Review Article

2D Chaos in the Interaction of Inflation and Unemployment: Moving Averages and the Modeling of High Frequency Macrodynamics

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Received 17 October 2013; Accepted 25 February 2014; Published 18 June 2014

Academic Editor: Grzegorz Litak

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The paper argues that applicable macro is high frequency macro and the data generating process is therefore to be modeled in continuous time. It exemplifies this with a misuse of a 2D period model of monetarist type which becomes extremely overshooting, allowing for routes to “chaos,” when iterated at low frequencies. Instead of such low frequency procedures, we augment the model by a Keynesian feedback chain (the real rate of interest channel) to introduce local instability into the model. We also introduce heterogeneous opinion dynamics into it. The implied 4D dynamics are made bounded thereby, but seem to allow only complex limit cycles, with no transition towards strange attractors anymore.

1. Introduction

The next several sections examine the behavior of a variety of models that differ mainly in how they model real and nominal stickiness. . . . They are formulated in continuous time to avoid the need to use the uninterpretable “one period” delays that plague the discrete time models in this literature [1, p.318].

This quotation can be considered as introducing the objective of this paper in a very pronounced way. We intend to demonstrate in addition to this interpretational riddle that macrodynamic period models are devoid of empirical content if their qualitative features differ from the ones of their continuous time analog. We will use Soliman’s [2] period model in order to demonstrate this from a different angle compared to how it was done in Flaschel and Proaño [3], but could have used equally well more recent approaches by Brianzoni et al. [4] or Roa et al. [5], and indeed many other papers as well. We have chosen the Soliman paper here, since it makes use of a monetarist baseline model—with an estimated wage Phillips curve—a model type which is known to

provide strong point attractors, and so we want to compare it with a Keynesian extension of it.

A basic empirical fact in the macrodynamic literature is, see Flaschel and Proaño [3], that the actual data generating process in macroeconomics (which is generally based on the use of annualized data) is by and large a daily one (and that the data collection frequency is now also much less than a year in the real markets of the economy). This suggests that empirically oriented or estimated macromodels should be iterated with a very short period length as far as actual processes are concerned and will then in general provide the same qualitative answer as their continuous-time analogues. Concerning expectation formation, the data collection process is however of importance and may give rise to certain (smaller) delays in the revision of expectations, which however may be overcome by the formulation of extrapolating expectation mechanisms and other ways by which agents smooth their expectation formation processes. We do not expect here that this implies a major difference between period and continuous time analysis if appropriately modelled, a situation which may however radically change if proper delays, as for example, considered in Invernizzi and

Medio [6], are taken into account. The point made in this paper therefore definitely needs further qualification outside the simple use of the period modelling of this paper.

We discuss in the next section a typical example from the literature (by far not the only one), where “chaos” results from an extremely stable continuous-time approach to inflation dynamics when reformulated as a “long-period” macro-model, and then exhibit a sufficient degree of locally destabilizing overshooting processes which are subsequently tamed by a wage Phillips curve nonlinearity as it was already suggested in Keynes [8] and which generates chaotic trajectories when the long-period based overshooting is made strong enough. Shortening the period lengths in such chaotic macro-models, that is, iterating them with a finer step size, removes by contrast on the one hand “chaos” from such model types, while it on the other hand (and at the same time) brings the model into closer contact with what happens in the data generating process of the real world. (Note in this respect that we focus in this paper on standard period models and therefore do not yet consider, as in Invernizzi and Medio [6] and Medio [9] the role of significant delays and exponential lags in economic activity.)

In contrast to the approach of Soliman [2], we do not believe that the quantity theory gives a direct theory of economic growth, by just deducting the inflation rate from the growth rate of the money supply. We therefore modify in the next section this equation from a Keynesian perspective and explain growth through the stabilizing influence of the Keynes effect (according to which wage level increases are stabilizing, since checked by decreases in effective demand) and the destabilizing influence of the Mundell effect (according to which increases in expected inflation are destabilizing, since they reduce—*ceteris paribus*—the expected real rate of interest and thus stimulate effective demand further). We then have a model type where the steady state is locally unstable if the Mundell effect dominates the Keynes effect, which in our case leads to persistent business cycles in a world where the above assumed wage Phillips curve applies and where inflationary expectations are heterogeneous and of the type as introduced by Charpe et al. [10].

We however cannot find much indication for irregular business fluctuations in the deterministic setup of this model type and thus have to add some noise in order to obtain such fluctuations from this still simple type of Keynesian macro-dynamics. In the framework of Keynesian macrodynamics where real wages matter for income distribution and where we distinguish between wage and price Phillips curves as in Chiarella et al. [11], however, the existence of period-doubling routes to complex business fluctuations can be shown in a 4D continuous-time framework, without the need for overshooting in an otherwise convergent environment. This—as it is well known—demands at least three laws of motion in order to make it possible, in striking contrast to the even 1D period approaches we have quoted previously.

2. The Emergence of Chaos in 2D Monetarist Baseline Analysis

Soliman [2] reconsiders a small textbook monetarist model of inflation dynamics with two laws of motion, one for the rate

of unemployment U_t and one for the expected inflation rate π_t . The model is presented as a period model (in a form, i.e., directly analogous to its familiar continuous-time versions, see Flaschel [12, 5.2]). It makes use of a uniform period length (of approximately one year, as the choice of the model's parameters suggests) in describing with this step size the adjustment of unemployment rate and expected inflation based on labor market disequilibrium and a vertical LM curve (the monetarist case of IS-LM equilibrium). This latter relationship in a stationary economy implies that real growth g_t is given by the discrepancy between nominal money supply growth μ and the actual inflation rate $\hat{p}_t$, to be inserted into Okun's law in order to relate this to the unemployment rate. The model is a nonlinear one due to its use of a nonlinear wage Phillips curve as measured by Lipsey [7] and makes use of constant markup pricing its subsequent derivation of the rate of price inflation.

In continuous time this model reads as follows: with β_π representing the speed of adjustment of inflationary expectations π , a parameter for which no empirical information exists,

$$\dot{U} = b(\hat{p} - \mu) = b(f(U) + \eta\pi - \mu), \quad \eta \in (0, 1), \quad (1)$$

$$\dot{\pi} = \beta_\pi(\hat{p} - \pi) = \beta_\pi(f(U) - (1 - \eta)\pi). \quad (2)$$

It is shown by Flaschel [12, 5.2] that the model is globally asymptotically stable in the domain of positive unemployment rates and can give rise to damped cycles in particular. This also holds for the nonlinear wage PC $f(U) = -1.14 + 5.53/U + 3.68/U^2$ as measured by Lipsey [7] which will also make use of in the following. Note that Soliman assumes a weakened accelerator term $\eta\pi$ in the PC of the model and thus a long-term PC which is negatively sloped, since the steady state of this system is given by

$$\pi^o = \mu = \hat{p}^o, \quad U^o = f^{-1}((1 - \eta)\pi^o), \quad (3)$$

and is thus uniquely determined. It is easy to show that the dynamics stay in the positive half plane if they are started there and that they are globally asymptotically stable in this domain; see Flaschel [12, ch.5] for details.

In period form (with period length h) the dynamics read

$$\begin{aligned} U_{t+h} &= U_t h b(f(U_t) + \eta\pi_t - \mu), \\ \pi_{t+h} &= \pi_t h \beta_\pi(f(U_t) + \eta\pi_t - \pi) \end{aligned} \quad (4)$$

which clearly shows that the impact of the right hand side disequilibria becomes the larger with respect to U_{t+h} and π_{t+h} when the period length h increases. Making the role of the parameter h invisible by setting it equal to one hides this fact and also generally avoids the question of the time length which is involved when iterating the model.

From an applied perspective this is however impossible and since Soliman [2] refers to empirical issues also needed in her case. In this respect one had to distinguish the data generating process (DGP) from the data collection process (DCP). The DGP is of a high-frequency nature. This is obvious for the financial markets but also holds for the real markets, since the

annualized inflation rate is just a moving average with respect to time. This is not to be confused with the DCP concerning the inflation rate which may still be a quarterly one ($h = 0.25$). But taking note of the frequency of the DCP and using such data for estimation do not mean that this frequency (or even a lower one) should be used when the model is simulated in order to study its dynamic implications. On the contrary, the iteration of the model with estimated parameters should come close to the DGP frequency and thus be less than a day $h = 1/365$, in particular if financial markets are present in the model.

In the above-considered case, this gives the result (with which Milton Friedman would have been totally in line) that the steady state of the natural rate inflation dynamics he had in mind is a global attractor, even if there is some downward rigidity involved in money wage formation.

The first equation of the model is implicitly based on Okun's law in the proximate form $\dot{U} = -a\hat{Y}$ and coupled with the quantity theory of money (in growth rate form): $\hat{Y} = \mu - \hat{p}$, where the parameter b was measured as approximately $1/3$ in Okun [13]. Since the quantity theory is a too strict relationship for measuring growth, we expect that the parameter b used in Soliman [2] is closer to $1/10$ than to $1/3$; that is, the time length in his period model is approximately one year (and thus correctly presented through the chosen step size).

Soliman [2] uses this monetarist model of inflation dynamics (where the long-run Phillips curve needs not to be vertical) in order to explore numerical transitions from stable equilibrium points to finally chaotic attractors. This latter result is simply impossible in continuous time, see Hirsch and Smale [14, p.240] for a classification of the limit sets for two-dimensional differential equation systems, and takes place in Soliman's analysis only due to the fact that his model is formulated in discrete time with a uniform and too large period length.

Already in 1975 Foley stated the following [15, p.310]:

The arguments of this section are based on a methodological precept concerning macroeconomic period models: No substantive prediction or explanation in a well-defined macroeconomic period model should depend on the real time length of the period.

Figures 1(a)–1(d) show the original simulation of Soliman [2, p.141]. Comparing the figures top left and right first of all strikingly reveals that there is not much difference between the variation of the growth rate of money supply (on the left) and the period length (on the right). This shows that Soliman needs significant reductions in the frequency of the updating of the macroeconomy in order to achieve what she calls transition into chaos; see Figure 1(d) for its exact measurement. As already stated this lowering of frequency is completely at odds with very basic observations on the working of actual macroeconomies. Moreover, it is very striking to learn that transition to chaos can occur in economics without any change in the behavior of economic agents.

Figure 1(c) stands in striking contrast to the above methodological precept, since it shows in a certain parameter space the areas where cycles of order 1, 2, 4, and 8 prevail in

the simulation example of Soliman if the period length of the employed model is varied. But which period length is then the empirically relevant one? Following Sims, we regard period models as counterfactual so this question cannot be answered empirically. We may use averages of hours or days and iterate them if we want, but this is not what period models are constructed for. And in view of such high frequencies we may of course state all μ less than 11 will provide a correct answer as far as the monetarist baseline model is concerned, but then we are back in a Friedman world and this is not the aim of the Soliman [2] paper. Soliman wants chaotic trajectories and these are on the right hand sides in Figures 1(a), 1(b), 1(c), and 1(d), which are however devoid of empirical content.

We conclude again that the data generating process is to be considered of a much finer step size than the data collection and data processing process (to be performed such that annualized data are established on a quarterly or maybe even monthly basis) and that this leads us to a dynamical system in discrete time (even if period synchronization is assumed) that in general (for most empirically relevant macrodynamic models) is not qualitatively distinguishable in its dynamic properties from its continuous time analogue.

The purpose of the monetarist baseline model considered in this section goes back to Friedman's famous statement:

At any moment of time there is some level of unemployment which has the property that it is consistent with equilibrium in the structure of real wage rates. At that level of unemployment real wages are tending on the average to rise at a "normal" secular rate, that is, at a rate, that can be indefinitely maintained so long as capital formation, technological improvements, and so forth, remain on their long-run trends. A lower level of unemployment is an indication that there is an excess demand for labor that will produce upward pressure on real wage rates. A higher level of unemployment is an indication that there is an excess supply of labor that will produce downward pressure on real wage rates. The "natural rate of unemployment" Friedman [16, p.8],

and it is to show, here largely on the textbook level, what type of dynamics can be used to test this statement theoretically as well as empirically (by means of the estimated nonlinear Phillips curve). It is therefore of no help in such a framework to use the counter-factual assumption of a perfectly synchronized period model with "pork cycle" period length (whether hidden in the parameter μ or as explicitly shown in Figure 1 top right) in order to create for this nonlinear globally asymptotically stable monetarist baseline model a situation where there is so much stress or overheating in the reaction patterns of the dynamics that a route to chaos can be found.

3. Tobinian Business Fluctuations in the Interaction of Inflation and Growth

"Where he deviated from Marshall, and it was a momentous deviation, was in reversing the roles

FIGURE 1: (a) The Soliman [2] bifurcation diagram for the money growth parameter μ . (b) The corresponding model period length h bifurcation diagram. (c) A corresponding parameter basin diagram (for the parameters μ, h). (d) The corresponding largest Lyapunov exponent (for the parameter h).

assigned to price and quantity. He assumed that, at least for changes in aggregate demand, quantity was the variable that adjusted rapidly, while price was the variable that adjusted slowly, at least in a downward direction.” Friedman is correct that this was a momentous deviation, and one way to appreciate the point is to look explicitly at the dynamic stability implications of Walrasian versus Marshallian assumptions about quantity adjustment. Marshallian adjustment in a particular market is that quantity adjusts to the difference between demand price and supply price for existing quantity. Walrasian adjustment is that quantity adjusts to the difference between demand and supply at existing price.

We do not want to discuss Tobin’s distinction concerning the two adjustment processes here but think that the major contribution of his paper is to discuss first the instability aspect of the real rate of interest effect which later became known as the Mundell effect. This effect is missing in the model of the preceding section and it introduces interesting baseline macrodynamics into this framework even on the textbook level, tacitly hidden in all prominent macroeconomic textbooks from Dornbusch and Fisher to Blanchard; see Asada et al. [17] for details. The global dynamic consequences of the destabilizing Mundell effect and its taming by means of downward money wage rigidity and an endogenous switch of the majority of economic agents from the accelerating adaptive expectations scheme of the preceding section to a regressive formation of inflationary expectations will be

the topic of this section which is therefore treating the issue of viability in a Keynesian model where the steady state is repelling.

We continue to use the wage PC of the preceding section which in graphical terms is given by what is shown in Figure 2 but will modify the expectational part of the model to a certain degree. We now use a parameter of unity in the expectational part of the PC (and thus move closer to Friedman again). We however now add heterogeneity in the behavior of agents who establish the wage increase claim π that workers want to achieve. Moreover we do not take the fact that the real rate of interest channel in aggregate demand, the most basic feedback channel of Keynesian IS-LM theory, provides a negative feedback of the price level on economic activity and a positive feedback of the expected rate of inflation on it (which differs from the wage claim inflation target). The latter target is formed by two groups of agents within workers' union, the aggressive fraction and the timid fraction where the timid fraction in first approaches asks for no (targeted) inflation compensation and the aggressive one for full compensation.

We denote by π_w the rate of inflation that is targeted for by workers or their union and by π_c the expected rate of inflation of investors in their calculation of the expected real rate of interest, which enters their investment behavior and thus aggregates demand. For the formation of the latter we postulate in terms of the growth rate g of the economy

$$g = -g_1(p - \bar{p}) - g_2(\hat{p} - (\mu - n)) + g_3(\pi_c - (\mu - n)) - g_4(\pi_w - (\mu - n)) + n, \quad (5)$$

where n is the natural rate of growth of the economy. The expressions behind g_1 , g_2 can be interpreted as representations of the working of the conventional Keynes effect on the growth rate of the economy and the g_3 expression as representing the stimulating Mundell effect; see Tobin [18] for its first formal investigation. The final term can be interpreted as moderation effect in the case where workers decrease their inflation compensation target π_w . Note that we assume $\mu = n$ for the growth rate of the money supply (Friedman's neutrality rule) so that there is no inflation or deflation in the steady state.

As Keynesian extension of the Friedmanian macromodel of the preceding section we now propose against this background

$$\begin{aligned} \hat{e} &= -g_1(p - \bar{p}) + g_2(\pi_c + n - \mu) \\ &\quad + g_3(\pi_c - (\mu - n)) - g_4(\pi_w + n - \mu), \\ \hat{p} &= f(e) + \pi_w, \\ \dot{\pi}_w &= \beta_w \left(\frac{1+x}{2} \hat{p} - \pi_w \right), \\ \dot{\pi}_c &= \beta_c (\hat{p} - \pi_c), \end{aligned} \quad (6)$$

for the dynamics of the rate of employment e , the rate of inflation $\hat{p}$, the inflation compensation π_w targeted for by workers, and the rate of inflation π_c expected by investors. The law of motion for the employment rate is obvious if a Leontief

FIGURE 2: An estimated wage Phillips curve (see [2, 7], with wage inflation on the vertical axis and the unemployment rate on the horizontal axis, both in %).

production function with given coefficients is assumed. The wage PC is the same as in the preceding section. The impact of the rate of inflation on the inflation compensation target of workers is scaled down with the portion of aggressive agents in the board which makes the decision on this target. Finally, inflationary expectations of investors are given as in the preceding section.

There remains the description of the opinion dynamics which determines the two fractions within the union for the determination of the rate π_w for the wage negotiations of the unions. We denote by $x = n_a - n_t$ the difference of their relative proportions (of aggressive and timid members of the union's board of decision makers) and treat this number as a continuum between -1 (only timid members) and $+1$ (all members behave aggressively). For the change in this population structure (classification of union members) we then postulate as law of motion

$$\begin{aligned} \dot{x} &= \beta_x ((1-x) \exp(as_w) - (1+x) \exp(-as_w)), \\ s_w &= s_x x + s_g(g - n) - s_p(p - \bar{p})^2 - s_\pi(\pi - (\mu - n))^2, \end{aligned} \quad (7)$$

where s is the state of confidence about the performance of the economy within the union. It is easily shown that $(1+x)/2$, $(1-x)/2$ are equal to n_a , n_t . The law of motion for their relative difference x therefore simply says that t -agents increase strongly in relative size to a -agents if x gets sufficiently close to -1 while the opposite occurs close to $+1$.

The only problematic feedback chain is the one between the employment rate e and the expected rate of inflation π_c . The Mundell effect stimulates growth through a decreasing expected real rate of interest, which is further decreased through the impact of the employment rate on inflation and thus on expected inflation. This positive feedback channel is of course only a partial one. But it is easy to show that it can be made dominant in the Routh-Hurwitz condition on the minors of order 2 if the parameter β_c is made sufficiently large

FIGURE 3: Limiting limit cycle for e , $\hat{p}$: no opinion dynamics.

FIGURE 5: Limit cycle of Figure 3 with opinion dynamics.

FIGURE 4: The dynamics of employment rates e after some transient phase, for variations of β_x .

FIGURE 6: Time series: employment rate and population share.

(and the sum of these minors made negative thereby). The determinant is also easily shown to be always positive; that is, the stability of the steady state e_o , p_o , 0, 0, 0 can only get lost by way of so-called Hopf bifurcation and thus in particular always in a cyclical fashion. We note that the law of motion for x allows for three steady state values; besides $x = 0$ a negative and a positive one, that is, one where timid behavior dominates and one where the opposite is true. It is shown by Charpe et al. [10] that the latter one can never be a locally asymptotically stable situation.

Turning now to numerical simulations again we will make use here of the (really user-friendly) software E&F Chaos which was developed at the Center for Nonlinear Dynamics and Finance (CeNDEF) at the Faculty of Economics and Business, University of Amsterdam. This tool for the numerical analysis of dynamic period models as well as continuous-time ODE systems can be downloaded from the webpage of the CeNDEF and is discussed in detail (from the viewpoint of period models primarily) by Diks et al. [19].

We first show in Figure 3 the phase plot of the Keynesian dynamics for a stationary population distribution in the board of the union (the program listing and the parameters are provided in the Appendix of this paper). The model is economically viable even then but approaches both boundaries for the rate of employment (zero and one). The ceiling for the employment rate is operative fairly soon—after a short phase of strong explosive behavior while the floor zero is only approached after a longer while (it cannot be reached in a continuous time framework due to the growth rate formulation of the employment dynamic). Figure 4 shows that making the population sizes in the board endogenous does improve the situation, but the increase of the parameter β_v even up to “1” does only remove the worst as Figure 5 exemplifies. Figure 6 finally shows that there are extremely large swings in the population dynamics accompanying the large fluctuations in the employment rate.

Figures 7–10 show bifurcation diagrams in e and v . Figures 7 and 8 do this for the working of the Mundell and the Keynes effect. Figures 9 and 10 do the same for what e call

FIGURE 7: Wage moderation and the return of aggressive wage negotiations through increases in parameter g_3 .

FIGURE 10: Persistent fluctuations in the population share x_t for varying values of parameter g_4 .

FIGURE 8: Decreased employment fluctuations through increases in parameter g_2 .

FIGURE 11: Limit cycle in inflation and employment with no noise.

FIGURE 9: Persistent fluctuations in employment for different values of parameter g_4 .

FIGURE 12: Limit cycle representation of Figure 11 with noise.

the moderation effect; that is, that growth is supported when the inflation rate targeted by workers is decreased. Figure 10 shows in this respect that the initial fluctuations between predominantly aggressive and predominantly timid behavior to only predominantly timid behavior as the corresponding parameter that impacts growth are increased. Surprisingly,

however, predominantly aggressive behavior returns around the threshold value 1; see Figure 10. Analogously, Figures 7 and 9 illustrate the effect of higher values of g_3 and g_4 for the dynamics of e_t , showing indeed the emergence of limit cycles as the one illustrated in Figures 11 and 12. As Figure 10 illustrates, such limit cycles are also associated with persistent

```

emplt=0.94, pt=1, piwt=0, pict=0, xt=0
mu=0.05, bpiw=0.5, bpic=1.5, b1=-1.14,
b2=5.53, b3=3.68, bx=0.5, ax=1, sx=1.2,
sg=4, sp=2, spi=100, g1=0.3, g2=0.8,
g3=0.8, g4=0.5, nn=0.05, barp=1

inflt
c 0.01
Ut = (1-emplt)*100
inflt = (b1+b2/Ut+b3/Ut^2)/100+piwt
ggt = nn-g1*(pt-barp)-g2*(inflt-(mu-nn))
      + g3*(pict-(mu-nn))-g4*(piwt-(mu-nn))
emplt1=emplt*(ggt-nn)
pt1=pt*inflt
piwt1 = bpiw*((1+xt)/2)*inflt-piwt
pict1 = bpic*(inflt-pict)
xt1 = bx*((1-xt)*math.exp(ax*(sx*xt+sg*(ggt-nn))
      - sp*(pt-barp)^2-spi*(pict-(mu-nn))^2))
      - (1+xt)*math.exp(-ax*(sx*xt+sg*(ggt-nn)-sp*(pt-barp)^2
      - spi*(pict-(mu-nn))^2)))
emplt=emplt1
pt=pt1
piwt=piwt1
pict=pict1
xt=xt1

```

ALGORITHM 1: Program listing, including parameter values.

FIGURE 13: Time series representation of the cycle with noise.

fluctuations in the population share x_t . Figure 13 shows time series representation of the cycle with noise.

4. Conclusions and Outlook

The paper has argued that applicable macro is high frequency macro and thus is to be modeled in continuous time from a theoretical point of view (which can be approximated by the Euler period iteration method with a small step size in general, if sensitivity to initial conditions is absent). It exemplified this requirement with a misuse of a baseline monetarist period model. Instead of enforcing low frequency in

FIGURE 14: Bifurcation Routes to Strange Attractors in Continuous Time.

its dynamic iteration, we add from the economic point of view a well-known Keynesian feedback chain (the real rate of interest channel) to introduce instability into such a Friedman world and extended this framework also adding asymptotically rational expectations (ARE) to the adaptive ones (AE) of the Soliman paper and introduced into such an environment with heterogeneous expectation formation an opinion dynamical process that switches between AE and ARE on the basis of certain observations made by the agents on the state of confidence they relate with the working of the economy. The implied dynamics are by and large of limit cycle type and thus represent persistent business cycles, with

no transition into complex dynamics in the now continuous time framework whatsoever, unless further destabilizing feedback channels are added, the wage-price spiral and income distribution dependent aggregate demand, from the Keynesian macrodynamic literature; see Chiarella et al. [11], from whom Figure 14 has been taken as an illustration. Concerning the effects of an increasing price level adjustment speed on the rate of profit obtained by firms, this may create sensitivity to initial conditions also in a framework where adjustment processes are smooth and not overheated by sufficiently strong overshooting dynamics.

As the model is formulated, the case of local explosiveness seems to be giving rise to limit cycle behavior only, but not really to mathematically complex dynamics. This is not exemplified here any further but must be left for future research.

Appendix

For more details see Algorithm 1.

Note that several lines must be single (long) lines in the program itself but that we needed a hard return within them here in order to show them properly. Note also that the first line represents initial conditions and the next ones give the parameter values. The line `inflt` defines an auxiliary variable. The program is started with `c` = continuous time and 0.01 step size (Runge-Kutta method).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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