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Research Article

On the Barycentric Labeling of Certain Graphs

Saeid Alikhani and Zeynab Amirzadeh

Department of Mathematics, Yazd University, Yazd 89195-741, Iran

Correspondence should be addressed to Saeid Alikhani; alikhani206@gmail.com

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Let A be an abelian group. A graph G is called A-magic if there exists edge labeling $l : E(G) \to A \setminus \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to A$, defined by $l^+(v) = \sum_{uv \in E(G)} l(uv)$, where the sum is over all edges in $E(G)$, is a constant map. A graph G is A-barycentric-magic (or has A-barycentric labeling) if G is A-magic and also satisfies $l^+(v) = \deg(v) l(u_v v)$ for all $v \in V$ and for some vertex u_v adjacent to v. In this paper we consider some graphs G and characterize all $m \in \mathbb{N}$ for which G is $\mathbb{Z}_m$-barycentric-magic.

1. Introduction

Let $G = (V, E)$ be a finite, simple, and undirected graph. Labeling for a graph is a map that takes graph elements to numbers (usually positive or nonnegative integers). Let A be an abelian group (written additively). The graph G is called A-magic if there exists labeling $l : E(G) \to A \setminus \{0\}$ such that, for each vertex v, the sum of values of all edges incident with v, denoted by $l^+(v)$, is a constant; that is, $l^+(v) = c$, for some $c \in A$. When this constant is 0, G is said to be A-zero-sum magic. The integer-magic spectrum of a graph G is the set $\mathrm{IM}(G) = \{k \in \mathbb{N} : G \text{ is } \mathbb{Z}_k\text{-magic}\}$.

As an example, Figure 1 shows a graph which is A-zero sum magic, for every group A (see [1]).

Let us state some easy lemmas (or observations). They are straightforward to verify and can be found in [2].

Lemma 1. *A graph G is $\mathbb{Z}_2$-magic if and only if every vertex of G is of the same parity.*

Lemma 2. *An Eulerian graph G with even size is A-magic.*

Lemma 3. *If A_1 is a subgroup of A and graph G is A_1-magic, then G is A-magic.*

Various authors have introduced labeling that generalizes the idea of magic square. Kotzig and Rosa [3] defined a magic labeling to be total labeling on the vertices and edges in which the labels are the integers from 1 to $|V(G)| + |E(G)|$. The sum of labels on an edge and its two endpoints is constant. In 1996 Ringel and Llado [4] redefined this type of labeling as edge-magic. Also, Enomoto et al. [5] have introduced the name super edge-magic for magic labeling in the sense of Kotzig and Rosa, with the added property that the n vertices receive the smaller labels, $\{1, 2, \ldots, n\}$. Lee et al. [6] defined the concept of k-edge magic graphs and studied it for certain graphs (see, e.g., [7]). Recently authors in [8] defined a new kind of group magicness graphs.

Here we recall the following definition.

Definition 4 (see [8]). If there exists labeling l for a graph G, whose induced vertex set labeling is a constant map and for all $v \in V(G)$ the sum $l^+(v)$ also satisfies $l^+(v) = \deg(v) l(u_v v)$ for some vertex u_v adjacent to v, G is said to be A-barycentric-magic.

Note that the motivation of Definition 4 is the following definition of k-barycentric sequence which was introduced in [9] and has already been used in graph labeling problems, specially in Ramsey theory [9–11].

Definition 5. Let $x_1, x_2, \ldots, x_k$ be k elements of an abelian group A. This sequence is k-barycentric if there exists j such that $x_1 + x_2 + \cdots + x_j + \cdots + x_k = k x_j$. The element x_j is called a barycenter.

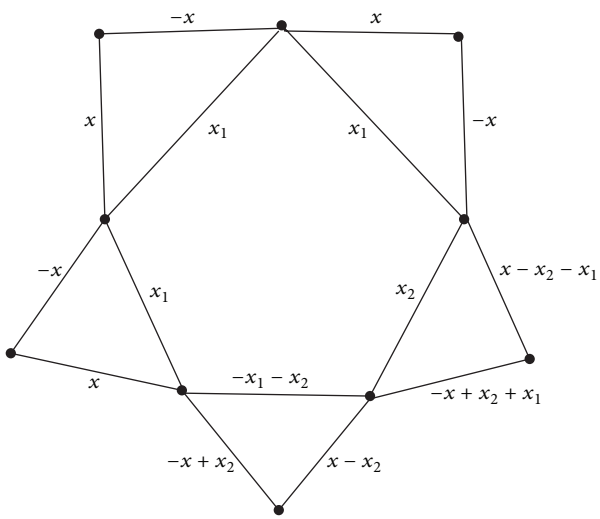

Figure 1: An example for A-zero sum magic graph.

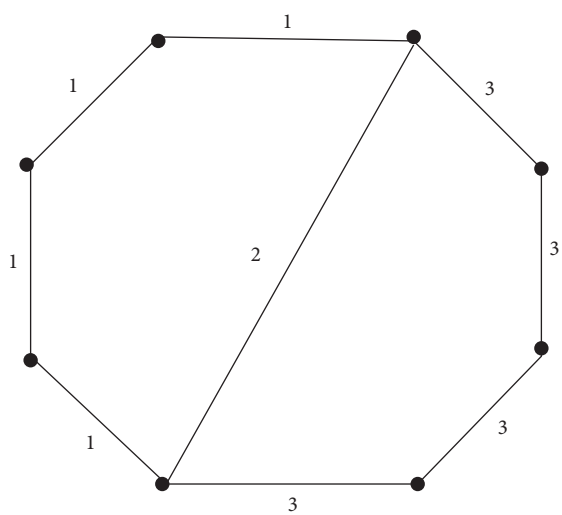

Figure 2: $\mathbb{Z}_4$-barycentric magic graph.

Example 6. Figure 2 shows a graph G with $|V| = 8$ and $|E| = 9$ which is $\mathbb{Z}_4$-barycentric-magic. Note that in this example the sum $l^+(v)$ is 2 (mod 4) for all $v \in V(G)$.

The study of the barycentric-magic graphs is motivated by the relationship between different types of magic labeling and the behavior of the sums of sequences in abelian groups. That is, A-magic labeling of a graph G is equivalent to sequences of nonzero elements of A with the same sum; A-zero-sum magic labeling is equivalent to zero-sum sequences in $A\setminus\{0\}$.

Note that A-barycentric-magic labeling is equivalent to sequences of nonzero elements of A that contain one element which is the "average" of its terms, called barycentric sequence.

Similar to the definition of integer magic spectrum of A-magic graphs we state the definition of barycenter-magic spectrum of graph.

Definition 7 (see [8]). For a given graph G the set of all positive integers m for which G is $\mathbb{Z}_m$-barycentric-magic is called the barycenter-magic spectrum of G and is denoted by BMS(G).

In this paper, we consider specific graphs G and characterize all $m \in \mathbb{N}$ for which G is $\mathbb{Z}_m$-barycentric-magic.

2. Barycentric-Magic Labeling of Certain Graphs

In this section, we characterize all $m \in \mathbb{N}$ for which G is $\mathbb{Z}_m$-barycentric-magic. First we consider some complete bipartite graphs.

First we state the following theorem.

Theorem 8 (see [8]). *The complete bipartite graph $K_{2,3}$ is not $\mathbb{Z}_m$-barycentric-magic for any m.*

We generalize the previous theorem.

Theorem 9. *For $n \geq 3$, $K_{2,n}$ is $\mathbb{Z}_m$-barycentric-magic if and only if $gcd(n-2, m) \neq 1$.*

Proof. Let $V(K_{2,n}) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of $K_{2,n}$. For each j, the edges incidents to v_j must have the same label. Suppose that $l(u_1, v_j) = \alpha_j$ and $l(u_2, v_j) = \alpha_j$. Then $2\alpha_1 \equiv 2\alpha_2 \equiv \cdots 2\alpha_n \pmod{m}$. We consider the two following cases.

Case 1 (m is odd). In this case, the condition $2\alpha_1 \equiv 2\alpha_2 \equiv \cdots 2\alpha_n \pmod{m}$ implies that all edges have the same label, say α. By the condition $l^+(u_i) = l^+(v_j)$ we have $(n-2)\alpha \equiv 0 \pmod{m}$ and this is impossible when $\gcd(n-2, m) = 1$. If $\gcd(n-2, m) = d \neq 1$, then using $\alpha = m/d$ gives a barycentric-magic labeling.

Case 2 (m is even). Here we give two different approaches. In this case, the condition $2\alpha_1 \equiv 2\alpha_2 \equiv \cdots 2\alpha_n \pmod{m}$ implies that there are at most two different labels α and $\beta = \alpha + m/2$, such that $2\alpha \equiv 2\beta \pmod{m}$. Now label $K_{2,n}$ as follows: $l(u_1, v_j) = \alpha$ for $1 \leq j \leq k$ and $l(u_1, v_j) = \beta$ for $k+1 \leq j \leq n$, for some $1 \leq k \leq n$. Then, the edges incidents to u_2 must be labeled in the same way. This labeling is barycentric-magic if and only if

$$k\alpha + (n-k)\beta \equiv n\alpha \equiv 2\alpha \equiv 2\beta \pmod{m} \tag{1}$$

or

$$k\alpha + (n-k)\beta \equiv n\beta \equiv 2\alpha \equiv 2\beta \pmod{m}. \tag{2}$$

Without loss of generality, we consider only the first relation. The condition $n\alpha \equiv 2\alpha \pmod{m}$ is satisfied only when $\gcd(n-2, m) \neq 1$. So suppose that $\gcd(n-2, m) = d \neq 1$. Choose

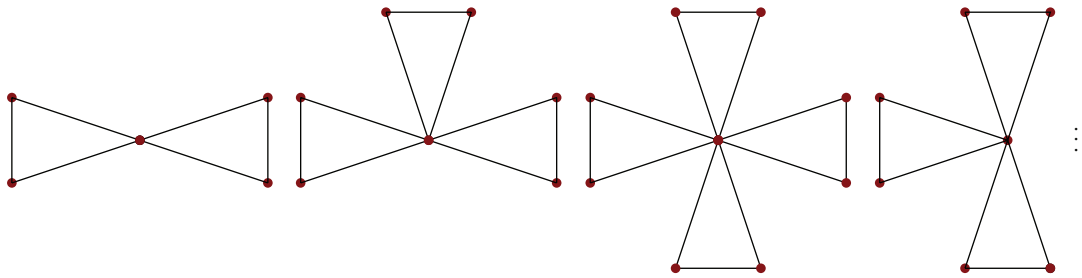

Figure 3: Friendship graphs F_2, F_3, F_4, and F_n, respectively.

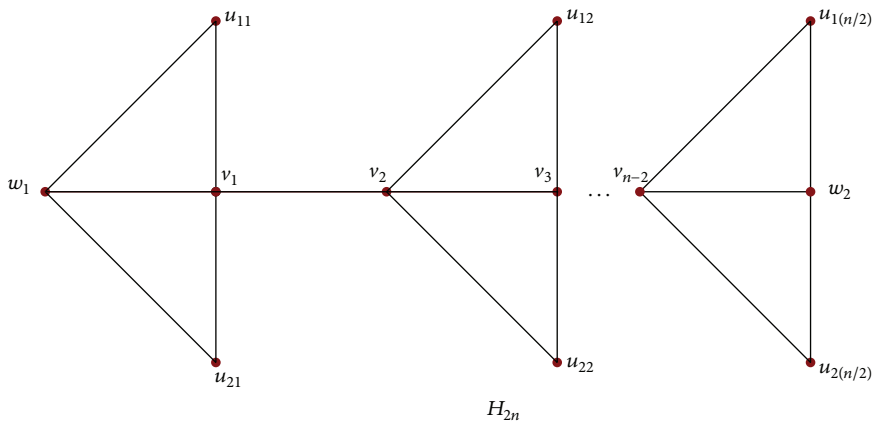

Figure 4: Graph H_{2n}.

$k = d$, $\alpha = m/d$, and $\beta = \alpha + m/2$ and since $n - d$ is even we get

$$k\alpha + (n-k)\beta \equiv n\alpha \pmod{m}, \qquad n\alpha \equiv 2\beta \equiv 2\alpha \pmod{m}. \tag{3}$$

Therefore with this labeling $K_{2,n}$ is $\mathbb{Z}_m$-barycentric-magic.

Now we state the second approach.

Since $n\alpha = 2\alpha \pmod{m}$, if $\gcd(m, n-2) = 1$, then $\alpha = 0$, a contradiction. Conversely, let $\alpha_1 = \alpha_2 = \cdots = \alpha_k = m/d$. This is also a $\mathbb{Z}_m$-barycentric-magic labeling for $K_{2,n}$. □

Example 10. Consider the graph $K_{2,10}$ and the group $\mathbb{Z}_{10}$. Let $V(K_{2,10}) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_{10}\}$ be the set of vertices of $K_{2,10}$. In this case, since $d = \gcd(n-2, m) = 2$ choose $\alpha = m/d = 5$ and $\beta = \alpha + m/2 = 10$ and $k = d = 2$. The labeling of $K_{2,10}$ is as follows:

$l(u_i v_j) = 5$ for $i = 1, 2$ and $j = 1, 2$;

$l(u_i v_j) = 10$ for $i = 1, 2$ and $j = 3, \ldots, 10$;

then $l^+(v_j) = 2 \times 5 = \deg(v_j) \times 5 \equiv 10 \pmod{10}$ for $j = 1, 2$;

$l^+(v_j) = 2 \times 10 = \deg(v_j) \times 10 \equiv 10 \pmod{10}$ for $j = 3, \ldots, 10$;

$l^+(u_i) = 2 \times 5 + 8 \times 10 \equiv 10 \equiv \deg(u_i) \times 5 \pmod{10}$ for $i = 1, 2$.

Then with this labeling we get barycentric-magic labeling.

Here we consider friendship graphs.

Let n be any positive integer and F_n Dutch-Windmill, or friendship graph with $2n + 1$ vertices and $3n$ edges. In other words, the friendship graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex. The friendship theorem of Erdös et al. [12] states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. Figure 3 shows some examples of friendship graphs.

Theorem 11. *Friendship graphs F_n are $\mathbb{Z}_{2m}$-barycentric-magic graphs for any $m \in \mathbb{N}$.*

Proof. Consider friendship graph F_n and suppose that $\deg(v) = 2n$ and $\deg(u_i) = 2$ for $i = 1, \ldots, n$. We label all the edges of F_n with $\alpha \in \mathbb{Z}_{2m} \setminus \{0\}$. Since $2n\alpha \equiv 2\alpha \pmod{2m}$ (e.g., put $m = \alpha$), the graphs F_n are $\mathbb{Z}_{2n}$-barycentric-magic. □

Here we consider another families of graphs denoted by H_{2n} (see Figure 4).

Theorem 12. *The graphs H_{2n} are not $\mathbb{Z}_k$-barycentric-magic for every k.*

Proof. Suppose that the set of vertices of H_{2n} is $V = \{u_{1i}, u_{2i}\} \cup \{w_1, w_2\} \cup \{v_j\}$, where $i = 1, \ldots, n/2$ and $j = 1, \ldots, n-2$ and $\deg(u_{1i}) = \deg(u_{2i}) = 2$, $\deg(w_1) = \deg(w_2) = 3$, and $\deg(v_j) = 4$. One can consider some cases to prove that there is no $\mathbb{Z}_k$ barycentric-magic labeling for H_{2n}. Here we state two cases. We label the graph H_{2n} as follows.

Case 1. Label all edges of H_{2n} with $\alpha \in \mathbb{Z}_k \setminus \{0\}$. In this case $3\alpha \equiv 2\alpha \pmod{k}$ or $\alpha \equiv 0 \pmod{k}$ which is not true.

Case 2. We label H_{2n} as follows: $l(w_1u_{11}) = l(v_ju_{1i}) = l(w_2u_{1(n/2)}) = \alpha + r$ for each $i = 1, \ldots, n/2$ and $l(w_1v_1) = l(v_{2j}v_{2j+1}) = l(w_2v_{n-2}) = \alpha$ for each $j = 1, \ldots, n-3$ and $l(v_{2j-1}v_{2j}) = \alpha + 3r$ for each $j = 1, \ldots, n-3$ and $l(w_1u_{21}) = l(v_ju_{2i}) = l(w_2u_{2(n/2)}) = \alpha + 2r$ ($i = 1, \ldots, n/2$ and $j = 1, \ldots, n-2$). From $l^+(w_1) = l^+(u_{11})$ we have $3\alpha + 3r \equiv 2\alpha + 2r \pmod{k}$ or $\alpha + r \equiv 0 \pmod{k}$, but $\alpha + r$ is the label of edges $u_{1i}v_j$, so this labeling is not barycentric-magic. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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