Research Article

On the Barycentric Labeling of Certain Graphs

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1. Introduction

Let $G = (V, E)$ be a finite, simple, and undirected graph. Labeling for a graph is a map that takes graph elements to numbers (usually positive or nonnegative integers). Let $G$ be an abelian group (written additively). The graph $G$ is called $A$-magic if there exists labeling $l : E(G) \rightarrow A \setminus \{0\}$ such that the induced vertex set labeling $l^* : V(G) \rightarrow A$, defined by $l^*(v) = \sum_{uv \in E(G)} l(uv)$, where the sum is over all edges in $E(G)$, is a constant map. A graph $G$ is $A$-barycentric-magic (or has $A$-barycentric labeling) if $G$ is $A$-magic and also satisfies $l^*(v) = \deg(v)(u, v)$ for all $v \in V$ and for some vertex $u$, adjacent to $v$. In this paper we consider some graphs $G$ and characterize all $m \in \mathbb{N}$ for which $G$ is $\mathbb{Z}_m$-barycentric-magic.

Theorem 1. A graph $G$ is $\mathbb{Z}_2$-magic if and only if every vertex of $G$ is of the same parity.

Theorem 2. An Eulerian graph $G$ with even size is $A$-magic.

Theorem 3. If $A_1$ is a subgroup of $A$ and graph $G$ is $A_1$-magic, then $G$ is $A$-magic.

Various authors have introduced labeling that generalizes the idea of magic square. Kotzig and Rosa [3] defined a magic labeling to be total labeling on the vertices and edges in which the labels are the integers from 1 to $|V(G)| + |E(G)|$. The sum of labels on an edge and its two endpoints is constant. In 1996 Ringel and Llado [4] redefined this type of labeling as edge-magic. Also, Enomoto et al. [5] have introduced the name super edge-magic for magic labeling in the sense of Kotzig and Rosa, with the added property that the $n$ vertices receive the smaller labels, $\{1, 2, \ldots, n\}$. Lee et al. [6] defined the concept of $k$-edge magic graphs and studied it for certain graphs (see, e.g., [7]). Recently authors in [8] defined a new kind of group magicness graphs.

Here we recall the following definition.

Definition 4 (see [8]). If there exists labeling $l$ for a graph $G$, whose induced vertex set labeling is a constant map and for all $v \in V(G)$ the sum $l^*(v)$ also satisfies $l^*(v) = \deg(v)(u, v)$ for some vertex $u$, adjacent to $v$, $G$ is said to be $A$-barycentric-magic.

Note that the motivation of Definition 4 is the following definition of $k$-barycentric sequence which was introduced in [9] and has already been used in graph labeling problems, specially in Ramsey theory [9–11].

Definition 5. Let $x_1, x_2, \ldots, x_k$ be $k$ elements of an abelian group $A$. This sequence is $k$-barycentric if there exists $j$ such that $x_1 + x_2 + \cdots + x_j + \cdots + x_k = kx_j$. The element $x_j$ is called a barycenter.
Similar to the definition of integer magic spectrum of $A$-magic graphs we state the definition of barycenter-magic spectrum of graph.

**Definition 7** (see [8]). For a given graph $G$ the set of all positive integers $m$ for which $G$ is $\mathbb{Z}_m$-barycentric-magic is called the barycenter-magic spectrum of $G$ and is denoted by $BMS(G)$.

In this paper, we consider specific graphs $G$ and characterize all $m \in \mathbb{N}$ for which $G$ is $\mathbb{Z}_m$-barycentric-magic.

### 2. Barycentric-Magic Labeling of Certain Graphs

In this section, we characterize all $m \in \mathbb{N}$ for which $G$ is $\mathbb{Z}_m$-barycentric-magic. First we consider some complete bipartite graphs.

First we state the following theorem.

**Theorem 8** (see [8]). The complete bipartite graph $K_{2,3}$ is not $\mathbb{Z}_m$-barycentric-magic for any $m$.

We generalize the previous theorem.

**Theorem 9.** For $n \geq 3$, $K_{2,n}$ is $\mathbb{Z}_m$-barycentric-magic if and only if $gcd(n - 2, m) \neq 1$.

*Proof.* Let $V(K_{2,n}) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of $K_{2,n}$. For each $j$, the edges incidents to $v_j$ must have the same label. Suppose that $l(u_1, v_j) = \alpha_j$ and $l(u_2, v_j) = \alpha_j$. Then $2\alpha_1 \equiv 2\alpha_2 \equiv \cdots \equiv 2\alpha_n \pmod{m}$. We consider the two following cases.

**Case 1** ($m$ is odd). In this case, the condition $2\alpha_1 \equiv 2\alpha_2 \equiv \cdots \equiv 2\alpha_n \pmod{m}$ implies that all edges have the same label, say $\alpha$. By the condition $l(u_i, v_j) = \alpha_j$ we have $(n - 2)\alpha \equiv 0 \pmod{m}$ and this is impossible when $gcd(n - 2, m) = 1$.

If $gcd(n - 2, m) = d \neq 1$, then using $\alpha = m/d$ gives a barycentric-magic labeling.

**Case 2** ($m$ is even). Here we give two different approaches. In this case, the condition $2\alpha_1 \equiv 2\alpha_2 \equiv \cdots \equiv 2\alpha_n \pmod{m}$ implies that there are at most two different labels $\alpha$ and $\beta = \alpha + m/2$, such that $2\alpha \equiv 2\beta \pmod{m}$. Now label $K_{2,n}$ as follows: $l(u_1, v_j) = \alpha$ for $1 \leq j \leq k$ and $l(u_1, v_j) = \beta$ for $k + 1 \leq j \leq n$, for some $1 \leq k \leq n$. Then, the edges incidents to $u_2$ must be labeled in the same way. This labeling is barycentric-magic if and only if

$$k\alpha + (n - k)\beta \equiv n\alpha \equiv 2\alpha \equiv 2\beta \pmod{m} \quad (1)$$

or

$$k\alpha + (n - k)\beta \equiv n\beta \equiv 2\alpha \equiv 2\beta \pmod{m} \quad (2)$$

Without loss of generality, we consider only the first relation. The condition $n\alpha \equiv 2\alpha \pmod{m}$ is satisfied only when $gcd(n - 2, m) \neq 1$. So suppose that $gcd(n - 2, m) = d \neq 1$. Choose
Figure 3: Friendship graphs $F_2, F_3, F_4$, and $F_n$, respectively.

Figure 4: Graph $H_{2n}$.

$k = d$, $\alpha = m/d$, and $\beta = \alpha + m/2$ and since $n - d$ is even we get

$$ka + (n - k)\beta \equiv na \pmod{m},$$

$$na \equiv 2\beta \equiv 2\alpha \pmod{m}.$$ (3)

Therefore with this labeling $K_{2n}$ is $\mathbb{Z}_m$-barycentric-magic.

Now we state the second approach.

Since $na = 2\alpha \pmod{m}$, if gcd($m, n - 2$) = 1, then $\alpha = 0$, a contradiction. Conversely, let $\alpha_1 = \alpha_2 = \cdots = \alpha_k = m/d$. This is also a $\mathbb{Z}_m$-barycentric-magic labeling for $K_{2n}$. \(\square\)

Example 10. Consider the graph $K_{2,10}$ and the group $\mathbb{Z}_{10}$. Let $V(K_{2,10}) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_{10}\}$ be the set of vertices of $K_{2,10}$. In this case, since $d = \gcd(n - 2, m) = 2$ choose $\alpha = m/d = 5$ and $\beta = \alpha + m/2 = 10$ and $k = d = 2$. The labeling of $K_{2,10}$ is as follows:

$$l(u_i, v_j) = 5 \text{ for } i = 1, 2 \text{ and } j = 1, 2;$$

$$l(u_1, v_j) = 10 \text{ for } i = 1, 2 \text{ and } j = 3, \ldots, 10;$$

then $l'(v_j) = 2 \times 5 = \deg(v_j) \times 5 \equiv 10 \pmod{10}$ for $j = 1, 2$;

$$l'(v_j) = 2 \times 10 = \deg(v_j) \times 10 \equiv 10 \pmod{10} \text{ for } j = 3, \ldots, 10;$$

Then with this labeling we get barycentric-magic labeling.

Here we consider friendship graphs.

Let $n$ be any positive integer and $F_n$ Dutch-Windmill, or friendship graph with $2n + 1$ vertices and $3n$ edges. In other words, the friendship graph $F_n$ is a graph that can be constructed by coalescence $n$ copies of the cycle graph $C_3$ of length 3 with a common vertex. The friendship theorem of Erdős et al. [12] states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. Figure 3 shows some examples of friendship graphs.

Theorem 11. Friendship graphs $F_n$ are $\mathbb{Z}_{2m}$-barycentric-magic graphs for any $m \in \mathbb{N}$.

Proof. Consider friendship graph $F_n$ and suppose that $\deg(v) = 2n$ and $\deg(u_i) = 2$ for $i = 1, \ldots, n$. We label all the edges of $F_n$ with $\alpha \in \mathbb{Z}_{2m} \setminus \{0\}$. Since $2na \equiv 2\alpha \pmod{2m}$ (e.g., put $m = \alpha$), the graphs $F_n$ are $\mathbb{Z}_{2m}$-barycentric-magic. \(\square\)

Here we consider another families of graphs denoted by $H_{2n}$ (see Figure 4).
Theorem 12. The graphs $H_{2n}$ are not $\mathbb{Z}_k$-barycentric-magic for every $k$.

Proof. Suppose that the set of vertices of $H_{2n}$ is $V = \{u_1, u_2\} \cup \{v_1, v_2\}$, where $i = 1, \ldots, n/2$ and $j = 1, \ldots, n - 2$ and \text{deg}(u_1) = \text{deg}(u_2) = 2$, \text{deg}(v_1) = 3$, and \text{deg}(v_2) = 4$. One can consider some cases to prove that there is no $\mathbb{Z}_k$ barycentric-magic labeling for $H_{2n}$. Here we state two cases. We label the graph $H_{2n}$ as follows.

Case 1. Label all edges of $H_{2n}$ with $\alpha \in \mathbb{Z}_k \setminus \{0\}$. In this case $3\alpha \equiv 2\alpha \pmod{k}$ or $\alpha \equiv 0 \pmod{k}$ which is not true.

Case 2. We label $H_{2n}$ as follows: $l(u_1, u_{11}) = l(v_1, u_{13}) = \alpha + r$ for each $i = 1, \ldots, n/2$ and $l(u_1, v_1) = l(v_2, v_{2n-2}) = \alpha$ for each $j = 1, \ldots, n - 3$ and $l(v_{2j-1}, v_{2j}) = \alpha + 3r$ for each $j = 1, \ldots, n - 3$ and $l(u_1, u_{21}) = l(v_1, u_{12}) = l(u_2, u_{2(n/2)}) = \alpha + 2r$ for each $i = 1, \ldots, n/2$ and $j = 1, \ldots, n - 2$). From $l^*(u_1) = l^*(u_{11})$ we have $3\alpha + 3r \equiv 2\alpha + 2r \pmod{k}$ or $\alpha + r \equiv 0 \pmod{k}$, but $\alpha + r$ is the label of edges $u_{11}, v_1$, so this labeling is not barycentric-magic. \hfill \Box

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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