Research Article

Hyers-Ulam Stability of Third Order Euler's Differential Equations

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We investigate the Hyers-Ulam stability of third order Euler's differential equations of the form $t^3 y''' + \alpha t^2 y'' + \beta t y' + \gamma y = 0$ on any open interval $I = (a, b)$, $0 < a < b \leq \infty$ or $-\infty < a < b < 0$, where α, β , and γ are complex constants.

1. Introduction

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems [1]. Among such problems is a problem concerning the stability of functional equations: *give conditions in order for a linear function near an approximately linear function to exist.*

In the following year, Hyers [2] gave an answer to the problem of Ulam for additive functions defined on Banach spaces.

Let X_1 and X_2 be two real Banach spaces and let $\epsilon > 0$. Then, for every function $f: X \rightarrow Y$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \quad (x, y \in X_1) \quad (1)$$

there exists a unique additive function $A: X_1 \rightarrow X_2$ with the property

$$\|f(x) - A(x)\| \leq \epsilon, \quad x \in X_1. \quad (2)$$

Furthermore, the result of Hyers has been generalized by Rassias [3, 4]. Since then, the stability problems of various functional equations have been investigated by many authors (see, e.g., [1, 5–8]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations.

The differential equation

$$\phi(t, y, y', \dots, y^{(n)}) = 0 \quad (3)$$

has Hyers-Ulam stability; if for given $\epsilon > 0$ and a function y such that

$$\phi(t, y, y', \dots, y^{(n)}) \leq \epsilon, \quad (4)$$

then there exists a solution y_a of the differential equation such that $|y(t) - y_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon \rightarrow 0} K(\epsilon) = 0$.

If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\psi(t)$, respectively, where ϕ and ψ are appropriate functions not depending on y and y_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see, e.g., [9, 10]). Thereafter, Alsina and Ger published their work [11], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$.

If a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \leq \epsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y'(t) - ce^t| \leq 3\epsilon$, for all $t \in (a, \infty)$.

In [1], Rezaei et al. have discussed the Hyers-Ulam stability of linear differential equations of first and n th order by applying Laplace transform which is comparable with the other methods available in the literature.

It is Jung et al. who have investigated the Hyers-Ulam stability of linear differential equations of different classes including the stability of the delay differential equation

$y'(t) = \lambda y(t - \tau)$, where $\tau > 0$ is a constant (see, e.g., [5–7, 12–14]). Among the works, we are motivated by the results of [13], where he has studied the Hyers-Ulam stability of the following Euler's differential equations:

$$ty'(t) + \alpha y(t) + \beta t^r x_0 = 0, \quad (5)$$

$$t^2 y''(t) + \alpha ty'(t) + \beta y(t) = 0, \quad (6)$$

where α, β , and γ are complex constants. We may note that the Hyers-Ulam stability of (6) depends on the Hyers-Ulam stability of (5) for every fixed $x_0 \neq 0$. We mentioned here the important results of [13].

Theorem 1. Let X be a complex Banach space and let $I = (a, b)$ be an open interval such that $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. Assume that a function $\phi : I \rightarrow [0, \infty)$ is given, that α, β , and γ are complex constants, and that x_0 is a fixed element of X . Furthermore, suppose a continuously differentiable function $f : I \rightarrow X$ satisfies

$$\|ty'(t) + \alpha y(t) + \beta t^r x_0\| \leq \phi(t), \quad t \in I. \quad (7)$$

If both $t^{\alpha+r-1}$ and $t^{\alpha-1}\phi(t)$ are integrable on (a, c) , for any c with $a < c \leq b$, then there exists a unique solution $f_0 : I \rightarrow X$ of the differential equation (5) such that

$$\|f(t) - f_0(t)\| \leq |t^{-\alpha}| \left| \int_t^b v^{\alpha-1} \phi(v) dv \right| \quad (8)$$

for any $t \in I$.

Indeed, the unique solution of (5) in the complex Banach space X is given by

$$f_0(t) = \begin{cases} \left(\frac{a}{t}\right)^\alpha x - \frac{\beta}{\alpha+r} \frac{1}{t^\alpha} (t^{\alpha+r} - a^{\alpha+r}) x_0, & (\alpha+r) \neq 0, \\ \left(\frac{a}{t}\right)^\alpha x - \frac{\beta}{t^\alpha} (\ln|t| - \ln|a|) x_0, & (\alpha+r) = 0. \end{cases} \quad (9)$$

Theorem 2. If a twice continuously differentiable function $f : I \rightarrow R$ satisfies the differential inequality

$$|t^2 f''(t) + \alpha t f'(t) + \beta f(t)| \leq \epsilon \quad (10)$$

for all $t \in I$ and for some $\epsilon > 0$, then there exists a solution $f_0 : I \rightarrow R$ of (6) such that

$$|f(t) - f_0(t)| \leq \frac{\epsilon}{|\beta|} \left| \left(\frac{b}{t}\right)^c - 1 \right| \quad (11)$$

for any $t \in I$, where c is a negative real number. In particular, if $I = (a, \infty)$, $a > 0$, then

$$|f(t) - f_0(t)| \leq \frac{\epsilon}{|\beta|}, \quad \forall t > a. \quad (12)$$

In this theorem, we may note that the unique solution of (6) is given by

$$f_0(t) = \left(c_3 + \frac{c_0}{d(c-d)}\right) \left(\frac{a}{t}\right)^c - \frac{c_0}{d(c-d)} \left(\frac{a}{t}\right)^d, \quad (13)$$

where

$$c = \frac{(\alpha-1) - [(1-\alpha)^2 - 4\beta]^{1/2}}{2}, \quad (14)$$

$$d = \frac{(\alpha-1) + [(1-\alpha)^2 - 4\beta]^{1/2}}{2}$$

are real constants and c_3 is any arbitrary constant.

Theorem 3. Let X be a complex Banach space and let $I = (a, b)$ be an open interval such that $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. Assume that a function $\phi : I \rightarrow [0, \infty)$ is given and that $h : I \rightarrow X$ is a continuous function. Furthermore, suppose a continuously differentiable function $f : I \rightarrow X$ satisfies

$$\|ty'(t) + \alpha y(t) + h(t)\| \leq \phi(t), \quad t \in I. \quad (15)$$

If both $t^{\alpha-1}\phi(t)$ and $t^{\alpha-1}h(t)$ are integrable on (a, c) , for any c with $a < c \leq b$, then there exists a unique solution $f_0 : I \rightarrow X$ of the differential equation

$$ty'(t) + \alpha y(t) + h(t) = 0 \quad (16)$$

such that (8) holds for any $t \in I$, where α is a complex constant.

Proof. The proof of the theorem follows from the proof of Theorem 1, if we set

$$z(t) = \left(\frac{t}{a}\right)^\alpha f(t) + \int_a^t \frac{u^{\alpha-1}}{a^\alpha} h(u) du, \quad (17)$$

where the unique solution is given by

$$f_0(t) = \left(\frac{a}{t}\right)^\alpha - \frac{1}{t^\alpha} \int_a^t u^{\alpha-1} h(u) du. \quad (18)$$

The details are omitted. $\square$

Remark 4. We may note that (5) is a particular case of (16). Hence, Theorems 1 and 3 are comparable.

The objective of this work is to investigate the generalized Hyers-Ulam stability of the following Euler's differential equations of the form

$$t^2 y''(t) + \alpha ty'(t) + \beta y(t) + \gamma t^r x_0 = 0, \quad (19)$$

$$t^3 y'''(t) + \alpha t^2 y''(t) + \beta ty'(t) + \gamma y(t) = 0, \quad (20)$$

where α, β, γ , and r are complex constants. Intuitively, we shall prove that if a twice continuously differentiable function $f : I \rightarrow X$ satisfies the differential inequality

$$\|t^2 y''(t) + \alpha ty'(t) + \beta y(t) + \gamma t^r x_0\| \leq \phi(t) \quad (21)$$

for all $t \in I$, where x_0 is a fixed element of the complex Banach space X and $I = (a, b)$ with $0 < a < b \leq \infty$ or $-\infty < a < b < 0$, then there exists a twice continuously differentiable solution $f_0(t)$ of (19) such that

$$\|f(t) - f_0(t)\| \leq |t|^l \left| \int_t^b u^{-l-1} \psi(u) du \right| \quad (22)$$

for any $t \in I$, where

$$\psi(t) = |t^m| \left| \int_t^b u^{-m-1} \phi(u) du \right|. \quad (23)$$

We also apply this result to the investigation of the Hyers-Ulam stability of (20).

2. Hyers-Ulam Stability of (19)

Throughout this section, we let $I = (a, b)$ with $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. For fixed $x_0 \neq 0$, the general solution of (19) in the class of real valued functions defined on I is given by

$$y(t) = \begin{cases} c_1 t^l + c_2 t^m - \frac{\gamma x_0 t^r}{g(r)}, & g(r) \neq 0, \quad l \neq m, \\ c_1 t^l + c_2 t^m - \frac{\gamma x_0 t^r \ln |t|}{g'(r)}, & g(r) = 0 \neq g'(r), \quad l \neq m, \\ (c_1 + c_2 \ln |t|) t^l - \frac{\gamma x_0 t^r}{g(r)}, & g(r) \neq 0, \quad l = m, \\ y(t) = (c_1 + c_2 \ln |t|) t^l - \frac{\gamma x_0 t^r (\ln |t|)^2}{2}, & g(r) = 0 = g'(r), \quad l = m, \end{cases} \quad (24)$$

where $g(r) = r^2 + (\alpha - 1)r + \beta$ and c_1, c_2 are arbitrary constants.

Remark 5. Indeed, $g'(r) = 2r + \alpha - 1$. If $g(r) = 0$, then either $r - l = 0$ or $r - m = 0$. Therefore, $r - l = 0$ implies that $2r + \alpha - 1 = 2r - l - m = r - m \neq 0$ and $r - m = 0$ implies that $2r + \alpha - 1 = r - l \neq 0$. Hence, the second solution could be any one of the following:

$$\begin{aligned} c_1 t^l + c_2 t^m - \frac{\gamma x_0 t^r \ln |t|}{r - m}; \quad & r - l = 0, \\ & r - m \neq 0, \quad l \neq m, \\ c_1 t^l + c_2 t^m - \frac{\gamma x_0 t^r \ln |t|}{r - l}; \quad & r - l \neq 0, \\ & r - m = 0, \quad l \neq m. \end{aligned} \quad (25)$$

Theorem 6. Let X be a complex Banach space and let $I = (a, b)$ be an open interval with either $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. Assume that a function $\phi : I \rightarrow [0, \infty)$ is given, that α, β, γ , and r are complex constants, and that x_0 is a fixed element of X . Furthermore, suppose a twice continuously differentiable function $f : I \rightarrow X$ satisfies the differential inequality (21). If $t^{r-l-1}, t^{r-m-1}, t^{m-l-1}, t^{-m-1}\phi(t)$, and $t^{-l-1}\psi(t)$ are integrable on (a, c) , for any c with $a < c \leq b$, then there exists a unique solution $f_0 : I \rightarrow X$ (which is twice continuously differentiable) of (19) such that (22) holds for any $t \in I$, where l and m are the roots of $g(r) = 0$.

Proof. We prove the theorem for five possible cases; namely,

- (i) $(r - l)(r - m) \neq 0, l \neq m$;
- (ii) $r - l = 0 \neq r - m, l \neq m$;
- (iii) $r - l \neq 0 = r - m, l \neq m$;
- (iv) $r - l \neq 0, l = m$;
- (v) $r - l = 0, l = m$.

Case (i). Suppose that X is a complex Banach space and $f : I \rightarrow X$ is a twice continuously differentiable function satisfying the differential inequality (21). Define $h : I \rightarrow X$ such that $h(t) = tf'(t) - lf(t)$. Then it is easy to verify that

$$\begin{aligned} & \|th'(t) - mh(t) + \gamma t^r x_0\| \\ &= \|t^2 f''(t) + (1 - l - m)tf'(t) + lm f(t) + \gamma t^r x_0\| \\ &= \|t^2 f''(t) + \alpha t f'(t) + \beta f(t) + \gamma t^r x_0\| \\ &\leq \phi(t). \end{aligned} \quad (26)$$

Hence, by Theorem 1, it follows that there exists a unique solution $h_0 : I \rightarrow X$ of the differential equation $ty' - my + \gamma t^r x_0 = 0$ such that

$$\|h(t) - h_0(t)\| \leq |t^m| \left| \int_t^b v^{-m-1} \phi(v) dv \right| \quad (27)$$

provided that t^{r-m-1} and $t^{-m-1}\phi(t)$ are integrable on (a, c) , for any c with $a < c \leq b$. Indeed,

$$h_0(t) = \left(\frac{a}{t}\right)^{-m} x - \frac{\gamma t^m}{r - m} (t^{r-m} - a^{r-m}) x_0, \quad (28)$$

where x is a limit point in X . If we denote

$$\psi(t) = \left| t^m \int_t^b v^{-m-1} \phi(v) dv \right|, \quad (29)$$

then (27) becomes

$$\|tf'(t) - lf(t) - h_0(t)\| \leq \psi(t). \quad (30)$$

Clearly, $\psi : I \rightarrow [0, \infty)$. By Theorem 3, there exists a unique solution $f_1 : I \rightarrow X$ of the differential equation $ty'(t) - ly(t) - h_0(t) = 0$ such that

$$\|f(t) - f_1(t)\| \leq |t^l| \left| \int_t^b u^{-l-1} \psi(u) du \right| \quad (31)$$

provided that $t^{-l-1}\psi(t)$ and $t^{-l-1}h_0(t)$ are integrable on (a, c) , for any c with $a < c \leq b$. Clearly, $t^{-l-1}h_0(t)$ is integrable if and only if t^{m-l-1} and t^{r-l-1} are integrable. According to Theorem 3,

$$f_1(t) = \left(\frac{t}{a}\right)^l x_1 + t^l \int_a^t u^{-l-1} h_0(u) du, \quad (32)$$

where x_1 is limit point in X . It is easy to verify that

$$\begin{aligned} \int_a^t u^{-l-1} h_0(u) du &= \left[\frac{x}{a^m} + \frac{\gamma x_0 a^{r-m}}{r-m} \right] \frac{t^{m-l}}{m-l} \\ &\quad - \frac{x}{a^l(m-l)} - \frac{\gamma x_0 t^{r-l}}{(r-l)(r-m)} \\ &\quad - \frac{\gamma x_0 a^{r-l}}{(r-l)(m-l)}. \end{aligned} \quad (33)$$

Consequently,

$$\begin{aligned} f_1(t) &= \left[\frac{x}{a^l} - \frac{\gamma x_0 a^{r-l}}{(r-l)(m-l)} - \frac{x}{a^l(m-l)} \right] t^l \\ &\quad + \left[\frac{x}{a^m(m-l)} + \frac{\gamma x_0 a^{r-m}}{(r-m)(m-l)} \right] t^m - \frac{\gamma x_0 t^r}{g(r)} \end{aligned} \quad (34)$$

which is a solution of (19).

Case (ii). Proceeding as in Case (i), we find a unique solution $f_2 : I \rightarrow X$ of the equation $ty'(t) - ly(t) - h_0(t) = 0$ such that

$$\|f(t) - f_2(t)\| \leq |t|^l \left| \int_t^b u^{-l-1} \psi(u) du \right|, \quad (35)$$

$$f_2(t) = \left(\frac{t}{a} \right)^l x_2 + t^l \int_a^t u^{-l-1} h_0(u) du,$$

where x_2 is a limit point in X . Clearly,

$$\begin{aligned} \int_a^t u^{-l-1} h_0(u) du &= \int_a^t \left[\frac{x}{a^m} u^{m-l-1} - \frac{\gamma x_0 u^{-1}}{r-m} + \frac{\gamma x_0 a^{r-m}}{r-m} u^{m-l-1} \right] du \\ &= \left[\frac{x}{a^m} + \frac{\gamma x_0 a^{r-m}}{r-m} \right] \frac{t^{m-l}}{m-l} + \frac{\gamma x_0}{(r-m)(l-m)} \\ &\quad - \frac{x}{a^l(m-l)} - \frac{\gamma x_0}{(r-m)} \ln \left| \frac{t}{a} \right| \end{aligned} \quad (36)$$

implies that

$$\begin{aligned} f_2(t) &= \left[\frac{x_2}{a^l} + \frac{\gamma x_0}{(r-m)^2} + \frac{x}{a^l(r-m)} \right] t^l \\ &\quad + \left[\frac{x}{a^m(m-l)} - \frac{\gamma x_0 a^{r-m}}{(r-m)^2} \right] t^m - \frac{\gamma x_0 t^r}{(r-m)} \ln \left| \frac{t}{a} \right| \end{aligned} \quad (37)$$

which is a solution of (19).

Case (iii). Interchanging the roles of l and m in Case (ii), we find the unique solution $f_3(t)$ as

$$\begin{aligned} f_3(t) &= \left[\frac{x_3}{a^l} - \frac{x}{a^l(m-l)} - \frac{\gamma x_0 a^{m-l}}{(m-l)^2} \right] t^l \\ &\quad + \left[\frac{x}{a^m(m-l)} + \frac{\gamma x_0}{(m-l)^2} \right] t^m - \frac{\gamma x_0 t^r}{(r-l)} \ln \left| \frac{t}{a} \right|, \end{aligned} \quad (38)$$

where $x_3 \in X$.

Case (iv). In this case, we proceed as in Case (i). Indeed, for $l = m$,

$$\begin{aligned} \int_a^t u^{-l-1} h_0(u) du &= \left[\frac{x}{a^l} + \frac{\gamma x_0 a^{r-l}}{r-l} \right] \ln \left| \frac{t}{a} \right| \\ &\quad + \frac{\gamma x_0}{(r-l)^2} [a^{r-l} - t^{r-l}] \end{aligned} \quad (39)$$

and the unique solution $f_4(t)$ is given by

$$\begin{aligned} f_4(t) &= \left[\left\{ \frac{x_4}{a^l} + \frac{\gamma x_0 a^{r-l}}{(r-l)^2} \right\} + \left\{ \frac{x}{a^l} + \frac{\gamma x_0 a^{r-l}}{(r-l)} \right\} \ln \left| \frac{t}{a} \right| \right] t^l \\ &\quad - \frac{\gamma x_0 t^r}{(r-l)^2}, \end{aligned} \quad (40)$$

where $x_4 \in X$.

Case (v). In this case,

$$h_0(t) = \left(\frac{t}{a} \right)^l x - \gamma x_0 t^l \ln \left| \frac{t}{a} \right|. \quad (41)$$

Using the same type of argument as in Case (i), we find the unique solution $f_5(t)$ as

$$\begin{aligned} f_5(t) &= \left(\frac{t}{a} \right)^l x_5 + t^l \int_a^t u^{-l-1} h_0(u) du \\ &= \left(\frac{t}{a} \right)^l x_5 + \frac{x}{a^l} t^l \ln \left| \frac{t}{a} \right| - \frac{\gamma x_0 t^l}{2} \left(\ln \left| \frac{t}{a} \right| \right)^2 \\ &= \left[\frac{x_5}{a^l} + \frac{x}{a^l} \ln \left| \frac{t}{a} \right| \right] t^l - \frac{\gamma x_0 t^r}{2} \left(\ln \left| \frac{t}{a} \right| \right)^2, \end{aligned} \quad (42)$$

where $x_5 \in X$. This completes the proof of the theorem. $\square$

3. Hyers-Ulam Stability of (20)

In this section, we shall investigate the Hyers-Ulam stability of (20) on any open interval $I = (a, b)$ with either $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. We may note that $\alpha = 3 - (l + m + n)$, $\beta = lm + mn + nl - l - m - n + 1$, and $\gamma = -lmn$, when l, m , and n are the characteristic roots of the associated characteristic equation of (20).

Theorem 7. Let X be a complex Banach space. Assume that $\theta : I \rightarrow [0, \infty)$ is given. Furthermore, assume that t^{n-l-1} , t^{n-m-1} , t^{m-l-1} , $t^{-n-1}\theta(t)$, $t^{-m-1}\lambda(t)$, and $t^{-l-1}\eta(t)$ are integrable on (a, c) with $a < c \leq b$, where

$$\begin{aligned} \lambda(t) &= \left| t^n \int_t^b u^{-n-1} \theta(u) du \right|, \\ \eta(t) &= \left| t^m \int_t^b v^{-m-1} \lambda(v) dv \right|. \end{aligned} \quad (43)$$

Suppose that $f \in C^3(I, X)$ and satisfies

$$\|t^3 f'''(t) + \alpha t^2 f''(t) + \beta t f'(t) + \gamma f(t)\| \leq \theta(t), \quad (44)$$

for all $t \in I$. Then, there exists a unique solution $f_0 \in C^3(I, X)$ of (20) such that

$$\|f(t) - f_0(t)\| \leq \left| t^l \int_t^b s^{-l-1} \eta(s) ds \right|. \quad (45)$$

Proof. Let $f : I \rightarrow X$ be such that (44) holds, for $t \in I$. Define $w : I \rightarrow X$ such that

$$w(t) = t^2 f''(t) + (1-l-m) t f'(t) + l m f(t). \quad (46)$$

Indeed,

$$\|t w'(t) - n w(t)\| \leq \theta(t), \quad t \in I. \quad (47)$$

Hence, using Theorem 3, it follows that there exists a unique $w_0 : I \rightarrow X$ such that

$$\|w(t) - w_0(t)\| \leq |t^n| \left| \int_t^b u^{-n-1} \theta(u) du \right| = \lambda(t), \quad (48)$$

where $w_0(t) = (t/a)^n x$ and x is a limit point in X . Ultimately, (48) becomes

$$\|t^2 f''(t) + (1-l-m) t f'(t) + l m f(t) - a^{-n} t^n x\| \leq \lambda(t), \quad t \in I. \quad (49)$$

Now, we can apply Theorem 6 to the above differential inequality for five possible cases. Consequently,

$$\begin{aligned} f_1(t) &= \left[x_1 - \frac{x_2}{m-l} + \frac{x}{(n-l)(n-m)} \right] \left(\frac{t}{a} \right)^l \\ &\quad + \left[\frac{x_2}{m-l} + \frac{x}{(n-m)(l-m)} \right] \left(\frac{t}{a} \right)^m \\ &\quad + \frac{x}{(n-l)(n-m)} \left(\frac{t}{a} \right)^n, \end{aligned}$$

$$\begin{aligned} f_2(t) &= \left[\frac{x_3}{a^l} + \frac{x_2}{a^l(n-m)} - \frac{x}{a^n(n-m)^2} + \frac{x}{a^n(n-m)} \ln \left| \frac{t}{a} \right| \right] t^l \\ &\quad + \left[\frac{x_2}{a^m(m-l)} + \frac{x}{a^m(n-m)^2} \right] t^m, \end{aligned}$$

$$\begin{aligned} f_3(t) &= \left[\frac{x_4}{a^l} - \frac{x_2}{a^l(m-l)} + \frac{x}{a^l(m-l)^2} \right] t^l \\ &\quad + \left[\frac{x_2}{a^m(m-l)} - \frac{x}{a^m(m-l)^2} + \frac{x}{a^m(m-l)} \ln \left| \frac{t}{a} \right| \right] t^m, \end{aligned}$$

$$\begin{aligned} f_4(t) &= \left[\frac{x_4}{a^l} - \frac{x}{a^l(n-l)^2} + \left\{ \frac{x_2}{a^l} - \frac{x}{a^l(n-l)} \right\} \ln \left| \frac{t}{a} \right| \right] t^l \\ &\quad + \frac{x}{(n-l)^2} \left(\frac{t}{a} \right)^n, \\ f_5(t) &= \left[\frac{x_5}{a^l} + \frac{x_2}{a^l} \ln \left| \frac{t}{a} \right| + \frac{x}{2a^n} \left(\ln \left| \frac{t}{a} \right| \right)^2 \right] t^l \end{aligned} \quad (50)$$

are the unique solutions with respect to the five possible cases, respectively. In fact, they are the possible solutions of (20). Hence, the theorem is proved. $\square$

Corollary 8. Let $X = \mathbb{R}$ be a real Banach space. Assume that t^{n-l-1} , t^{n-m-1} , and t^{m-l-1} are integrable on $I = (a, b)$ with $0 < a < b \leq \infty$ or $-\infty < a < b < 0$, where l, m , and n are positive roots of the associated characteristic equation of (20). Suppose that $f \in C^3(I, X)$ and satisfies

$$\|t^3 f'''(t) + \alpha t^2 f''(t) + \beta t f'(t) + \gamma f(t)\| \leq \epsilon, \quad (51)$$

for all $t \in I$. Then, there exists a unique solution $f_0 \in C^3(I, X)$ of (20) such that

$$\|f(t) - f_0(t)\| \leq \frac{|\epsilon|}{l m n}. \quad (52)$$

Proof. Let $f : I \rightarrow X$ be such that (51) holds, for all $t \in I$. Since all the conditions of Theorem 7 are satisfied, then there exists a unique solution $f_0 : I \rightarrow X$ of (20) such that

$$\lambda(t) = \left| t^n \int_t^b u^{-n-1} \epsilon du \right| = |\epsilon| |t^n| \left| \left[\frac{u^{-n}}{-n} \right]_t^b \right| = \left| \frac{\epsilon}{n} \right| \quad (53)$$

when $I = (-\infty, \infty)$. Then, for $I = (-\infty, \infty)$,

$$\eta(t) = \left| t^m \int_t^b v^{-m-1} \lambda(v) dv \right| = \left| \frac{\epsilon}{m n} \right| \left| \left(\frac{t}{b} \right)^m - 1 \right| = \left| \frac{\epsilon}{m n} \right|. \quad (54)$$

Hence,

$$\|f(t) - f_0(t)\| \leq \left| t^l \int_t^b s^{-l-1} \eta(s) ds \right| = \left| \frac{\epsilon}{l m n} \right| \left| \left(\frac{t}{b} \right)^l - 1 \right|. \quad (55)$$

Consequently,

$$\|f(t) - f_0(t)\| \leq \frac{|\epsilon|}{l m n} \quad \text{as } b \rightarrow \infty, \quad (56)$$

where $f_0(t)$ can be chosen from Theorem 7. $\square$

Example 9. Let $X = \mathbb{R}$ be a Banach space and let $I = (a, \infty)$, $a > 0$. Consider Euler's equation

$$t^3 y'''(t) - 6t^2 y''(t) + 18t y'(t) - 24y(t) = 0 \quad (57)$$

with $l = 4$, $m = 3$, and $n = 2$. Suppose that $f : I \rightarrow X$ satisfies the differential inequality

$$\|t^3 f'''(t) - 6t^2 f''(t) + 18tf'(t) - 24f(t)\| \leq \epsilon, \quad (58)$$

for any $t \in I$. By Corollary 8, there exists a unique solution $f_0 : I \rightarrow X$ of (57) such that

$$\|f(t) - f_0(t)\| \leq \frac{\epsilon}{24}, \quad (59)$$

for any $t \in I$ and for unique $x, x_1, x_2 \in X$,

$$f_0(t) = \left\{ \frac{x}{2} + x_1 + x_2 \right\} \left(\frac{t}{a} \right)^4 + \{x - x_2\} \left(\frac{t}{a} \right)^3 + \left(\frac{x}{2} \right) \left(\frac{t}{a} \right)^2. \quad (60)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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