mathfrak{e}(\Omega)$ and $(\tilde{v}_n) \in \Delta_2(\mathbb{R}_+)$ be such that $f(\alpha, \zeta) \bullet \tilde{v}_n(\zeta) = g(\beta, \zeta) \bullet \tilde{v}_n(\zeta)$; then*

$$f(\alpha, \zeta) = g(\beta, \zeta), \quad \alpha, \beta \in \mathbb{R}_{+1}. \tag{24}$$

Proof. Proof of (i). For all $(\tilde{v}_n), (\tilde{\psi}_n) \in \Delta_2(\mathbb{R}_+)$ we can find their related sequences $(v_n), (\psi_n) \in \Delta_1(\mathbb{R}_+)$. Since $v_n \circledast \psi_n \in \Delta_1(\mathbb{R}_+)$, $\forall n$, it follows

$$\mu(v_n \circledast \psi_n) = \zeta \tilde{v}_n(\zeta)\, \tilde{\psi}_n(\zeta) = \tilde{v}_n \bullet \tilde{\psi}_n \in \Delta_2(\mathbb{R}_+), \tag{25}$$

$\forall n \in \mathbb{N}$, by (23).

This proves Part (i) of the lemma.

Proof of (ii). For each delta sequence $(\tilde{v}_n) \in \Delta_2(\mathbb{R}_+)$ we see that

$$\mathtt{d}_\beta \tilde{v}_n \longrightarrow \frac{1}{\zeta}, \quad \tilde{v}_n \longrightarrow \frac{1}{\zeta} \quad \text{as } n \longrightarrow \infty. \tag{26}$$

Hence

$$\begin{aligned} f(\alpha, \zeta) \bullet \tilde{v}_n(\zeta) &= \zeta f(\alpha, \zeta)\, \mathtt{d}_\beta \tilde{v}_n(\zeta) \longrightarrow f(\alpha, \zeta) \\ &\qquad \text{as } n \longrightarrow \infty, \\ g(\beta, \zeta) \bullet \tilde{v}_n(\zeta) &= \zeta g(\beta, \zeta)\, \mathtt{d}_\alpha \tilde{v}_n(\zeta) \longrightarrow g(\beta, \zeta) \\ &\qquad \text{as } n \longrightarrow \infty. \end{aligned} \tag{27}$$

This implies

$$\begin{aligned} f(\alpha, \zeta) &= g(\beta, \zeta), \\ \forall \zeta \in \mathbb{R}_+,\ \alpha, \beta &\in \mathbb{R}_{+1}. \end{aligned} \tag{28}$$

This completes the proof of the lemma. □

Hence, from Lemma 4, $(\Delta_2(\mathbb{R}_+), \circledast)$ can be regarded as delta sequence in $\mathfrak{n}(\mathbb{R}_+)$.

Lemma 5. *The mapping $\mathfrak{e}(\Omega) \times \mathfrak{n}(\mathbb{R}_+) \to \mathfrak{n}(\mathbb{R}_+)$ defined by*

$$f(\alpha, \zeta) \bullet \tilde{v}(\zeta) = \zeta f(\alpha, \zeta)\, \mathtt{d}_\beta \tilde{v}(\zeta). \tag{29}$$

Satisfies the following properties.

(i) *$\tilde{v}_n \bullet \tilde{\psi}_n = \tilde{\psi}_n \bullet \tilde{v}_n$, for every $(v_n), (\psi_n) \in \Delta_1$, $n \in \mathbb{N}$.*

(ii) *If $f, g \in \mathfrak{e}(\Omega)$ and $(\tilde{v}_n) \in \Delta_2(\mathbb{R}_+)$, then $(f + g) \bullet \tilde{v}_n = f \bullet \tilde{v}_n + g \bullet \tilde{v}_n$.*

(iii) *If $f \in \mathfrak{e}(\Omega)$, $(\tilde{v}_n), (\tilde{\psi}_n) \in \Delta_2(\mathbb{R}_+)$, then $(f \bullet \tilde{v}_n) \bullet \tilde{\psi}_n = f \bullet (\tilde{v}_n \bullet \tilde{\psi}_n)$.*

Proof is straightforward from definitions.

For similar proofs see [15] and other cited papers.

Lemma 6. *The following hold true.*

(i) *If $f_n \to f \in \mathfrak{e}(\Omega)$ and $\tilde{v} \in \mathfrak{n}(\mathbb{R}_+)$ then $f_n \bullet \tilde{v} \to f \bullet \tilde{v}$ as $n \to \infty$.*

(ii) *If* $f_n \to f \in \mathfrak{e}(\Omega)$ *and* $(\tilde{v}_n) \in \Delta_2(\mathbb{R}_+)$ *then* $f_n \bullet \tilde{v}_n \to f$ *as* $n \to \infty$.

Proof. Proof of (i). Let $\tilde{v} \in \mathfrak{n}(\mathbb{R}_+)$, $f_n, f \in \mathfrak{e}(\Omega)$, $\forall n$. The hypothesis of the lemma implies

$$\begin{aligned}&\left|\mathscr{D}_\zeta^k\left(f_n(\alpha,\zeta)\bullet\tilde{v}(\zeta)-f(\alpha,\zeta)\bullet\tilde{v}(\zeta)\right)\right|\\&\quad=\left|\mathscr{D}_\zeta^k\zeta\mathtt{d}_\beta\tilde{v}(\zeta)\left(f_n-f\right)(\alpha,\zeta)\right|\longrightarrow 0\end{aligned}\tag{30}$$

as $n \to \infty$ in $\mathfrak{e}(\Omega)$.

Proof of (ii). Let $(\tilde{v}_n) \in \Delta_2(\mathbb{R}_+)$; then the fact that $\mathtt{d}_\beta\tilde{v}_n(\zeta) \to 1/\zeta$ as $n \to \infty$ implies

$$\begin{aligned}&\left|\mathscr{D}_\zeta^k\left(f_n\bullet\tilde{v}_n(\zeta)-f(\alpha,\zeta)\right)\right|\\&\quad\longrightarrow\left|\mathscr{D}_\zeta^k\left(f_n(\alpha,\zeta)-f(\alpha,\zeta)\right)\right|\longrightarrow 0\end{aligned}\tag{31}$$

as $n \to \infty$ in $\mathfrak{e}(\Omega)$.

This completes the proof of the lemma. □

The general Boehmian space $\pounds_{\mathrm{ge}}(\mathfrak{e},\mathfrak{n},\Delta_2,\bullet)$, or $\pounds_{\mathrm{ge}}$, is, therefore, constructed.

The sum and multiplication by a scalar of two Boehmians in $\pounds_{\mathrm{ge}}$ are defined in a natural way as

$$\begin{aligned}&\left[\frac{f_n}{\tilde{v}_n}\right]+\left[\frac{g_n}{\tilde{\psi}_n}\right]=\left[\frac{f_n\bullet\tilde{\psi}_n+g_n\bullet\tilde{v}_n}{\tilde{v}_n\bullet\tilde{\psi}_n}\right],\\&\alpha\left[\frac{f_n}{\tilde{v}_n}\right]=\left[\frac{\alpha f_n}{\tilde{v}_n}\right],\quad\alpha\in\mathbb{C}.\end{aligned}\tag{32}$$

The operation $\bullet$ and differentiation in $\pounds_{\mathrm{ge}}$ are, respectively, defined by

$$\left[\frac{f_n}{\tilde{v}_n}\right]\bullet\left[\frac{g_n}{\tilde{\psi}_n}\right]=\left[\frac{f_n\bullet g_n}{\tilde{v}_n\bullet\tilde{\psi}_n}\right],\qquad\mathscr{D}^\alpha\left[\frac{f_n}{\tilde{v}_n}\right]=\left[\frac{\mathscr{D}^\alpha f_n}{\tilde{v}_n}\right].\tag{33}$$

On the other hand, we are concerned with the strong space of Boehmians that can be described by the set $(\mathfrak{e},\otimes)$ and the subset $(\mathfrak{u},\circledast)$, injected with the family Δ_1 of delta sequences. Such a space is denoted by $\pounds_{\mathrm{st}}$ $(\mathfrak{e},(\mathfrak{u},\circledast),\Delta_1,\otimes)$, or, simply, by $\pounds_{\mathrm{st}}$, for more convenience. With this injection, the space $\pounds_{\mathrm{st}}$ preserves the operations of addition, scalar multiplication, differentiation, and the convolution given to general Boehmians. Theorem 1 then suggests the following definition.

Definition 7. Let $f \in \mathfrak{e}(\Omega)$ and $v \in \mathfrak{u}(\mathbb{R}_+)$; then we define the Sumudu transform of the strong Boehmian $[f_n/v_n]$ in $\pounds_{\mathrm{st}}$ by

$$\eth\left[\frac{f_n}{v_n}\right]=\left[\frac{\widetilde{f_n}}{\mathtt{d}_\alpha\tilde{v}_n}\right]\in\pounds_{\mathrm{ge}},\tag{34}$$

where $\mathtt{d}_\alpha$ has its usual meaning.

Theorem 8. *The Sumudu transform* $\eth : \pounds_{st} \to \pounds_{ge}$ *is well-defined.*

Proof. Let $[f_n/v_n],[g_n/\psi_n] \in \pounds_{\mathrm{st}}$ be such that $[f_n/v_n] = [g_n/\psi_n]$; then using (12) we get

$$f_n(\alpha,x)\otimes\mathtt{d}_\beta\psi_n(x)=g_n(\beta,x)\otimes\mathtt{d}_\alpha v_n(x).\tag{35}$$

Applying the convolution theorem on both sides of (35) yields

$$\frac{\tilde{g}_n}{\mathtt{d}_\beta\tilde{\psi}_n}=\frac{\widetilde{f_n}}{\mathtt{d}_\alpha\tilde{v}_n}.\tag{36}$$

Hence

$$\eth\left[\frac{f_n}{v_n}\right]=\eth\left[\frac{g_n}{\psi_n}\right].\tag{37}$$

This completes the proof of the theorem. □

Theorem 9. *Let* $(\psi_n),(v_n) \in \Delta_1(\mathbb{R}_+)$ *and* $f,g \in \mathfrak{e}(\Omega)$*; then the mapping* $\eth : \pounds_{st} \to \pounds_{ge}$ *is one-one.*

Proof. Assume $\eth[f_n/v_n] = \eth[g_n/\psi_n]$ in $\pounds_{\mathrm{ge}}$. Using (34) we get

$$\widetilde{f_n}(\alpha,\zeta)\bullet\tilde{\psi}_n(\zeta)=\tilde{g}_n(\beta,\zeta)\bullet\tilde{v}_n(\zeta).\tag{38}$$

By (23) we get

$$\zeta\widetilde{f_n}(\alpha,\zeta)\,\mathtt{d}_\beta\tilde{\psi}_n(\zeta)=\zeta\tilde{g}_n(\beta,\zeta)\,\mathtt{d}_\alpha\tilde{v}_n(\zeta).\tag{39}$$

Using Theorem 1, (39) becomes

$$\mu\left(f_n(\alpha,x)\otimes\mathtt{d}_\beta\psi_n(x)\right)=\mu\left(g_n(\beta,x)\otimes\mathtt{d}_\alpha v_n(x)\right).\tag{40}$$

Since μ is one-one we get

$$f_n(\alpha,x)\otimes\mathtt{d}_\beta\psi_n(x)=g_n(\beta,x)\otimes\mathtt{d}_\alpha v_n(x).\tag{41}$$

The property of equivalence classes in $\pounds_{\mathrm{st}}$, therefore, implies

$$\frac{f_n(\alpha,x)}{v_n(x)}\sim\frac{g_n(\beta,x)}{\psi_n(x)}.\tag{42}$$

Hence $[f_n/v_n] = [g_n/\psi_n]$.

This completes the proof of the theorem □

Theorem 10. $\eth : \pounds_{st} \to \pounds_{ge}$ *is continuous with respect to* $\mathfrak{u}$ *convergence.*

Proof. Let $x_n \to x \in \pounds_{\mathrm{st}}$. By using $\mathfrak{u}$ convergence in $\pounds_{\mathrm{st}}$ and [9, Theorem 2.6] we can find common v for all x_n such that $x_n = [f_n/v]$, $x = [f/v]$ and $f_n \to f$ as $n \to \infty$. Hence $\widetilde{f_n} \to \tilde{f}$ as $n \to \infty$. Therefore

$$\frac{\widetilde{f_n}}{\mathtt{d}_\alpha\tilde{v}}\longrightarrow\frac{\tilde{f}}{\mathtt{d}_\alpha\tilde{v}}\quad\text{as } n\longrightarrow\infty.\tag{43}$$

Hence, $\eth x_n \to \eth x$ as $n \to \infty$ in $\pounds_{\mathrm{ge}}$.

This completes the proof of the theorem. □

Definition 11. Let $y = [\widetilde{f_n}/\mathtt{d}_\alpha \widetilde{\upsilon}_n]$ in $\pounds_{\mathrm{ge}}$; then we define the inverse $\eth^{-1}$ of $\eth$ by the formula

$$\eth^{-1} y = \left[\frac{f_n}{\upsilon_n}\right], \tag{44}$$

in $\pounds_{\mathrm{st}}$.

Theorem 12. *The mapping $\eth^{-1} : \pounds_{ge} \to \pounds_{st}$ is well-defined.*

Proof. Let $[\widetilde{f_n}/\mathtt{d}_\alpha \widetilde{\upsilon}_n] = [\widetilde{g}_n/\mathtt{d}_\beta \widetilde{\psi}_n]$; then $\widetilde{f_n}(\alpha,\zeta) \bullet \widetilde{\psi}_m(\zeta) = \widetilde{g_m}(\beta,\zeta) \bullet \widetilde{\upsilon}_n(\zeta)$. Using (23) we get

$$\zeta \widetilde{f_n}(\alpha,\zeta)\, \mathtt{d}_\beta \widetilde{\psi}_m(\zeta) = \zeta \widetilde{g_m}(\beta,\zeta)\, \mathtt{d}_\alpha \widetilde{\upsilon}_n(\zeta). \tag{45}$$

Theorem 1 implies $\mu(f_n(\alpha,x) \otimes \mathtt{d}_\beta \psi m_n(x)) = \mu(g_m(\beta,x) \otimes \mathtt{d}_\alpha \upsilon_n(x))$. Hence

$$f_n(\alpha,x) \otimes \mathtt{d}_\beta \psi_m(x) = g_m(\beta,x) \otimes \mathtt{d}_\alpha \upsilon_n(x). \tag{46}$$

This completes the proof of the Theorem. □

Theorem 13. *The mapping $\eth^{-1} : \pounds_{ge} \to \pounds_{st}$ is linear.*

Proof. Let $[\widetilde{f_n}/\mathtt{d}_\alpha \widetilde{\upsilon}_n], [\widetilde{g}_n/\mathtt{d}_\beta \widetilde{\psi}_n] \in \pounds_{\mathrm{ge}}$ and $k \in \mathbb{R}_+$; then using (23) we write

$$\begin{aligned}
&\eth^{-1}\left[\frac{\widetilde{f_n}}{\mathtt{d}_\alpha \widetilde{\upsilon}_n}\right] + \left[\frac{\widetilde{g}_n}{\mathtt{d}_\beta \widetilde{\psi}_n}\right] \\
&\quad = \eth^{-1}\left[\frac{\zeta \widetilde{f_n}(\alpha,\zeta)\, \mathtt{d}_\beta \widetilde{\psi}_n(\zeta) + \zeta \widetilde{g}_n(\beta,\zeta)\, \mathtt{d}_\alpha \widetilde{\upsilon}_n(\zeta)}{\mathtt{d}_\alpha \widetilde{\upsilon}_n(\zeta) \circledast \mathtt{d}_\beta \widetilde{\psi}_n(\zeta)}\right] \\
&\quad = \left[\frac{f_n(\alpha,x) \otimes \mathtt{d}_\beta \psi_n(x) + g_n(\beta,x) \otimes \mathtt{d}_\alpha \upsilon_n(x)}{\mathtt{d}_\alpha \upsilon_n(x) \circledast \mathtt{d}_\beta \psi_n(x)}\right] \\
&\quad = \left[\frac{f_n}{\upsilon_n}\right] + \left[\frac{g_n}{\psi_n}\right].
\end{aligned} \tag{47}$$

Also, $k\eth^{-1}([\widetilde{f_n}/\mathtt{d}_\alpha \widetilde{\upsilon}_n]) = \eth^{-1}([k\widetilde{f_n}/\mathtt{d}_\alpha \widetilde{\upsilon}_n])$ is obvious.

The proof is completed. □

Theorem 14. *The mapping $\eth^{-1} : \pounds_{ge} \to \pounds_{st}$ is continuous with respect to δ convergence.*

Proof. Let $y_n \to y \in \pounds_{\mathrm{ge}}$, $y_n = [\widetilde{f_{n,k}}/\mathtt{d}_\alpha \widetilde{\upsilon}_k]$, $y = [\widetilde{f_k}/\mathtt{d}_\alpha \widetilde{\upsilon}_k]$, and $\widetilde{f_{n,k}} \to \widetilde{f_k}$ as $n \to \infty$. Applying the inverse Sumudu transform yields $f_{n,k} \to f_k$ as $n \to \infty$. Thus $f_n/\upsilon_n \to f/\upsilon_n$ as $n \to \infty$.

This completes the proof of the theorem. □

If the family Δ_1 of delta sequences is not described in the strong Boehmian space we have considered in this note, it will be interesting to pay attention to the following remarks.

Useful Remark. (i) The injection $\mathfrak{i} : (\mathfrak{e}, \mathfrak{u}, \otimes) \to \pounds_{\mathrm{st}}$ defined by

$$\mathfrak{i}(f) = \left[\frac{f \otimes \mathtt{d}_\alpha \upsilon}{\upsilon}\right], \tag{48}$$

is continuous and independent of the choice of υ.

(ii) The injection $\mathfrak{j} : \pounds_{\mathrm{st}} \to \pounds_{\mathrm{ge}}$ defined by

$$\mathfrak{j}\left[\frac{f}{\upsilon}\right] = \left[\frac{f(n,\cdot)}{\mathtt{d}_n \upsilon}\right], \tag{49}$$

is continuous.

Proof. Proof of (i) is same as that of [9, Theorem 3.1].

Detailed proof of (ii) is as follows.

Suppose $y_n = [f_n/\upsilon]$, $y = [f/\upsilon]$, and $f_n(n,\cdot) \to f(n,\cdot)$ as $n \to \infty$ uniformly on compact subsets of Ω. Then

$$\begin{aligned}
\left[\frac{f_n}{\upsilon}\right] \otimes \mathtt{d}_k \upsilon &= \left[\frac{f_n(k,\cdot)}{\mathtt{d}_k \upsilon}\right] \\
&= \left[\frac{f_n(k,\cdot) \otimes \mathtt{d}_k \upsilon}{\mathtt{d}_k \upsilon}\right] \sim f_n(k,\cdot) \longrightarrow f(k,\cdot),
\end{aligned} \tag{50}$$

as $n \to \infty$.

Employing (i) we get

$$\begin{aligned}
\left[\frac{f_n}{\upsilon}\right] \otimes \mathtt{d}_k \upsilon \longrightarrow f(k,\cdot) &= \left[\frac{f(k,\cdot) \otimes \mathtt{d}_k \upsilon}{\upsilon}\right] \\
&= \left[\frac{f(k,\cdot)}{\upsilon}\right] \otimes \mathtt{d}_k \upsilon.
\end{aligned} \tag{51}$$

That is

$$\left[\frac{f_n}{\upsilon}\right] \otimes \mathtt{d}_k \upsilon \longrightarrow y \otimes \mathtt{d}_k \upsilon, \tag{52}$$

as $n \to \infty$. Therefore

$$y_n \otimes \mathtt{d}_k \upsilon \longrightarrow y \otimes \mathtt{d}_k \upsilon, \tag{53}$$

as $n \to \infty$.

This completes the proof of the remark. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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