Research Article

Representing and Counting the Subgroups of the Group $\mathbb{Z}_m \times \mathbb{Z}_n$

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We deduce a simple representation and the invariant factor decompositions of the subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$, where m and n are arbitrary positive integers. We obtain formulas for the total number of subgroups and the number of subgroups of a given order.

1. Introduction

Let $\mathbb{Z}_m$ be the group of residue classes modulo m and consider the direct product $G = \mathbb{Z}_m \times \mathbb{Z}_n$, where m and n are arbitrary positive integers. This paper aims to deduce a simple representation and the invariant factor decompositions of the subgroups of the group G . As consequences we derive formulas for the number of certain types of subgroups of G , including the total number $s(m, n)$ of its subgroups and the number $s_k(m, n)$ of its subgroups of order k ($k \mid mn$).

Subgroups of $\mathbb{Z} \times \mathbb{Z}$ (sublattices of the two-dimensional integer lattice) and associated counting functions were considered by several authors in pure and applied mathematics. It is known, for example, that the number of subgroups of index n in $\mathbb{Z} \times \mathbb{Z}$ is $\sigma(n)$, the sum of the (positive) divisors of n . See, for example, [1, 2], [3, item A001615]. Although features of the subgroups of G not only are interesting by their own but also have applications, one of them is described below, it seems that a synthesis on subgroups of G cannot be found in the literature.

In the case $m = n$, the subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$ play an important role in numerical harmonic analysis, more specifically in the field of applied time-frequency analysis. Time-frequency analysis attempts to investigate function behavior via a phase space representation given by the short-time Fourier transform [4]. The short-time Fourier coefficients of a function f are given by inner products

with translated modulations (or time-frequency shifts) of a prototype function g , assumed to be well localized in phase space, for example, a Gaussian. In applications, the phase space corresponding to discrete, finite functions (or vectors) belonging to $\mathbb{C}^n$ is exactly $\mathbb{Z}_n \times \mathbb{Z}_n$. Concerned with the question of reconstruction from samples of short-time Fourier transforms, it has been found that when sampling on lattices, that is, subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$, the associated analysis and reconstruction operators are particularly rich in structure, which, in turn, can be exploited for efficient implementation (cf. [5–7] and references therein). It is of particular interest to find subgroups in a certain range of cardinality; therefore, a complete characterization of these groups helps choose the best one for the desired application.

We recall that a finite Abelian group of order > 1 has rank r if it is isomorphic to $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$, where $n_1, \dots, n_r \in \mathbb{N} \setminus \{1\}$ and $n_j \mid n_{j+1}$ ($1 \leq j \leq r-1$), which is the invariant factor decomposition of the given group. Here the number r is uniquely determined and represents the minimal number of generators of the group. For general accounts on finite Abelian groups see, for example, [8, 9].

It is known that for every finite Abelian group, the problem of counting all subgroups and the subgroups of a given order reduces to p -groups, which follows from the properties of the subgroup lattice of the group (see [10, 11]). In particular, for $G = \mathbb{Z}_m \times \mathbb{Z}_n$, this can be formulated as follows. Assume that $\gcd(m, n) > 1$. Then G is an Abelian

group of rank two, since $G \simeq \mathbb{Z}_u \times \mathbb{Z}_v$, where $u = \gcd(m, n)$ and $v = \text{lcm}(m, n)$. Let $u = p_1^{a_1} \cdots p_r^{a_r}$ and $v = p_1^{b_1} \cdots p_r^{b_r}$ be the prime power factorizations of u and v , respectively, where $0 \leq a_j \leq b_j$ ($1 \leq j \leq r$). Then

$$s(m, n) = \prod_{j=1}^r s(p_j^{a_j}, p_j^{b_j}), \quad (1)$$

$$s_k(m, n) = \prod_{j=1}^r s_{k_j}(p_j^{a_j}, p_j^{b_j}), \quad (2)$$

where $k = k_1 \cdots k_r$ and $k_j = p_j^{c_j}$ with some exponents $0 \leq c_j \leq a_j + b_j$ ($1 \leq j \leq r$).

Now consider the p -group $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$, where $0 \leq a \leq b$. This is of rank two for $1 \leq a \leq b$. One has the simple explicit formulae:

$$\begin{aligned} s(p^a, p^b) &= \frac{(b-a+1)p^{a+2} - (b-a-1)p^{a+1} - (a+b+3)p + (a+b+1)}{(p-1)^2}, \end{aligned} \quad (3)$$

$$s_{p^c}(p^a, p^b) = \begin{cases} \frac{p^{c+1} - 1}{p - 1}, & c \leq a \leq b, \\ \frac{p^{a+1} - 1}{p - 1}, & a \leq c \leq b, \\ \frac{p^{a+b-c+1} - 1}{p - 1}, & a \leq b \leq c \leq a + b. \end{cases} \quad (4)$$

Formula (3) was derived by Călugăreanu [12, Section 4] and recently by Petrillo [13, Proposition 2] using Goursat's lemma for groups. Tărnăuceanu [14, Proposition 2.9] and [15, Theorem 3.3] deduced (3) and (4) by a method based on properties of certain attached matrices.

Therefore, $s(m, n)$ and $s_k(m, n)$ can be computed using (1), (3) and (2), (4), respectively. We deduce other formulas for $s(m, n)$ and $s_k(m, n)$ (Theorems 3 and 4), which generalize (3) and (4) and put them in more compact forms. These are consequences of a simple representation of the subgroups of $G = \mathbb{Z}_m \times \mathbb{Z}_n$, given in Theorem 1. This representation might be known, but the only source we could find is the paper [5], where only a special case is treated in a different form. More exactly, in [5, Lemma 4.1] a representation for lattices in $\mathbb{Z}_n \times \mathbb{Z}_n$ of redundancy 2, that is, subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$ having index $n/2$, is given, using matrices in Hermite normal form. Theorem 2 gives the invariant factor decompositions of the subgroups of G . We also consider the number of cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ (Theorem 5) and the number of subgroups of a given exponent in $\mathbb{Z}_n \times \mathbb{Z}_n$ (Theorem 8).

Our approach is elementary, using only simple group-theoretic and number-theoretic arguments. The proofs are given in Section 4.

Throughout the paper we use the following notations: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\tau(n)$ and $\sigma(n)$ are the number and the sum, respectively, of the positive divisors of n , $\psi(n) = n \prod_{p|n} (1 + 1/p)$ is the Dedekind function, $\omega(n)$ stands for the number of distinct prime factors of n , μ is the Möbius function, ϕ denotes Euler's totient function, and ζ is the Riemann zeta function.

FIGURE 1

2. Subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$

The subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ can be identified and visualized in the plane with sublattices of the lattice $\mathbb{Z}_m \times \mathbb{Z}_n$. Every two-dimensional sublattice is generated by two basis vectors. For example, Figure 1 shows the subgroup of $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ having the basis vectors $(3, 0)$ and $(1, 2)$.

This suggests the following representation of the subgroups.

Theorem 1. For every $m, n \in \mathbb{N}$, let

$$I_{m,n} := \left\{ (a, b, t) \in \mathbb{N}^2 \times \mathbb{N}_0 : a \mid m, b \mid n, \right. \\ \left. 0 \leq t \leq \gcd\left(a, \frac{n}{b}\right) - 1 \right\} \quad (5)$$

and, for $(a, b, t) \in I_{m,n}$, define

$$H_{a,b,t} := \left\{ \left(ia + \frac{jta}{\gcd(a, n/b)}, jb \right) : \right. \\ \left. 0 \leq i \leq \frac{m}{a} - 1, 0 \leq j \leq \frac{n}{b} - 1 \right\}. \quad (6)$$

Then $H_{a,b,t}$ is a subgroup of order mn/ab of $\mathbb{Z}_m \times \mathbb{Z}_n$ and the map $(a, b, t) \mapsto H_{a,b,t}$ is a bijection between the set $I_{m,n}$ and the set of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$.

Note that for the subgroup $H_{a,b,t}$, the basis vectors mentioned above are $(a, 0)$ and (s, b) , where

$$s = \frac{ta}{\gcd(a, n/b)}. \quad (7)$$

This notation for s will be used also in the rest of the paper. Note also that in the case $a \neq m, b \neq n$ the area of

the parallelogram spanned by the basis vectors is ab , exactly the index of $H_{a,b,t}$.

We say that a subgroup $H = H_{a,b,t}$ is a subproduct of $\mathbb{Z}_m \times \mathbb{Z}_n$ if $H = H_1 \times H_2$, where H_1 and H_2 are subgroups of $\mathbb{Z}_m$ and $\mathbb{Z}_n$, respectively.

Theorem 2. (i) The invariant factor decomposition of the subgroup $H_{a,b,t}$ is given by

$$H_{a,b,t} \simeq \mathbb{Z}_\alpha \times \mathbb{Z}_\beta, \quad (8)$$

where

$$\alpha = \left(\frac{m}{a}, \frac{n}{b}, \frac{ns}{ab} \right), \quad \beta = \frac{mn}{ab\alpha} \quad (9)$$

satisfying $\alpha \mid \beta$.

(ii) The exponent of the subgroup $H_{a,b,t}$ is β .

(iii) The subgroup $H_{a,b,t}$ is cyclic if and only if $\alpha = 1$.

(iv) The subgroup $H_{a,b,t}$ is a subproduct if and only if $t = 0$ and $H_{a,b,0} = \mathbb{Z}_{m/a} \times \mathbb{Z}_{n/b}$. Here $H_{a,b,0}$ is cyclic if and only if $\gcd(m/a, n/b) = 1$.

For example, for the subgroup represented by Figure 1 one has $m = n = 12$, $a = 3$, $b = 2$, $s = 1$, $\alpha = 2$, and $\beta = 12$, and this subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{12}$. It is not cyclic and is not a subproduct.

According to Theorem 1, the number $s(m, n)$ of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ can be obtained by counting the elements of the set $I_{m,n}$. We deduce the following.

Theorem 3. For every $m, n \in \mathbb{N}$, $s(m, n)$ is given by

$$s(m, n) = \sum_{a \mid m, b \mid n} \gcd(a, b) \quad (10)$$

$$= \sum_{d \mid \gcd(m, n)} \phi(d) \tau\left(\frac{m}{d}\right) \tau\left(\frac{n}{d}\right) \quad (11)$$

$$= \sum_{d \mid \gcd(m, n)} d \tau\left(\frac{mn}{d^2}\right). \quad (12)$$

Formula (10) is a special case of a formula representing the number of all subgroups of a class of groups formed as cyclic extensions of cyclic groups, deduced by Calhoun [16] and having a laborious proof. Note that formula (10) is given, without proof in [3, item A054584].

Note also that the function $(m, n) \mapsto s(m, n)$ is representing a multiplicative arithmetic function of two variables; that is, $s(mm', nn') = s(m, n)s(m', n')$ holds for any $m, n, m', n' \in \mathbb{N}$ such that $\gcd(mm', nn') = 1$. This property, which is in concordance with (1), is a direct consequence of formula (10). See Section 5.

Let $N(a, b, c)$ denote the number of solutions $(x, y, z, t) \in \mathbb{N}^4$ of the system of equations $xy = a$, $zt = b$, $xz = c$.

Theorem 4. For every $k, m, n \in \mathbb{N}$, such that $k \mid mn$,

$$s_k(m, n) = \sum_{\substack{a \mid m, b \mid n \\ mb/a = k}} \gcd(a, b) \quad (13)$$

$$= \sum_{\substack{d \mid \gcd(k, m) \\ e \mid \gcd(k, n) \\ k \mid de}} \phi\left(\frac{de}{k}\right) \quad (14)$$

$$= \sum_{d \mid \gcd(m, n, k)} \phi(d) N\left(\frac{m}{d}, \frac{n}{d}, \frac{k}{d}\right). \quad (15)$$

The identities (3) and (4) can be easily deduced from each of the identities given in Theorems 3 and 4, respectively.

Theorem 5. Let $m, n \in \mathbb{N}$.

(i) The number $c(m, n)$ of cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ is given by

$$c(m, n) = \sum_{\substack{a \mid m, b \mid n \\ \gcd(m/a, n/b) = 1}} \gcd(a, b) \quad (16)$$

$$= \sum_{a \mid m, b \mid n} \phi(\gcd(a, b)) \quad (17)$$

$$= \sum_{d \mid \gcd(m, n)} (\mu * \phi)(d) \tau\left(\frac{m}{d}\right) \tau\left(\frac{n}{d}\right) \quad (18)$$

$$= \sum_{d \mid \gcd(m, n)} \phi(d) \tau\left(\frac{mn}{d^2}\right). \quad (19)$$

(ii) The number of subproducts of $\mathbb{Z}_m \times \mathbb{Z}_n$ is $\tau(m)\tau(n)$ and the number of its cyclic subproducts is $\tau(mn)$.

Formula (17), as a special case of an identity valid for arbitrary finite Abelian groups, was derived by the third author [17, 18] using different arguments. The function $(m, n) \mapsto c(m, n)$ is also multiplicative.

3. Subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$

In the case $m = n$, which is of special interest in applications, the results given in the previous section can be easily used. We point out that $n \mapsto s(n) := s(n, n)$ and $n \mapsto c(n) := c(n, n)$ are multiplicative arithmetic functions of a single variable (sequences [3, items A060724, A060648]). They can be written in the form of Dirichlet convolutions as shown by the next corollaries.

Corollary 6. For every $n \in \mathbb{N}$,

$$\begin{aligned} s(n) &= \sum_{de=n} \phi(d) \tau^2(e) \\ &= \sum_{de=n} d \tau(e^2). \end{aligned} \quad (20)$$

Corollary 7. For every $n \in \mathbb{N}$,

$$c(n) = \sum_{de=n} d 2^{\omega(e)} \quad (21)$$

$$= \sum_{de=n} \phi(d) \tau(e^2). \quad (22)$$

Further convolutional representations can also be given; for example,

$$s(n) = \sum_{de=n} \tau(d) \psi(e), \quad c(n) = \sum_{d|n} \psi(d); \quad (23)$$

all of these follow from the Dirichlet-series representations

$$\sum_{n=1}^{\infty} \frac{s(n)}{n^z} = \frac{\zeta^3(z) \zeta(z-1)}{\zeta(2z)}, \quad (24)$$

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^z} = \frac{\zeta^2(z) \zeta(z-1)}{\zeta(2z)}, \quad (25)$$

valid for $z \in \mathbb{C}$, $\Re(z) > 2$.

Observe that

$$s(n) = \sum_{d|n} c(d) \quad (n \in \mathbb{N}), \quad (26)$$

which is a simple consequence of (23) or of (24) and (25). It also follows from the next result.

Theorem 8. For every $n, \delta \in \mathbb{N}$ with $\delta \mid n$, the number of subgroups of exponent δ of $\mathbb{Z}_n \times \mathbb{Z}_n$ equals the number of cyclic subgroups of $\mathbb{Z}_\delta \times \mathbb{Z}_\delta$.

4. Proofs

Proof of Theorem 1. Let H be a subgroup of $G = \mathbb{Z}_m \times \mathbb{Z}_n$. Consider the natural projection $\pi_2 : G \rightarrow \mathbb{Z}_n$ given by $\pi_2(x, y) = y$. Then $\pi_2(H)$ is a subgroup of $\mathbb{Z}_n$ and there is a unique divisor b of n such that $\pi_2(H) = \langle b \rangle := \{jb : 0 \leq j \leq n/b - 1\}$. Let $s \geq 0$ be minimal such that $(s, b) \in H$.

Furthermore, consider the natural inclusion $\iota_1 : \mathbb{Z}_m \rightarrow G$ given by $\iota_1(x) = (x, 0)$. Then $\iota_1^{-1}(H)$ is a subgroup of $\mathbb{Z}_m$ and there exists a unique divisor a of m such that $\iota_1^{-1}(H) = \langle a \rangle$.

We show that $H = \{(ia + js, jb) : i, j \in \mathbb{Z}\}$. Indeed, for every $i, j \in \mathbb{Z}$, $(ia + js, jb) = i(a, 0) + j(s, b) \in H$. On the other hand, for every $(u, v) \in H$ one has $v \in \pi_2(H)$ and hence there is $j \in \mathbb{Z}$ such that $v = jb$. We obtain $(u - js, 0) = (u, v) - j(s, b) \in H$, $u - js \in \iota_1^{-1}(H)$ and there is $i \in \mathbb{Z}$ with $u - js = ia$.

Here a necessary condition is that $(sn/b, 0) \in H$ (obtained for $i = 0$ and $j = n/b$), that is, $a \mid sn/b$, equivalent to $a/\gcd(a, n/b) \mid s$. Clearly, if this is verified, then for the above representation of H it is enough to take the values $0 \leq i \leq m/a - 1$ and $0 \leq j \leq n/b - 1$.

Also, dividing s by a we have $s = aq + r$ with $0 \leq r < a$ and $(r, b) = (s, b) - q(a, 0) \in H$, showing that $s < a$, by its minimality. Hence $s = ta/\gcd(a, n/b)$ with $0 \leq t \leq \gcd(a, n/b) - 1$. Thus we obtain the given representation.

Conversely, every $(a, b, t) \in I_{m,n}$ generates a subgroup $H_{a,b,t}$ of order $mn/(ab)$ of $\mathbb{Z}_m \times \mathbb{Z}_n$ and the proof is complete. $\square$

Proof of Theorem 2. (i)-(ii) We first determine the exponent of the subgroup $H_{a,b,t}$. $H_{a,b,t}$ is generated by $(a, 0)$ and (s, b) ; hence, its exponent is the least common multiple of the orders of these two elements. The order of $(a, 0)$ is m/a . To compute the order of (s, b) note that $m \mid rs$ if and only if $m/\gcd(m, s) \mid r$. Thus the order of (s, b) is $\text{lcm}(m/\gcd(m, s), n/b)$. We deduce that the exponent of $H_{a,b,t}$ is

$$\begin{aligned} & \text{lcm}\left(\frac{m}{a}, \frac{m}{\gcd(m, s)}, \frac{n}{b}\right) \\ &= \text{lcm}\left(\frac{mn}{na}, \frac{mn}{ngcd(m, s)}, \frac{mn}{mb}\right) \\ &= \frac{mn}{\gcd(na, nm, ns, mb)} = \frac{mn}{\gcd(mb, na, ns)} = \beta. \end{aligned} \quad (27)$$

For every finite Abelian group the rank of a nontrivial subgroup is at most the rank of the group. Therefore, the rank of $H_{a,b,t}$ is 1 or 2. That is, $H_{a,b,t} \simeq \mathbb{Z}_A \times \mathbb{Z}_B$ with certain $A, B \in \mathbb{N}$ such that $A \mid B$. Here the exponent of $H_{a,b,t}$ equals that of $\mathbb{Z}_A \times \mathbb{Z}_B$, which is $\text{lcm}(A, B) = B$. Using (ii) already proved we deduce that $B = \beta$. Since the order of $H_{a,b,t}$ is $AB = mn/(ab)$ we have $A = mn/(ab\beta) = \alpha$.

(iii) According to (i) $H_{a,b,t} \simeq \mathbb{Z}_\alpha \times \mathbb{Z}_\beta$, where $\alpha \mid \beta$. Hence $H_{a,b,t}$ is cyclic if and only if $\alpha = 1$.

(iv) The subgroups of $\mathbb{Z}_m$ are of form $\{ia : 0 \leq i \leq m/a - 1\}$, where $a \mid m$, and the properties follow from (6) and (iii). $\square$

Proof of Theorem 3. By its definition, the number of elements of the set $I_{m,n}$ is

$$\begin{aligned} & \sum_{a|m, b|n} \sum_{0 \leq t \leq \gcd(a, n/b) - 1} 1 \\ &= \sum_{a|m, b|n} \gcd\left(a, \frac{n}{b}\right) = \sum_{a|m, b|n} \gcd(a, b), \end{aligned} \quad (28)$$

representing $s(m, n)$. This is formula (10).

To obtain formula (11) apply the Gauss formula $n = \sum_{d|n} \phi(d)$ ($n \in \mathbb{N}$) by writing the following:

$$\begin{aligned} s(m, n) &= \sum_{a|m, b|n} \sum_{d|\gcd(a, b)} \phi(d) = \sum_{\substack{ax=m \\ by=n}} \sum_{\substack{di=a \\ dj=b}} \phi(d) = \sum_{\substack{dix=m \\ djiy=n}} \phi(d) \\ &= \sum_{\substack{du=m \\ dv=n}} \phi(d) \sum_{\substack{ix=u \\ jy=v}} 1 = \sum_{\substack{du=m \\ dv=n}} \phi(d) \tau(u) \tau(v) \\ &= \sum_{d|\gcd(m, n)} \phi(d) \tau\left(\frac{m}{d}\right) \tau\left(\frac{n}{d}\right). \end{aligned} \quad (29)$$

Now (12) follows from (11) by the Busche-Ramanujan identity (cf. [19, Chapter 1]):

$$\tau(m)\tau(n) = \sum_{d|\gcd(m,n)} \tau\left(\frac{mn}{d^2}\right) \quad (m, n \in \mathbb{N}). \quad (30)$$

Proof of Theorem 4. According to Theorem 1, $\square$

$$s_k(m, n) = \sum_{\substack{a|m, b|n \\ mn/ab=k}} \gcd\left(a, \frac{n}{b}\right), \quad (31)$$

giving (13), which can be written, by Gauss' formula again, as

$$\begin{aligned} s_k(m, n) &= \sum_{\substack{a|m, b|n \\ mb/a=k}} \sum_{c|a, c|b} \phi(c) = \sum_{\substack{cix=m \\ c jy=n \\ c jx=k}} \phi(c) \\ &= \sum_{\substack{di=mx=d \\ ey=n \\ ey=n \\ c jx=k}} \phi(c), \end{aligned} \quad (32)$$

where in the inner sum one has $c = de/k$ and obtains (14). Now, to get (15) write (32) as

$$\begin{aligned} s_k(m, n) &= \sum_{\substack{cu=m \\ cv=n \\ cw=k}} \phi(c) \sum_{\substack{ix=u \\ jy=v \\ jx=w}} 1 = \sum_{\substack{cu=m \\ cv=n \\ cw=k}} \phi(c) N(u, v, w) \\ &= \sum_{c|\gcd(m,n,k)} \phi(c) N\left(\frac{m}{c}, \frac{n}{c}, \frac{k}{c}\right), \end{aligned} \quad (33)$$

and the proof is complete. $\square$

Proof of Theorem 5. (i) According to Theorems 1 and 2/(iii) and using that $\sum_{d|n} \mu(d) = 1$ or 0 and according to $n = 1$ or $n > 1$,

$$\begin{aligned} c(m, n) &= \sum_{a|m, b|n} \sum_{\substack{1 \leq s \leq a \\ ab|ns}} 1 = \sum_{\substack{ax=m \\ by=n}} \sum_{\substack{1 \leq s \leq a \\ ar=ys}} 1 \\ &= \sum_{\substack{ax=m \\ by=n}} \sum_{\substack{1 \leq s \leq a \\ ar=ys}} \sum_{e|\gcd(x, y, r)} \mu(e) = \sum_{\substack{aei=m \\ bej=n}} \mu(e) \sum_{\substack{1 \leq s \leq a \\ a/\gcd(a, j)|s}} 1, \end{aligned} \quad (34)$$

where the inner sum is $\gcd(a, j)$. Hence

$$c(m, n) = \sum_{\substack{aei=m \\ bej=n}} \mu(e) \gcd(a, j). \quad (35)$$

Now regrouping the terms according to $ei = z$ and $be = t$ we obtain

$$\begin{aligned} c(m, n) &= \sum_{\substack{az=m \\ jt=n}} \gcd(a, j) \sum_{\substack{ei=z \\ be=t}} \mu(e) = \sum_{\substack{az=m \\ jt=n}} \gcd(a, j) \sum_{e|\gcd(z, t)} \mu(e) \\ &= \sum_{\substack{az=m \\ jt=n \\ \gcd(z, t)=1}} \gcd(a, j), \end{aligned} \quad (36)$$

which is (16).

The next results follow applying Gauss' formula and the Busche-Ramanujan formula, similar to the proof of Theorem 3.

(ii) For the subproducts $H_{a,b,0}$ the values $a | m$ and $b | n$ can be chosen arbitrary and it follows at once that the number of subproducts is $\tau(m)\tau(n)$. The number of cyclic subproducts is

$$\begin{aligned} \sum_{\substack{a|m \\ b|n \\ \gcd(m/a, n/b)=1}} 1 &= \sum_{\substack{ax=m \\ by=n \\ \gcd(x, y)=1}} 1 = \sum_{\substack{ax=m \\ by=n}} \sum_{e|\gcd(x, y)} \mu(e) \\ &= \sum_{\substack{eA=m \\ eB=n}} \mu(e) \tau(A) \tau(B) \\ &= \sum_{e|\gcd(m, n)} \mu(e) \tau\left(\frac{m}{e}\right) \tau\left(\frac{n}{e}\right) = \tau(mn), \end{aligned} \quad (37)$$

by the inverse Busche-Ramanujan identity. $\square$

Proof of Theorem 8. According to Theorem 2/(ii), the number of subgroups of exponent δ of $\mathbb{Z}_n \times \mathbb{Z}_n$ is

$$E_\delta(n) = \sum_{a|n, b|n} \sum_{\substack{1 \leq s \leq a \\ ab|ns \\ n/\gcd(a, b, s)=\delta}} 1 = \sum_{\substack{ax=n \\ by=n}} \sum_{\substack{1 \leq s \leq a \\ ar=ys \\ \gcd(a, b, s)=n/\delta}} 1. \quad (38)$$

Write $a = a_1 n / \delta$, $b = b_1 n / \delta$, and $s = s_1 n / \delta$ with $\gcd(a_1, b_1, s_1) = 1$. We deduce, similar to the proof of Theorem 5/(i), that

$$E_\delta(n) = \sum_{\substack{eix=\delta \\ ejy=\delta}} \mu(e) \gcd(i, y), \quad (39)$$

which is exactly $c(\delta, \delta) = c(\delta)$ (cf. (35)). $\square$

5. Further Remarks

- (1) As mentioned in Section 2 the functions $(m, n) \mapsto s(m, n)$ and $(m, n) \mapsto c(m, n)$ are multiplicative functions of two variables. This follows easily from formulae (10) and (17), respectively. Namely, according to those formulae $s(m, n)$ and $c(m, n)$ are two variables Dirichlet convolutions of the functions $(m, n) \mapsto \gcd(m, n)$ and $(m, n) \mapsto \phi(\gcd(m, n))$, respectively, with the constant 1 function, all multiplicative. Since convolution preserves the multiplicativity we deduce that $s(m, n)$ and $c(m, n)$ are also multiplicative. See [17, Section 2] for details.
- (2) Asymptotic formulas with sharp error terms for the sums $\sum_{m, n \leq x} s(m, n)$ and $\sum_{m, n \leq x} c(m, n)$ were given in the paper [20].
- (3) For any finite groups A and B a subgroup C of $A \times B$ is cyclic if and only if $\iota_1^{-1}(C)$ and $\iota_2^{-1}(C)$ have coprime orders, where ι_1 and ι_2 are the natural inclusions ([21, Theorem 4.2]). In the case $A = \mathbb{Z}_m$, $B = \mathbb{Z}_n$, and $C = H_{a,b,t}$ one has $\#\iota_1^{-1}(C) = m/a$ and $\#\iota_2^{-1}(C) = \gcd(n/b, ns/ab)$ and the characterization of the cyclic subgroups $H_{a,b,t}$ given in Theorem 2/(iii) can be

obtained also in this way. It turns out that regarding the sublattice, $H_{a,b,t}$ is cyclic if and only if the numbers of points on the horizontal and vertical axes, respectively, are relatively prime. Note that in the case $m = n$ the above condition reads $\text{ngcd}(a, b, s) = ab$. Thus it is necessary that $n \mid ab$. The subgroup on Figure 1 is not cyclic.

- (4) Note also the next formula for the number of cyclic subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$, derived in [22, Example 2]:

$$c(n) = \sum_{\text{lcm}(d,e)=n} \gcd(d,e) \quad (n \in \mathbb{N}), \quad (40)$$

where the sum is over all ordered pairs (d, e) such that $\text{lcm}(d, e) = n$. For a short direct proof of (40) write $d = \ell a$, $e = \ell b$ with $\gcd(a, b) = 1$. Then $\gcd(d, e) = \ell$, $\text{lcm}(d, e) = \ell ab$, and obtain

$$\begin{aligned} \sum_{\text{lcm}(d,e)=n} \gcd(d,e) &= \sum_{\substack{\ell ab=n \\ \gcd(a,b)=1}} \ell = \sum_{\ell k=n} \ell \sum_{\substack{ab=k \\ \gcd(a,b)=1}} 1 = \sum_{\ell k=n} \ell 2^{\omega(k)} = c(n), \end{aligned} \quad (41)$$

according to (21).

- (5) Every subgroup K of $\mathbb{Z} \times \mathbb{Z}$ has the representation $K = \{(ia + js, jb) : i, j \in \mathbb{Z}\}$, where $0 \leq s \leq a$ and $0 \leq b$ are unique integers. This follows like in the proof of Theorem 1. Furthermore, in the case $a, b \geq 1$, $0 \leq s \leq a-1$ the index of K is ab and one obtains at once that the number of subgroups K having index n ($n \in \mathbb{N}$) is $\sum_{ab=n} \sum_{0 \leq s \leq a-1} 1 = \sum_{ab=n} a = \sigma(n)$, mentioned in Section 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] M. J. Grady, "A group theoretic approach to a famous partition formula," *The American Mathematical Monthly*, vol. 112, no. 7, pp. 645–651, 2005.
- [2] Y. M. Zou, "Gaussian binomials and the number of sublattices," *Acta Crystallographica*, vol. 62, no. 5, pp. 409–410, 2006.
- [3] "The On-Line Encyclopedia of Integer Sequences," <http://oeis.org/>.
- [4] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, Mass, USA, 2001.
- [5] G. Kutyniok and T. Strohmer, "Wilson bases for general time-frequency lattices," *SIAM Journal on Mathematical Analysis*, vol. 37, no. 3, pp. 685–711 (electronic), 2005.
- [6] A. J. van Leest, *Non-separable Gabor schemes: their design and implementation [Ph.D. thesis]*, Technische Universiteit Eindhoven, Eindhoven, The Netherlands, 2001.
- [7] T. Strohmer, "Numerical algorithms for discrete Gabor expansions," in *Gabor Analysis and Algorithms: Theory and Applications*, H. G. Feichtinger and T. Strohmer, Eds., pp. 267–294, Birkhäuser, Boston, Mass, USA, 1998.
- [8] A. Machi, *Groups: An Introduction to Ideas and Methods of the Theory of Groups*, Springer, 2012.
- [9] J. J. Rotman, *An Introduction to the Theory of Groups*, vol. 148 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 4th edition, 1995.
- [10] R. Schmidt, *Subgroup Lattices of Groups*, de Gruyter Expositions in Mathematics 14, de Gruyter, Berlin, Germany, 1994.
- [11] M. Suzuki, "On the lattice of subgroups of finite groups," *Transactions of the American Mathematical Society*, vol. 70, pp. 345–371, 1951.
- [12] G. Călugăreanu, "The total number of subgroups of a finite abelian group," *Scientiae Mathematicae Japonicae*, vol. 60, no. 1, pp. 157–167, 2004.
- [13] J. Petrillo, "Counting subgroups in a direct product of finite cyclic groups," *College Mathematics Journal*, vol. 42, no. 3, pp. 215–222, 2011.
- [14] M. Tărnăuceanu, "A new method of proving some classical theorems of abelian groups," *Southeast Asian Bulletin of Mathematics*, vol. 31, no. 6, pp. 1191–1203, 2007.
- [15] M. Tărnăuceanu, "An arithmetic method of counting the subgroups of a finite abelian group," *Bulletin Mathématiques de la Société des Sciences Mathématiques de Roumanie*, vol. 53, no. 101, pp. 373–386, 2010.
- [16] W. C. Calhoun, "Counting the subgroups of some finite groups," *The American Mathematical Monthly*, vol. 94, no. 1, pp. 54–59, 1987.
- [17] L. Tóth, "Menon's identity and arithmetical sums representing functions of several variables," *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, vol. 69, no. 1, pp. 97–110, 2011.
- [18] L. Tóth, "On the number of cyclic subgroups of a finite Abelian group," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 55, no. 103, pp. 423–428, 2012.
- [19] P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer, New York, NY, USA, 1986.
- [20] W. G. Nowak and L. Tóth, "On the average number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$," *International Journal of Number Theory*, vol. 10, pp. 363–374, 2014.
- [21] K. Bauer, D. Sen, and P. Zvengrowski, "A generalized Goursat lemma," <http://arxiv.org/abs/1109.0024>.
- [22] A. Pakapongpun and T. Ward, "Functorial orbit counting," *Journal of Integer Sequences*, vol. 12, Article ID 09.2.4, 20 pages, 2009.