Research Article

Second Order Ideal-Ward Continuity

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The main aim of the paper is to introduce a concept of second order ideal-ward continuity in the sense that a function f is second order ideal-ward continuous if $I - \lim_{n \rightarrow \infty} \Delta^2 f(x_n) = 0$ whenever $I - \lim_{n \rightarrow \infty} \Delta^2 x_n = 0$ and a concept of second order ideal-ward compactness in the sense that a subset E of $\mathbf{R}$ is second order ideal-ward compact if any sequence $x = (x_n)$ of points in E has a subsequence $z = (z_k) = (x_{n_k})$ of the sequence x such that $I - \lim_{k \rightarrow \infty} \Delta^2 z_k = 0$ where $\Delta^2 z_k = z_{k+2} - 2z_{k+1} + z_k$. We investigate the impact of changing the definition of convergence of sequences on the structure of ideal-ward continuity in the sense of second order ideal-ward continuity and compactness of sets in the sense of second order ideal-ward compactness and prove related theorems.

1. Introduction

Let us start with basic definitions from the literature. Let $K \subseteq \mathbb{N}$, the set of all natural numbers, and $K_n = \{k \leq n : k \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set.

Fast [1] presented the following definition of statistical convergence for sequences of real numbers. The sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\epsilon > 0$, the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero; that is, for each $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \{j \leq n : |x_j - L| \geq \epsilon\} \right| = 0. \quad (1)$$

In this case, we write $S - \lim x = L$ or $x_k \rightarrow L(S)$ and S denotes the set of all statistically convergent sequences. Note that every convergent sequence is statistically convergent but not conversely.

Some basic properties related to the concept of statistical convergence were studied in [2, 3]. In 1985, Fridy [4] presented the notion of statistically Cauchy sequence and determined that it is equivalent to statistical convergence. Caserta et al. [5] studied statistical convergence in function spaces, while Caserta and Kočinac [6] investigated statistical exhaustiveness.

Kostyrko et al. [7] introduced the notion of ideal convergence. It is a generalization of statistical convergence. For details on ideal convergence we refer to [8–13].

Let X be a nonempty set; then a family of sets $I \subset P(X)$ (power sets of X) is called an *ideal* on X if and only if

- (a) $\phi \in I$,
- (b) for each $A, B \in I$, we have $A \cup B \in I$,
- (c) for each $A \in I$ and each $B \subset A$, we have $B \in I$.

A nonempty family of sets $F \subset P(X)$ is a *filter* on X if and only if

- (a) $\phi \notin F$,
- (b) for each $A, B \in F$, we have $A \cap B \in F$,
- (c) each $A \in F$ and each $B \supset A$, we have $B \in F$.

An ideal I is called *nontrivial* ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset P(X)$ is a nontrivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X . A nontrivial ideal $I \subset P(X)$ is called *admissible* if and only if $\{\{x\} : x \in X\} \subset I$. A nontrivial ideal I is maximal if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset.

Definition 1 (see [7]). A sequence $x = (x_n)$ of points in $\mathbf{R}$ is said to be I -convergent to the number ℓ if, for every $\epsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \epsilon\} \in I$. One writes $I - \lim x_n = \ell$. One

sees that a sequence $x = (x_n)$ being I -convergent implies that $I - \lim_{n \rightarrow \infty} \Delta x_n = 0$.

Burton and Coleman [14] introduced the concept of quasi-Cauchy sequences as a sequence (x_n) of points of $\mathbf{R}$ is said to be a quasi-Cauchy sequence if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. Çakallı and Hazarika [15] introduced the concept of ideal quasi-Cauchy sequences. Recall from [15] that a sequence (x_n) of points of $\mathbf{R}$ is called ideal quasi-Cauchy if $I - \lim_{n \rightarrow \infty} \Delta x_n = 0$.

We say that a sequence $x = (x_n)$ is ward convergent to a number ℓ if $\lim_{n \rightarrow \infty} \Delta x_n = \ell$ where $\Delta x_n = x_{n+1} - x_n$. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, Çakallı [16] introduced the concept of ward continuity in the sense that a function f is ward continuous if it transforms ward convergent to 0 sequences to ward convergent to 0 sequences; that is, $(f(x_n))$ is ward convergent to 0 whenever (x_n) is ward convergent to 0, and Çakallı [17] introduced the concept of ward compactness in the sense that a subset E of $\mathbf{R}$ is ward compact if any sequence $x = (x_n)$ of points in E has a subsequence $z = (z_k) = (x_{n_k})$ of the sequence x such that $\lim_{k \rightarrow \infty} \Delta z_k = 0$ where $\Delta z_k = z_{k+1} - z_k$. Throughout the paper c , S , I , and ΔI will denote the set of all convergent sequences, statistically convergent sequences, I -convergent sequences, and the set of all I -ward convergent to 0 sequences of points in $\mathbf{R}$ where a sequence $x = (x_n)$ is called I -ward convergent to 0 if $I - \lim_{n \rightarrow \infty} \Delta x_n = 0$.

Throughout the paper we assume I is a nontrivial admissible ideal of $\mathbf{N}$.

2. Second Order Ideal-Ward Continuity

We introduce the notion of second order ward convergent sequences as follows.

Definition 2. A sequence $x = (x_n)$ is said to be second order ward convergent to a number ℓ if $\lim_{n \rightarrow \infty} \Delta^2 x_n = \ell$ where $\Delta^2 x_n = x_{n+2} - 2x_{n+1} + x_n$. For the special case $\ell = 0$, x is called second order ward convergent to 0.

We note that any ward convergent to 0 sequence is also second order ward convergent to 0, but the opposite is not always true as it can be considering the sequence (n) .

Definition 3. A sequence $x = (x_n)$ is said to be second order ideal-ward convergent to a number ℓ if $I - \lim_{n \rightarrow \infty} \Delta^2 x_n = \ell$ where $\Delta^2 x_n = x_{n+2} - 2x_{n+1} + x_n$. For the special case $\ell = 0$, x is called second order ideal-ward convergent to 0. One denotes by $\Delta^2 I$ the set of all second order ideal-ward convergent sequences.

Now we give the definition of second order ideal-ward continuous function on a subset of $\mathbf{R}$.

Definition 4. A function f is called ward continuous on E if the sequence $(f(x_n))$ is ward convergent to 0 whenever $x = (x_n)$ is a ward convergent to 0 sequence of terms in E .

Definition 5. A function f is called second order ideal-ward continuous on E if $I - \lim_{n \rightarrow \infty} \Delta^2 f(x_n) = 0$ whenever $I - \lim_{n \rightarrow \infty} \Delta^2 x_n = 0$, for a sequence $x = (x_n)$ of terms in E .

Theorem 6. If f is second order ideal-ward continuous on a subset E of $\mathbf{R}$, then it is an ideal-ward continuous on E .

Proof. Suppose that f is a second order ideal-ward continuous function on a subset E of $\mathbf{R}$. Let (x_n) be a sequence with $I - \lim_{n \rightarrow \infty} \Delta x_n = 0$. Then we have the sequence

$$(x_1, x_1, x_2, x_2, x_3, x_3, \dots, x_{n-1}, x_{n-1}, x_n, x_n, \dots) \quad (2)$$

such that $I - \lim_{n \rightarrow \infty} \Delta^2 x_n = 0$. Since f is second order ideal-ward continuous, then we get the sequence

$$(y_n) = (f(x_1), f(x_1), f(x_2), f(x_2), \dots, f(x_n), f(x_n), \dots) \quad (3)$$

such that $I - \lim_{n \rightarrow \infty} \Delta^2 y_n = 0$. It follows that

$$(f(x_1), f(x_1), f(x_2), f(x_2), \dots, f(x_n), f(x_n), \dots) \quad (4)$$

is second order ideal-ward convergent to 0. This completes the proof of the theorem. $\square$

The converse is not always true for this we consider the function $f(x) = \sin x$ which is ideal-ward continuous but not second order ideal-ward continuous.

Corollary 7. If f is second order ward continuous, then it is an ideal continuous.

Theorem 8. If f is second order ideal-ward continuous, then it is second order ward continuous.

Proof. The proof is easy, so omitted. $\square$

Definition 9. A subset E of $\mathbf{R}$ is called second order ward compact if $x = (x_n)$ is a sequence of points in E and there is a subsequence $z = (z_k) = (x_{n_k})$ of x such that $\lim_{k \rightarrow \infty} \Delta^2 z_k = 0$.

Now we give the definition of second order ideal-ward compactness of a subset of $\mathbf{R}$.

Definition 10. A subset E of $\mathbf{R}$ is called second order ideal-ward compact if $x = (x_n)$ is a sequence of points in E and there is a subsequence $z = (z_k) = (x_{n_k})$ of x such that $I - \lim_{k \rightarrow \infty} \Delta^2 z_k = 0$.

Theorem 11. A second order ideal-ward continuous image of any second order ideal-ward compact subset of $\mathbf{R}$ is second order ideal-ward compact.

Proof. Suppose that f is a second order ideal-ward continuous function on a subset E of $\mathbf{R}$ and E is a second order ideal-ward compact subset of $\mathbf{R}$. Let (y_n) be a sequence of points in $f(E)$. Write $y_n = f(x_n)$ where $x_n \in E$ for each

$n \in \mathbf{N}$. A second order ideal-ward compactness of E implies that there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n) with $I - \lim_{k \rightarrow \infty} \Delta^2 z_k = 0$. Write $(t_k) = (f(z_k))$. Since f is second order ideal-ward continuous, so we have $I - \lim_{k \rightarrow \infty} \Delta^2 f(z_k) = 0$. Thus we have obtained a subsequence (t_k) of the sequence $(f(x_n))$ with $I - \lim_{k \rightarrow \infty} \Delta^2 t_k = 0$. Thus $f(E)$ is second order ideal-ward compact. This completes the proof of the theorem. $\square$

Corollary 12. *A second order ideal-ward continuous image of any compact subset of $\mathbf{R}$ is compact.*

Proof. The proof of this theorem follows from the preceding theorem. $\square$

Theorem 13. *If (f_n) is a sequence of second order ideal-ward continuous functions defined on a subset E of $\mathbf{R}$ and (f_n) is uniformly convergent to a function f , then f is second order ideal-ward continuous on E .*

Proof. Let $\varepsilon > 0$ and (x_n) be a sequence of points in E such that $I - \lim_{n \rightarrow \infty} \Delta^2 x_n = 0$. By the uniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \varepsilon/8$ for all $x \in E$ whenever $n \geq N$. Since f_N is second order ideal-ward continuous on E , then we have

$$\left\{ n \in \mathbf{N} : |f_N(x_{n+2}) - 2f_N(x_{n+1}) + f_N(x_n)| \geq \frac{\varepsilon}{8} \right\} \in I. \quad (5)$$

But

$$\begin{aligned} & \{n \in \mathbf{N} : |f(x_{n+2}) - 2f(x_{n+1}) + f(x_n)| \geq \varepsilon\} \\ & \subseteq \left\{ \left\{ n \in \mathbf{N} : |f(x_{n+2}) - f_N(x_{n+2})| \geq \frac{\varepsilon}{8} \right\} \right. \\ & \quad \cup \left\{ n \in \mathbf{N} : |f_N(x_{n+1}) - f(x_{n+1})| \geq \frac{\varepsilon}{8} \right\} \\ & \quad \cup \left\{ n \in \mathbf{N} : |f_N(x_n) - f(x_n)| \geq \frac{\varepsilon}{8} \right\} \\ & \quad \left. \cup \left\{ n \in \mathbf{N} : |f_N(x_{n+2}) - 2f_N(x_{n+1}) + f_N(x_n)| \geq \frac{\varepsilon}{8} \right\} \right\}. \end{aligned} \quad (6)$$

Since I is an admissible ideal, the right-hand side of relation (6) belongs to I , and we have

$$\{n \in \mathbf{N} : |f(x_{n+2}) - 2f(x_{n+1}) + f(x_n)| \geq \varepsilon\} \in I. \quad (7)$$

This completes the proof of the theorem. $\square$

Theorem 14. *The set of all second order ideal-ward continuous functions on a subset E of $\mathbf{R}$ is a closed subset of the set of all continuous functions on E ; that is, $\overline{\Delta^2 iwc(E)} = \Delta^2 iwc(E)$ where $\Delta^2 iwc(E)$ is the set of all second order ideal-ward continuous functions on E , $\overline{\Delta^2 iwc(E)}$ denotes the set of all cluster points of $\Delta^2 iwc(E)$.*

Proof. Let f be an element in $\overline{\Delta^2 iwc(E)}$. Then there exists sequence (f_n) of points in $\Delta^2 iwc(E)$ such that $\lim_{n \rightarrow \infty} f_n = f$.

To show that f is second order ideal-ward continuous, consider a sequence (x_n) of points in E such that $I - \lim_{n \rightarrow \infty} \Delta^2 x_n = 0$. Since (f_n) converges to f , there exists a positive integer N such that, for all $x \in E$ and for all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon/8$. Since f_N is second order ideal-ward continuous on E we have

$$\left\{ n \in \mathbf{N} : |f_N(x_{n+2}) - 2f_N(x_{n+1}) + f_N(x_n)| \geq \frac{\varepsilon}{4} \right\} \in I. \quad (8)$$

But

$$\begin{aligned} & \{n \in \mathbf{N} : |f(x_{n+2}) - 2f(x_{n+1}) + f(x_n)| \geq \varepsilon\} \\ & \subseteq \left\{ \left\{ n \in \mathbf{N} : |f(x_{n+2}) - f_N(x_{n+2})| \geq \frac{\varepsilon}{8} \right\} \right. \\ & \quad \cup \left\{ n \in \mathbf{N} : |f_N(x_{n+1}) - f(x_{n+1})| \geq \frac{\varepsilon}{8} \right\} \\ & \quad \cup \left\{ n \in \mathbf{N} : |f(x_n) - f_N(x_n)| \geq \frac{\varepsilon}{8} \right\} \\ & \quad \left. \cup \left\{ n \in \mathbf{N} : |f_N(x_{n+2}) - 2f_N(x_{n+1}) + f_N(x_n)| \geq \frac{\varepsilon}{8} \right\} \right\}. \end{aligned} \quad (9)$$

Since I is an admissible ideal, the right-hand side of the relation (9) belongs to I , and we have

$$\{n \in \mathbf{N} : |f(x_{n+2}) - 2f(x_{n+1}) + f(x_n)| \geq \varepsilon\} \in I. \quad (10)$$

This completes the proof of the theorem. $\square$

Corollary 15. *The set of all second order ideal-ward continuous functions on a subset E of $\mathbf{R}$ is a complete subspace of the space of all continuous functions on E .*

Proof. The proof follows from the preceding theorem. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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