Research Article

A Reaction-Diffusion System with Nonlinear Nonlocal Boundary Conditions

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We consider initial boundary value problem for a reaction-diffusion system with nonlinear and nonlocal boundary conditions and nonnegative initial data. We prove local existence, uniqueness, and nonuniqueness of solutions.

1. Introduction

In this paper we consider the following semilinear reaction-diffusion system with nonlinear nonlocal boundary conditions:

$$\begin{aligned} u_t &= \Delta u + c_1(x, t) v^p, & v_t &= \Delta v + c_2(x, t) u^q, \\ x &\in \Omega, & 0 < t < T, \\ u(x, t) &= \int_{\Omega} k_1(x, y, t) u^m(y, t) dy, \\ x &\in \partial\Omega, & 0 < t < T, \\ v(x, t) &= \int_{\Omega} k_2(x, y, t) v^n(y, t) dy, \\ x &\in \partial\Omega, & 0 < t < T, \\ u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), \quad x \in \Omega, \end{aligned} \quad (1)$$

where $p, q, m, n > 0$, Ω is a bounded domain in $\mathbb{R}^N$ for $N \geq 1$ with smooth boundary $\partial\Omega$, $T > 0$. Here, $c_1(x, t)$, $c_2(x, t)$ are nonnegative Hölder continuous functions defined for $x \in \overline{\Omega}$, and $t \in [0, T]$ and $k_1(x, y, t)$, $k_2(x, y, t)$ are nonnegative continuous functions defined for $x \in \partial\Omega$, $y \in \overline{\Omega}$, and $t \in [0, T]$. The initial data $u_0(x)$, $v_0(x)$ are nonnegative continuous functions satisfying the boundary conditions at $t = 0$.

In the past several decades, many physical phenomena have been formulated into nonlocal mathematical models. Initial boundary value problem for semilinear reaction-diffusion equations and systems with nonlocal boundary conditions has been analyzed by many authors (see, e.g., [1–12] and the references therein). Local and global existence, comparison principle, and various qualitative properties have been discussed.

We note that for $\max(p, q, m, n) < 1$ the nonlinearities in (1) are non-Lipschitzian. The problem of uniqueness and nonuniqueness for different nonlinear parabolic equations and systems with non-Lipschitzian data has been addressed by several authors. See, for example, [13–16] for equations and [17–21] for systems. In particular, the authors of [16] have investigated the uniqueness of solution for a problem (1) with a single equation.

In [7] the authors have considered a problem (1) with $c_i(x, t) = 1$ and $k_i(x, y, t) = k_i(x, y)$, $i = 1, 2$. They have proved a comparison principle and investigated the blowup properties of the positive solutions. The aim of this paper is to study the uniqueness of the nonnegative solution of the problem (1) for any $p, q, m, n > 0$.

The plan of this paper is as follows. In the next section we prove a comparison principle; an existence theorem of a local solution is given in Section 3; uniqueness of solutions with nontrivial initial data, uniqueness of solution with trivial initial datum for $\min(p, q, m, n) \geq 1$; nonuniqueness of

solution with trivial initial datum for $\min(p, q, m, n) < 1$ are proved in Section 4.

2. Comparison Principle

Let us introduce the definitions of a subsolution and a supersolution. For the remainder of this paper we denote $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$.

Definition 1. A pair of nonnegative functions $(u, v) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is called a subsolution of problem (1) in Q_T if

$$\begin{aligned} u_t &\leq \Delta u + c_1(x, t) v^p, & v_t &\leq \Delta v + c_2(x, t) u^q, \\ x &\in \Omega, & 0 < t < T, \\ u(x, t) &\leq \int_{\Omega} k_1(x, y, t) u^m(y, t) dy, \\ x &\in \partial\Omega, & 0 < t < T, \\ v(x, t) &\leq \int_{\Omega} k_2(x, y, t) v^n(y, t) dy, \\ x &\in \partial\Omega, & 0 < t < T, \\ u(x, 0) &\leq u_0(x), & v(x, 0) &\leq v_0(x), \quad x \in \Omega, \end{aligned} \quad (2)$$

and a pair of nonnegative functions $(u, v) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is called a supersolution of problem (1) in Q_T if the reversed inequalities hold in (2).

Definition 2. A pair of functions (u, v) is called a solution of (1) in Q_T if it is both a subsolution and a supersolution of problem (1) in Q_T .

Definition 3. We say that solution (u, v) of (1) is positive in Q_T if $u > 0$ and $v > 0$ in Q_T .

To establish the uniqueness results we need a comparison principle. We prove it in a different way, not as in the work of [7].

Theorem 4. Let $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ be a nonnegative supersolution and a nonnegative subsolution of problem (1) in Q_T , respectively. Suppose that $\bar{u} > 0$ and $\bar{v} > 0$ or $\underline{u} > 0$ and $\underline{v} > 0$ in $\overline{Q}_T$ if $\min(p, q, m, n) < 1$. If $\bar{u}(x, 0) \geq \underline{u}(x, 0)$ and $\bar{v}(x, 0) \geq \underline{v}(x, 0)$ for $x \in \overline{\Omega}$, then $\bar{u} \geq \underline{u}$ and $\bar{v} \geq \underline{v}$ in $\overline{Q}_T$.

Proof. Let $\xi(x, t) \in C^{2,1}(\overline{Q}_T)$ be a nonnegative function such that $\xi|_{S_T} = 0$. Then $\partial\xi/\partial\nu|_{S_T} \leq 0$, where ν is the unit outward normal to the lateral boundary of Q_T . By the definition of a subsolution we have

$$u_t \leq \Delta u + c_1(x, t) v^p, \quad x \in \Omega, \quad 0 < t < T. \quad (3)$$

If we multiply (3) by ξ and then integrate over Q_t for $0 < t < T$, we get

$$\begin{aligned} &\int_{\Omega} u(x, t) \xi(x, t) dx \\ &\leq \int_{\Omega} u(x, 0) \xi(x, 0) dx \\ &\quad + \int_0^t \int_{\Omega} (u \xi_\tau + u \Delta \xi + c_1(x, \tau) v^p \xi) dx d\tau \\ &\quad - \int_0^t \int_{\partial\Omega} \frac{\partial \xi}{\partial \nu} \left(\int_{\Omega} k_1(x, y, \tau) u^m(y, \tau) dy \right) ds d\tau. \end{aligned} \quad (4)$$

On the other hand, the supersolution $\bar{u}$ satisfies (4) with reversed inequality. Set $w(x, t) = \underline{u}(x, t) - \bar{u}(x, t)$, $z(x, t) = \underline{v}(x, t) - \bar{v}(x, t)$. Then we have

$$\begin{aligned} &\int_{\Omega} w(x, t) \xi(x, t) dx \\ &\leq \int_{\Omega} w(x, 0) \xi(x, 0) dx + \int_0^t \int_{\Omega} w(\xi_\tau + \Delta \xi) dx d\tau \\ &\quad - \int_0^t \int_{\partial\Omega} \frac{\partial \xi}{\partial \nu} \\ &\quad \times \left(\int_{\Omega} k_1(x, y, \tau) m \Theta_2^{m-1}(y, \tau) w(y, \tau) dy \right) ds d\tau \\ &\quad + \int_0^t \int_{\Omega} z(x, t) p c_1(x, t) \Theta_1^{p-1} \xi dx d\tau, \end{aligned} \quad (5)$$

where Θ_1, Θ_2 are nonnegative continuous functions if $\min(p, q, m, n) \geq 1$ and positive continuous functions if $\min(p, q, m, n) < 1$ in $\overline{Q}_T$ which satisfy the following equalities:

$$\begin{aligned} \underline{v}^p - \bar{v}^p &= p \Theta_1^{p-1} (\underline{v} - \bar{v}), \\ \underline{u}^m - \bar{u}^m &= m \Theta_2^{m-1} (\underline{u} - \bar{u}). \end{aligned} \quad (6)$$

Obviously there exists a positive constant M_1 such that $\Theta_1^{p-1} \leq M_1$, $\Theta_2^{m-1} \leq M_1$ in $\overline{Q}_T$. Since $c_1(x, t)$, $k_1(x, y, t)$ are nonnegative and continuous functions, then there exists a constant $M_2 > 0$ such that $0 \leq c_1(x, t) \leq M_2$, $0 \leq k_1(x, y, t) \leq M_2$ in $\overline{Q}_T$ and $\partial\Omega \times \overline{Q}_T$, respectively.

Consider the following backward problem in Q_t :

$$\begin{aligned} \xi_\tau + \Delta \xi &= 0, \quad (x, \tau) \in Q_t, \\ \xi &= 0, \quad (x, \tau) \in S_t, \\ \xi(x, t) &= \psi(x), \quad x \in \Omega, \end{aligned} \quad (7)$$

where $\psi(x) \in C_0^\infty(\Omega)$, $0 \leq \psi(x) \leq 1$. By the maximum principle for the heat equation $0 \leq \xi \leq 1$, it is easy to show that $-M_3 \leq \partial\xi/\partial\nu \leq 0$ on S_t for some $M_3 \geq 0$.

Let $s_+ = \max(s, 0)$. Then from (5), we get

$$\begin{aligned} \int_{\Omega} w(x, t) \psi(x) dx &\leq K_1 \int_0^t \int_{\Omega} z_+(x, t) dx dt \\ &\quad + K_2 \int_0^t \int_{\Omega} w_+(y, \tau) dy d\tau, \end{aligned} \quad (8)$$

where $K_1 = pM_1M_2$, $K_2 = mM_1M_2M_3|\partial\Omega|$, and $|\partial\Omega|$ is the Lebesgue measure of $\partial\Omega$. Consider the sequence of functions $\{\phi_n(x)\}$, $\phi_n(x) \in C_0^\infty(\bar{\Omega})$, $0 \leq \phi_n \leq 1$, converging in $L^1(\Omega)$ to $\phi(x)$, defined as follows:

$$\begin{aligned}\phi(x) &= 1, & \text{if } w(x, t) > 0, \\ \phi(x) &= 0, & \text{if } w(x, t) \leq 0.\end{aligned}\quad (9)$$

Replacing $\psi(x)$ by $\phi_n(x)$ in (8) and passing to the limit as $n \rightarrow \infty$, we have

$$\begin{aligned}\int_{\Omega} w_+(x, t) dx &\leq K_1 \int_0^t \int_{\Omega} z_+(x, t) dx dt \\ &+ K_2 \int_0^t \int_{\Omega} w_+(y, \tau) dy d\tau.\end{aligned}\quad (10)$$

Using a similar argument for the inequality $v_t \leq \Delta v + c_2(x, t)u^q$, $x \in \Omega$, $0 < t < T$, we get

$$\begin{aligned}\int_{\Omega} z_+(x, t) dx &\leq K_3 \int_0^t \int_{\Omega} w_+(x, t) dx dt \\ &+ K_4 \int_0^t \int_{\Omega} z_+(y, \tau) dy d\tau,\end{aligned}\quad (11)$$

where K_3, K_4 are positive constants. Adding (10) and (11), we have

$$\begin{aligned}\int_{\Omega} (w_+(x, t) + z_+(x, t)) dx \\ \leq K_5 \int_0^t \int_{\Omega} (w_+(x, t) + z_+(x, t)) dx dt,\end{aligned}\quad (12)$$

where $K_5 = \max(K_1 + K_4, K_2 + K_3)$.

Applying now Gronwall's inequality, we conclude that

$$\int_{\Omega} (w_+(x, t) + z_+(x, t)) dx \leq 0. \quad (13)$$

Since $w_+ + z_+ \geq 0$, then $\underline{u}(x, t) - \bar{u}(x, t) \leq 0$, $\underline{v}(x, t) - \bar{v}(x, t) \leq 0$. $\square$

3. Local Existence

Let $\{\varepsilon_l\}$ be decreasing to 0 sequence such that $0 < \varepsilon_l < 1$. For $\varepsilon = \varepsilon_l$ let $u_{0\varepsilon}(x)$, $v_{0\varepsilon}(x)$ be the functions with the following properties: $u_{0\varepsilon}(x), v_{0\varepsilon}(x) \in C(\bar{\Omega})$, $u_{0\varepsilon}(x) \geq \varepsilon$, $v_{0\varepsilon}(x) \geq \varepsilon$; $u_{0\varepsilon_i}(x) \geq u_{0\varepsilon_j}(x)$, $v_{0\varepsilon_i}(x) \geq v_{0\varepsilon_j}(x)$ for $i < j$; $u_{0\varepsilon}(x) \rightarrow u_0(x)$, $v_{0\varepsilon}(x) \rightarrow v_0(x)$ as $\varepsilon \rightarrow 0$; $u_{0\varepsilon}(x) = \int_{\Omega} k_1(x, y, 0)u_{0\varepsilon}^m(y)dy + \varepsilon$, $v_{0\varepsilon}(x) = \int_{\Omega} k_2(x, y, 0)v_{0\varepsilon}^n(y)dy + \varepsilon$ for $x \in \partial\Omega$.

Due to the nonlinearities in (1), the Lipschitz condition is not satisfied if $\min(p, q, m, n) < 1$, and thus we need to consider the following auxiliary problem:

$$\begin{aligned}u_t &= \Delta u + c_1(x, t)v^p, & v_t &= \Delta v + c_2(x, t)u^q, \\ x &\in \Omega, & 0 < t < T,\end{aligned}$$

$$\begin{aligned}u(x, t) &= \int_{\Omega} k_1(x, y, t)u^m(y, t)dy + \varepsilon, \\ x &\in \partial\Omega, & 0 < t < T,\end{aligned}\quad (14)$$

$$\begin{aligned}v(x, t) &= \int_{\Omega} k_2(x, y, t)v^n(y, t)dy + \varepsilon, \\ x &\in \partial\Omega, & 0 < t < T,\end{aligned}$$

$$u(x, 0) = u_{0\varepsilon}(x), \quad v(x, 0) = v_{0\varepsilon}(x), \quad x \in \Omega.$$

Theorem 5. For small values of T , (14) has a unique solution in Q_T .

Proof. We start the proof with the construction of a supersolution of (14). Let $\max(\sup_{\Omega} u_{0\varepsilon}(x), \sup_{\Omega} v_{0\varepsilon}(x)) \leq C$, $C > 0$. Denote $M = \max(\sup_{Q_T} c_1(x, t), \sup_{Q_T} c_2(x, t), \sup_{\partial\Omega \times Q_T} k_1(x, y, t), \sup_{\partial\Omega \times Q_T} k_2(x, y, t))$. Introduce an auxiliary function $\varphi(x)$ with the following properties:

$$\begin{aligned}\varphi(x) &\in C^2(\bar{\Omega}), \\ \inf_{\Omega} \varphi(x) &\geq 1,\end{aligned}\quad (15)$$

$$\inf_{\partial\Omega} \varphi(x) \geq \bar{M} \int_{\Omega} \max(\varphi^m(y), \varphi^n(y)) dy + C^{-1},$$

where $\bar{M} = M \max(C^{m-1}, C^{n-1}) \max(1, \exp(m-1), \exp(n-1))$. Let α, β be positive constants such that $\alpha q - \beta = \beta p - \alpha$ and

$$\begin{aligned}\alpha &\geq \sup_{\Omega} \left(\frac{\Delta \varphi}{\varphi} + MC^{p-1} \exp(1) \varphi^{p-1} \right), \\ \beta &\geq \sup_{\Omega} \left(\frac{\Delta \varphi}{\varphi} + MC^{q-1} \exp(1) \varphi^{q-1} \right).\end{aligned}\quad (16)$$

Note that $\alpha q - \beta > 0$ for $pq > 1$ and $\alpha q - \beta \leq 0$ for $pq \leq 1$. Obviously the pair of functions $(\varepsilon, \varepsilon)$ is a subsolution of problem (14). We show that

$$\begin{aligned}f(x, t) &= C \exp(\alpha t) \varphi(x), \\ g(x, t) &= C \exp(\beta t) \varphi(x)\end{aligned}\quad (17)$$

is a supersolution of (14) in Q_T for $T \leq \min(1/\alpha, 1/\beta, 1/(\alpha q - \beta))$ if $pq > 1$ and $T \leq \min(1/\alpha, 1/\beta)$ if $pq \leq 1$. We have

$$\begin{aligned} f_t(x, t) - \Delta f(x, t) - c_1(x, t) g^p(x, t) \\ = \alpha C \exp(\alpha t) \varphi(x) - \Delta \varphi C \exp(\alpha t) \\ - c_1(x, t) C^p \exp(\beta p t) \varphi^p(x) \\ \geq f(x, t) \left(\alpha - \frac{\Delta \varphi(x)}{\varphi(x)} - M C^{p-1} \exp((\beta p - \alpha)t) \varphi^{p-1} \right) \\ \geq 0, \end{aligned} \quad (18)$$

for $(x, t) \in Q_T$. On the other hand, we get that

$$\begin{aligned} C \exp(\alpha t) \varphi(x) &\geq C^m \exp(\alpha m t) M \int_{\Omega} \varphi^m(y) dy + 1 \\ &\geq \int_{\Omega} k_1(x, y, t) f^m(y, t) dy + \varepsilon, \end{aligned} \quad (19)$$

for $(x, t) \in S_T$. Similarly, we can show that

$$\begin{aligned} g_t(x, t) - \Delta g(x, t) - c_2(x, t) f^q(x, t) &\geq 0, \quad \text{for } (x, t) \in Q_T, \\ g(x, t) &\geq \int_{\Omega} k_2(x, y, t) g^n(y, t) dy + \varepsilon, \quad \text{for } (x, t) \in S_T. \end{aligned} \quad (20)$$

To prove the existence of a solution of the problem (14) we introduce the set

$$\begin{aligned} B = \{ (h_1, h_2) \in C(\overline{Q}_T) \times C(\overline{Q}_T) : \varepsilon \leq h_1 \leq f, \\ \varepsilon \leq h_2 \leq g, h_1(x, 0) = u_{0\varepsilon}(x), \\ h_2(x, 0) = v_{0\varepsilon}(x) \}. \end{aligned} \quad (21)$$

Clearly B is a nonempty convex subset of $C(\overline{Q}_T) \times C(\overline{Q}_T)$.

Consider the following problem:

$$\begin{aligned} u_t &= \Delta u + c_1(x, t) v^p, \quad v_t = \Delta v + c_2(x, t) u^q, \\ x &\in \Omega, \quad 0 < t < T, \\ u(x, t) &= \int_{\Omega} k_1(x, y, t) s_1^m(y, t) dy + \varepsilon, \\ x &\in \partial\Omega, \quad 0 < t < T, \\ v(x, t) &= \int_{\Omega} k_2(x, y, t) s_2^n(y, t) dy + \varepsilon, \\ x &\in \partial\Omega, \quad 0 < t < T, \\ u(x, 0) &= u_{0\varepsilon}(x), \quad v(x, 0) = v_{0\varepsilon}(x), \quad x \in \Omega, \end{aligned} \quad (22)$$

where $(s_1, s_2) \in B$. Problem (22) has a nontrivial positive solution. Let us call $A(s_1, s_2) = (u, v)$. In order to show that A has a fixed point in B we verify that A is a continuous mapping from B into itself such that AB is relatively compact. Thanks

to the comparison principle for (22) we have that A maps B into itself.

Let $G(x, y, t)$ denote the Green's function for the heat equation given by

$$u_t - \Delta u = 0, \quad x \in \Omega, \quad t > 0 \quad (23)$$

with zero boundary condition. Then (u, v) is a solution of (22) if and only if

$$\begin{aligned} u(x, t) &= \int_{\Omega} G(x, y, t) u_{0\varepsilon}(y) dy \\ &+ \int_0^t \int_{\Omega} G(x, y, t - \tau) c_1(y, \tau) v^p(y, \tau) dy d\tau \\ &- \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi, t - \tau) \\ &\times \left(\int_{\Omega} k_1(\xi, y, \tau) s_1^m(y, \tau) dy + \varepsilon \right) d\xi d\tau, \end{aligned} \quad (24)$$

$$\begin{aligned} v(x, t) &= \int_{\Omega} G(x, y, t) v_{0\varepsilon}(y) dy \\ &+ \int_0^t \int_{\Omega} G(x, y, t - \tau) c_2(y, \tau) u^q(y, \tau) dy d\tau \\ &- \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi, t - \tau) \\ &\times \left(\int_{\Omega} k_2(\xi, y, \tau) s_2^n(y, \tau) dy + \varepsilon \right) d\xi d\tau. \end{aligned} \quad (25)$$

We claim that A is continuous. In fact, let $\{(s_{1k}, s_{2k})\}$ be a sequence in B converging to $(s_1, s_2) \in B$ in $C(\overline{Q}_T) \times C(\overline{Q}_T)$. Denote $(u_k, v_k) = A(s_{1k}, s_{2k})$. Then we see that

$$\begin{aligned} |u - u_k| + |v - v_k| \\ \leq \int_0^t \int_{\Omega} G(x, y, t - \tau) c_1(y, \tau) |v^p - v_k^p| dy d\tau \\ + \int_0^t \int_{\Omega} G(x, y, t - \tau) c_2(y, \tau) |u^q - u_k^q| dy d\tau \\ - \sup_{Q_T} |s_1^m - s_{1k}^m| \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi, t - \tau) \\ \times \int_{\Omega} k_1(\xi, y, \tau) dy d\xi d\tau \\ - \sup_{Q_T} |s_2^n - s_{2k}^n| \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi, t - \tau) \\ \times \int_{\Omega} k_2(\xi, y, \tau) dy d\xi d\tau \end{aligned}$$

$$\begin{aligned}
&\leq r \left(\sup_{Q_T} |u - u_k| + \sup_{Q_T} |v - v_k| \right) \\
&\quad - \sup_{Q_T} |s_1^m - s_{1k}^m| \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi; t - \tau) \\
&\quad \times \int_{\Omega} k_1(\xi, y, \tau) dy d\xi d\tau \\
&\quad - \sup_{Q_T} |s_2^n - s_{2k}^n| \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi; t - \tau) \\
&\quad \times \int_{\Omega} k_2(\xi, y, \tau) dy d\xi d\tau,
\end{aligned} \tag{26}$$

where $r = \max(\Theta_1, \Theta_2)$ and

$$\begin{aligned}
\Theta_1 &= p \max \left(\varepsilon^{p-1}, \sup_{Q_T} h_1^{p-1}(x, t) \right) \\
&\quad \times \sup_{Q_T} \int_0^t \int_{\Omega} G(x, y; t - \tau) c_1(y, \tau) dy d\tau, \\
\Theta_2 &= q \max \left(\varepsilon^{q-1}, \sup_{Q_T} h_2^{q-1}(x, t) \right) \\
&\quad \times \sup_{Q_T} \int_0^t \int_{\Omega} G(x, y; t - \tau) c_2(y, \tau) dy d\tau.
\end{aligned} \tag{27}$$

Choosing T so small that $r < 1$, we conclude that $(u_k, v_k) \rightarrow (u, v)$ in $C(\overline{Q_T}) \times C(\overline{Q_T})$ as $k \rightarrow \infty$. The equicontinuity of AB follows from (24), (25) and the properties of the Green's function (see, e.g. [22]). The Ascoli-Arzelà theorem guarantees the relative compactness of AB . Thus, we are able to apply the Schauder-Tychonoff fixed point theorem and conclude that A has a fixed point in B if T is small. Since (u, v) is a fixed point of A , it is a solution of (14). Uniqueness of solution follows from a comparison principle for (14) which can be proved in a similar way as in the previous section. $\square$

Using Theorem 5, we can prove the following local existence theorem of a solution of problem (1).

Theorem 6. *For small values of T (1) has a maximal solution in Q_T .*

Proof. Let $\varepsilon_2 > \varepsilon_1$. It is easy to show that $(u_{\varepsilon_2}(x, t), v_{\varepsilon_2}(x, t))$ is a supersolution of the problem (14) with $\varepsilon = \varepsilon_1$. Then $u_{\varepsilon_1}(x, t) \leq u_{\varepsilon_2}(x, t)$, $v_{\varepsilon_1}(x, t) \leq v_{\varepsilon_2}(x, t)$. Using these inequalities and the continuation principle of solutions we deduce that the existence time of $(u_{\varepsilon}(x, t), v_{\varepsilon}(x, t))$ does not decrease as $\varepsilon \rightarrow 0$. Let $\varepsilon \rightarrow 0$, then

$$\begin{aligned}
u_{\max}(x, t) &= \lim_{\varepsilon \rightarrow 0} u_{\varepsilon}(x, t) \geq 0, \\
v_{\max}(x, t) &= \lim_{\varepsilon \rightarrow 0} v_{\varepsilon}(x, t) \geq 0,
\end{aligned} \tag{28}$$

and $(u_{\max}(x, t), v_{\max}(x, t))$ exist in Q_T for some $T > 0$.

Moreover, by dominated convergence theorem, $(u_{\max}(x, t), v_{\max}(x, t))$ satisfies the following equations:

$$\begin{aligned}
u_{\max}(x, t) &= \int_{\Omega} G(x, y; t) u_0(y) dy \\
&\quad + \int_0^t \int_{\Omega} G(x, y; t - \tau) c_1(y, \tau) \\
&\quad \times v_{\max}^p(y, \tau) dy d\tau \\
&\quad - \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi; t - \tau) \\
&\quad \times \left(\int_{\Omega} k_1(\xi, y, \tau) u_{\max}^m(y, \tau) dy \right) d\xi d\tau, \\
v_{\max}(x, t) &= \int_{\Omega} G(x, y; t) v_0(y) dy \\
&\quad + \int_0^t \int_{\Omega} G(x, y; t - \tau) c_2(y, \tau) \\
&\quad \times u_{\max}^q(y, \tau) dy d\tau \\
&\quad - \int_0^t \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, \xi; t - \tau) \\
&\quad \times \left(\int_{\Omega} k_2(\xi, y, \tau) v_{\max}^n(y, \tau) dy \right) d\xi d\tau.
\end{aligned} \tag{29}$$

The interior regularity of $(u_{\max}(x, t), v_{\max}(x, t))$ follows from the continuity of $(u_{\max}(x, t), v_{\max}(x, t))$ in Q_T and the properties of the Green's function. Obviously $(u_{\max}(x, t), v_{\max}(x, t))$ satisfies (1). Let $(y_1(x, t), y_2(x, t))$ be any other solution of (1). Then by the comparison principle $u_{\varepsilon}(x, t) \geq y_1(x, t)$, $v_{\varepsilon}(x, t) \geq y_2(x, t)$. Taking $\varepsilon \rightarrow 0$, we conclude that $u_{\max} \geq y_1(x, t)$, $v_{\max}(x, t) \geq y_2(x, t)$. $\square$

To prove the positiveness of nontrivial solutions we need the following definition.

Definition 7. We say that a function $g(x, t)$ has the property (N) if there exist $x_k \in \Omega$ and $t_k > 0$, $k \in \mathbb{N}$ such that $g(x_k, t_k) > 0$, $k \in \mathbb{N}$, and $t_k \rightarrow 0$ as $k \rightarrow \infty$.

Remark 8. Note that if a nonnegative function $g(x, t)$ has no the property (N) then $g(x, t) \equiv 0$ in Q_T for some $\tau > 0$.

Theorem 9. *Let either $u_0(x)$ or $v_0(x)$ be a nontrivial function in Ω . Supposing that $k_1(x, \cdot, t)$ and $k_2(x, \cdot, t)$ are nontrivial functions for any $x \in \partial\Omega$ and $t \in (0, T)$, $c_1(x, t)$ has the property (N) if $u_0(x) \equiv 0$, and $c_2(x, t)$ has the property (N) if $v_0(x) \equiv 0$. Let $(u(x, t), v(x, t))$ be a supersolution of (1) in Q_T . Then $(u(x, t), v(x, t))$ is positive in $\overline{Q_T}$ for $0 < t < T$.*

Proof. Suppose for definiteness that $u_0(x)$ is a nontrivial function. We show at first that $u(x, t) > 0$ in $\overline{Q_T}$ for $0 < t < T$. We have

$$u_t - \Delta u \geq c_1(x, t) v^p \geq 0; \tag{30}$$

then by strong maximum principle a minimum of $u(x, t)$ should be attained in $\overline{Q_T}$ on a parabolic boundary. Thus,

$u(x, t) > 0$ in Q_T , otherwise, there would be a contradiction with the initial datum. We show that $u(x, t) > 0$ on $\partial\Omega \times (0, T)$. Let there exist a point $(x_0, t_0) \in S_T$ such that $u(x_0, t_0) = 0$. But $u(x, t_0) > 0$ for $x \in \Omega$. By boundary conditions (1) and assumption for $k_1(x, y, t)$ we have $u(x, t) > 0$ for $(x, t) \in S_T$. This contradiction shows that $u(x, t) > 0$ on S_T , and therefore $u(x, t) > 0$ in $\bar{Q}_T$ for $0 < t < T$.

Now we show the positiveness of $v(x, t)$. If $v_0(x)$ is a nontrivial function, then $v(x, t) > 0$ in $\bar{Q}_T$ for $0 < t < T$ by previous arguments. If $v_0(x) \equiv 0$ we suppose that there exists a constant $\tau > 0$ such that $v(x, t) \equiv 0$ in Q_τ since otherwise we can use the arguments from the beginning of the proof again. But this is a contradiction with the second equation in (1) since $u(x, t) > 0$ in Q_τ and $c_2(x, t)$ has the property (N). Hence, we conclude that $v(x, t) > 0$ in $\bar{Q}_T$ for $0 < t < T$. $\square$

4. Uniqueness and Nonuniqueness

As a simple consequence of Theorem 4 and Theorem 9 we get the first uniqueness result for problem (1).

Theorem 10. *Let problem (1) have a solution in Q_T with nonnegative initial datum for $\min(p, q, m, n) \geq 1$ and with positive initial datum under conditions $\min(p, q, m, n) < 1$ and $k_1(x, \cdot, t)$, $k_2(x, \cdot, t)$ are nontrivial functions for any $x \in \partial\Omega$ and $t \in (0, T)$. Then solution of (1) is unique in Q_T .*

Now we show nonuniqueness of solutions of problem (1) with trivial initial datum for $\min(pq, m, n) < 1$.

Theorem 11. *Let $\min(pq, m, n) < 1$, $u_0(x) = v_0(x) \equiv 0$. Suppose that the maximal solution of problem (1) exists in Q_T . Assume that at least one of the following conditions is fulfilled:*

$$c_1(x_0, t_0) > 0, \quad c_2(x_0, t_0) > 0 \quad (31)$$

for some $x_0 \in \Omega$, $t_0 \in [0, T)$ if $pq < 1$,

$$k_1(x, y_1, t_1) > 0$$

for any $x \in \partial\Omega$, and some $y_1 \in \partial\Omega$, $t_1 \in [0, T)$ if $m < 1$, (32)

$$k_2(x, y_2, t_2) > 0$$

for any $x \in \partial\Omega$, and some $y_2 \in \partial\Omega$, $t_2 \in [0, T)$ if $n < 1$. (33)

Then the maximal solution of problem (1) is nontrivial in Q_T .

Proof. In the local existence theorem we constructed a maximal solution $(u_{\max}(x, t), v_{\max}(x, t))$ of (1) in the following way: $u_{\max}(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$, $v_{\max}(x, t) = \lim_{\varepsilon \rightarrow 0} v_\varepsilon(x, t)$, where $(u_\varepsilon(x, t), v_\varepsilon(x, t))$ is some positive supersolution of (1). To prove the theorem we construct a nontrivial nonnegative subsolution $(\underline{u}(x, t), \underline{v}(x, t))$ of some problem with trivial initial datum. By the comparison principle we conclude that $u_\varepsilon(x, t) \geq \underline{u}(x, t)$, $v_\varepsilon(x, t) \geq \underline{v}(x, t)$, and therefore maximal solution is a nontrivial solution.

Consider at first the case when $pq < 1$ and $c_1(x_0, t_0) > 0$, $c_2(x_0, t_0) > 0$ for some $x_0 \in \Omega$ and $0 \leq t_0 < T$. Then

there exists a neighborhood $U(x_0)$ of x_0 in Ω and $\bar{t} \in (t_0, T)$ such that $c_1(x, t) \geq c_0$, $c_2(x, t) \geq c_0$, $c_0 > 0$ for $x \in U(x_0)$ and $t_0 \leq t \leq \bar{t}$.

Consider the following problem:

$$\begin{aligned} w_t &= \Delta w, \quad x \in U(x_0), \quad t_0 < t < \bar{t}, \\ w(x, t) &= 0, \quad x \in \partial U(x_0), \quad t_0 < t < \bar{t}, \\ w(x, t_0) &= w_0(x), \quad x \in U(x_0), \end{aligned} \quad (34)$$

where $w_0(x)$ is a bounded nontrivial nonnegative continuous function which satisfies a boundary condition. By the strong maximum principle $0 < w(x, t) < M_0$ for $t \in (t_0, \bar{t})$, $x \in U(x_0)$, where $M_0 = \sup_{x \in U(x_0)} w_0(x)$.

Let $\alpha = (1 + p)/(1 - pq) > 1$, $\beta = (1 + q)(1 - pq) > 1$. Denote that

$$\begin{aligned} \underline{u}(x, t) &= C^\alpha (t - t_0)^\alpha w^\alpha(x, t), \\ \underline{v}(x, t) &= C^\beta (t - t_0)^\beta w^\beta(x, t), \end{aligned} \quad (35)$$

where $C = \min(c_0/(\alpha M_0), c_0/(\beta M_0))$. Note that $\underline{u}(x, t) = 0$, $\underline{v}(x, t) = 0$ if $t = t_0$ or $x \in \partial U(x_0)$. After simple calculations we obtain

$$\begin{aligned} \underline{u}_t - \Delta \underline{u} - c_1(x, t) \underline{v}^p &= \alpha C^\alpha (t - t_0)^{\alpha-1} w^\alpha \\ &+ \alpha C^\alpha (t - t_0)^\alpha w^{\alpha-1} w_t - \alpha \Delta w w^{\alpha-1} C^\alpha (t - t_0)^\alpha \\ &- \alpha(\alpha - 1) w^{\alpha-2} |\nabla w|^2 C^\alpha (t - t_0)^\alpha \\ &- c_1(x, t) C^{\beta p} (t - t_0)^{\beta p} w^{\beta p} \\ &\leq C^{\beta p} (t - t_0)^{\beta p} w^{\beta p} (\alpha C w - c_1(x, t)) \leq 0. \end{aligned} \quad (36)$$

Similarly we can get that

$$\underline{v}_t - \Delta \underline{v} - c_2(x, t) \underline{u}^q \leq 0. \quad (37)$$

Then $(\underline{u}(x, t), \underline{v}(x, t))$ and $(u_\varepsilon(x, t), v_\varepsilon(x, t))$ are subsolution and supersolution, respectively, of the following problem:

$$\begin{aligned} u_t &= \Delta u + c_1(x, t) v^p, \\ v_t &= \Delta v + c_2(x, t) u^q, \\ x &\in U(x_0), \quad t_0 < t < \bar{t}, \\ u(x, t) &= v(x, t) = 0, \\ x &\in \partial U(x_0), \quad t_0 < t < \bar{t}, \\ u(x, t_0) &= v(x, t_0) = 0, \\ x &\in U(x_0). \end{aligned} \quad (38)$$

By comparison principle for (38) we conclude that $u_\varepsilon(x, t) \geq \underline{u}(x, t)$, $v_\varepsilon(x, t) \geq \underline{v}(x, t)$ and hence $u_{\max}(x, t) \geq \underline{u}(x, t)$, $v_{\max}(x, t) \geq \underline{v}(x, t)$, for $(x, t) \in U(x_0) \times (t_0, \bar{t})$.

Now consider the case when $m < 1$ and $k_1(x, y_1, t_1) > 0$ for any $x \in \partial\Omega$ and some $y_1 \in \partial\Omega$, $0 \leq t_1 < T$. We will consider the following problem:

$$\begin{aligned} u_t &= \Delta u + c_1(x, t) v^p, \\ v_t &= \Delta v + c_2(x, t) u^q, \\ x &\in \Omega, \quad t_1 < t < T_0, \\ u(x, t) &= \int_{\Omega} k_1(x, y, t) u^m(y, t) dy, \\ x &\in \partial\Omega, \quad t_1 < t < T_0, \\ v(x, t) &= 0, \quad x \in \partial\Omega, \quad t_1 < t < T_0, \\ u(x, t_1) &= 0, \quad v(x, t_1) = 0, \quad x \in \Omega, \end{aligned} \quad (39)$$

where $T_0 > t_1$ will be defined later. Construct a subsolution of (39) using the change of variables in a neighborhood of $\partial\Omega$ as in [23]. Let $\bar{x} \in \partial\Omega$. We denote by $\bar{n}$ the inner unit normal to $\partial\Omega$ at the point $\bar{x}$. Since $\partial\Omega$ is smooth it is well known that there exists $\delta > 0$ such that the mapping $\psi : \partial\Omega \times [0, \delta] \rightarrow \mathbb{R}^N$ given by $\psi(\bar{x}, s) = \bar{x} + s\bar{n}(\bar{x})$ defines new coordinates $(\bar{x}, s)$ in a neighborhood of $\partial\Omega$ in $\bar{\Omega}$. A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\bar{x}, s) = g(s)$ which is independent of the variable $\bar{x}$, evaluated at a point $(\bar{x}, s)$ is given by

$$\Delta g(\bar{x}, s) = \frac{\partial^2 g}{\partial s^2}(\bar{x}, s) - \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \frac{\partial g}{\partial s}(\bar{x}, s), \quad (40)$$

where $H_j(\bar{x})$, $j = 1, \dots, N-1$ denote the principal curvatures of $\partial\Omega$ at $\bar{x}$.

Under the made assumption there exists $\tilde{t} > 0$ such that $k_1(x, y, t) > 0$ for $t_1 \leq t \leq t_1 + \tilde{t}$, $x \in \partial\Omega$, $y \in V(y_1)$, where $V(y_1)$ is some neighborhood of y_1 in $\bar{\Omega}$. Let $\alpha > 1/(2(1-m))$ and assume that $0 < \xi_0 \leq 1$ and $t_1 < T_0 \leq t_1 + \min(\tilde{t}, \delta^2, T - t_1)$. For points in $\partial\Omega \times [0, \delta] \times (t_1, T_0]$ of coordinates $(\bar{x}, s, t)$ define

$$\begin{aligned} \underline{u}(\bar{x}, s, t) &= (t - t_1)^\alpha \left(\xi_0 - \frac{s}{\sqrt{t - t_1}} \right)_+^3, \\ \underline{v}(\bar{x}, s, t) &= 0 \end{aligned} \quad (41)$$

and extend $\underline{u}, \underline{v}$ as zero to the whole $\bar{\Omega} \times [t_1, T_0]$. Using (40), we get that

$$\begin{aligned} \underline{u}_t(\bar{x}, s, t) - \Delta \underline{u}(\bar{x}, s, t) - c_1(x, t) \underline{v}^p(\bar{x}, s, t) \\ = \frac{3s(t - t_1)^{\alpha-3/2}}{2} \left(\xi_0 - \frac{s}{\sqrt{t - t_1}} \right)_+^2 \\ + \alpha(t - t_1)^{\alpha-1} \left(\xi_0 - \frac{s}{\sqrt{t - t_1}} \right)_+^3 \end{aligned}$$

$$\begin{aligned} &- 6(t - t_1)^{\alpha-1} \left(\xi_0 - \frac{s}{\sqrt{t - t_1}} \right)_+ \\ &- 3(t - t_1)^{\alpha-1/2} \left(\xi_0 - \frac{s}{\sqrt{t - t_1}} \right)_+^2 \\ &\times \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \leq 0 \end{aligned} \quad (42)$$

for sufficiently small values of ξ_0 . It is clear that

$$\underline{v}_t(\bar{x}, s, t) - \Delta \underline{v}(\bar{x}, s, t) - c_2(x, t) \underline{u}^q(\bar{x}, s, t) \leq 0. \quad (43)$$

It remains to verify the validity of the inequality

$$\begin{aligned} (t - t_1)^\alpha \xi_0^3 &\leq (t - t_1)^{\alpha m} \\ &\times \int_{\partial\Omega \times [0, \delta]} k_1(x, (\bar{y}, s), t) |J(\bar{y}, s)| \\ &\times \left(\xi_0 - \frac{s}{\sqrt{t - t_1}} \right)_+^{3m} d\bar{y} ds \end{aligned} \quad (44)$$

for $x \in \partial\Omega$ and $t_1 < t < T_0$. Here $J(\bar{y}, s)$ is Jacobian of the change of variables. Estimating the integral I in the right-hand side of (44) we get

$$\begin{aligned} I &= (t - t_1)^{1/2} \\ &\times \int_{\partial\Omega} d\bar{y} \int_0^{\xi_0} k_1(x, (\bar{y}, z\sqrt{t - t_1}), t) \\ &\times |J(\bar{y}, z\sqrt{t - t_1})| (\xi_0 - z)_+^{3m} dz \\ &\geq C(t - t_1)^{1/2}, \end{aligned} \quad (45)$$

where constant C does not depend on t ; we obtain that (44) is true if $T_0 - t_1$ is small enough.

By comparison principle for (39) we conclude that $u_\varepsilon(x, t) \geq \underline{u}(x, t)$, $v_\varepsilon(x, t) \geq \underline{v}(x, t)$ and, respectively, $u_{\max}(x, t) \geq \underline{u}(x, t)$, $v_{\max}(x, t) \geq \underline{v}(x, t)$ for $(x, t) \in \Omega \times (t_1, T_0]$.

The proof in the case $n < 1$ and $k_2(x, y_2, t_2) > 0$ for any $x \in \partial\Omega$ and some $y_2 \in \partial\Omega$, $0 \leq t_2 < T$ is similar. $\square$

It is easy to get from Theorem 9 and the proof of Theorem 11 the following statement.

Corollary 12. *Let the conditions of Theorem 11 hold with $t_i = 0$, $i = 0, 1, 2$. Suppose also that $k_1(x, \cdot, t)$ and $k_2(x, \cdot, t)$ are nontrivial functions for any $x \in \partial\Omega$ and $t \in (0, T)$, $c_2(x, t)$ has the property (N) if (32) is realized, and $c_1(x, t)$ has the property (N) if (33) is realized. Then maximal solution is positive in $\bar{Q}_T$ for $0 < t < T$.*

Under the conditions of Corollary 12 for some class of the coefficients $c_i(x, t)$ and $k_i(x, y, t)$, $i = 1, 2$, we can prove the uniqueness of solution for (1) with trivial initial datum which is positive for all positive times as long as it exists.

Theorem 13. Let the conditions of Corollary 12 hold. Suppose also that there exists $t_0 > 0$ such that for $0 \leq t \leq t_0$ the functions $c_1(x, t)$, $c_2(x, t)$, $k_1(x, y, t)$, and $k_2(x, y, t)$ are nondecreasing with respect to t .

Then there exists exactly one solution of problem (1) which is positive in $\overline{Q_T}$ for $0 < t < T$.

Proof. Suppose that there exists different from $(u_{\max}(x, t), v_{\max}(x, t))$ solution $(u(x, t), v(x, t))$ of (1) with trivial initial datum which is a positive in $\overline{Q_T}$ for $0 < t < T$. Denote $t_* = \min(t_0, T)$. Due to the conditions of the theorem it is easy to see that $u(x, t + \tau), v(x, t + \tau)$ is positive supersolution of (1) with trivial initial datum in $Q_{t_* - \tau}$ for any $\tau \in (0, t_*)$. By Theorem 4 we have $u_{\max}(x, t) \leq u(x, t + \tau), v_{\max}(x, t) \leq v(x, t + \tau)$ for every $0 \leq t \leq t_* - \tau$. Passing to the limit as $\tau \rightarrow 0$ we get $u_{\max}(x, t) \leq u(x, t), v_{\max}(x, t) \leq v(x, t)$ for $0 \leq t \leq t_*$. Hence $u_{\max}(x, t) = u(x, t), v_{\max}(x, t) = v(x, t)$ in Q_T . $\square$

Note that by Theorem 10, the solution $(u(x, t), v(x, t))$ of (1) is unique if $\min(p, q, m, n) \geq 1$. Now we specify our uniqueness result in the case $\min(pq, m, n) < 1$.

Theorem 14. Let the conditions of Corollary 12 fulfill only $u_0(x) \not\equiv 0$ or $v_0(x) \not\equiv 0$. Then the solution of (1) is unique.

Proof. To prove the uniqueness of the solution if $\min(pq, m, n) < 1$, it suffices to show that if $(u(x, t), v(x, t))$ is any solution of (1), then

$$u_{\max}(x, t) \leq u(x, t), \quad v_{\max}(x, t) \leq v(x, t), \quad (46)$$

where $(u_{\max}(x, t), v_{\max}(x, t))$ is a maximal solution of (1).

First, consider the case when $0 < p < 1, 0 < q \leq 1, 0 < m < 1$, and $0 < n \leq 1$. Let

$$\begin{aligned} w_1(x, t) &= u_{\max}(x, t) - u(x, t), \\ w_2(x, t) &= v_{\max}(x, t) - v(x, t). \end{aligned} \quad (47)$$

Then (w_1, w_2) satisfies

$$\begin{aligned} w_{1t} &\leq \Delta w_1 + c_1(x, t) w_2^p, \\ w_{2t} &\leq \Delta w_2 + c_2(x, t) w_1^q, \\ x &\in \Omega, \quad 0 < t < T_1, \\ w_1(x, t) &\leq \int_{\Omega} k_1(x, y, t) w_1^m(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_1, \\ w_2(x, t) &\leq \int_{\Omega} k_2(x, y, t) w_2^n(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_1, \\ w_1(x, 0) &\equiv 0, \quad w_2(x, 0) \equiv 0, \quad x \in \Omega \end{aligned} \quad (48)$$

for some $T_1 > 0$. By Theorem 13, there exists a unique solution $(h_1(x, t), h_2(x, t))$ of the following problem:

$$\begin{aligned} h_{1t} &= \Delta h_1 + c_1(x, t) h_2^p, \\ h_{2t} &= \Delta h_2 + c_2(x, t) h_1^q, \\ x &\in \Omega, \quad 0 < t < T_2, \\ h_1(x, t) &= \int_{\Omega} k_1(x, y, t) h_1^m(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_2, \\ h_2(x, t) &= \int_{\Omega} k_2(x, y, t) h_2^n(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_2, \\ h_1(x, 0) &\equiv 0, \quad h_2(x, 0) \equiv 0, \quad x \in \Omega, \end{aligned} \quad (49)$$

such that $h_1(x, t) > 0, h_2(x, t) > 0$ for $x \in \overline{\Omega}, 0 < t < T_2$ for some $T_2 > 0$. Let $T_0 = \min(T_1, T_2)$. In a similar way as in Theorems 13 and 4 we can prove that $u_{\max}(x, t) \geq h_1(x, t) \geq w_1(x, t), v_{\max}(x, t) \geq h_2(x, t) \geq w_2(x, t)$. Put $a_1 = h_1 - w_1, a_2 = h_2 - w_2$. Now, use an elementary inequality, which is recalled for instance in [20],

$$a^k - b^k + c^k \geq (a - b + c)^k, \quad (50)$$

where $0 < k \leq 1, \min(a, b, c) > 0$, and $\max(a, c) \leq b \leq a + c$. Then we obtain

$$\begin{aligned} a_{1t} &\geq \Delta a_1 + c_1(x, t) a_2^p, \quad a_{2t} \geq \Delta a_2 + c_2(x, t) a_1^q, \\ x &\in \Omega, \quad 0 < t < T_0, \\ a_1(x, t) &\geq \int_{\Omega} k_1(x, y, t) a_1^m(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_0, \end{aligned} \quad (51)$$

$$\begin{aligned} a_2(x, t) &\geq \int_{\Omega} k_2(x, y, t) a_2^n(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_0, \end{aligned}$$

$$a_1(x, 0) \equiv 0, \quad a_2(x, 0) \equiv 0, \quad x \in \Omega.$$

We show that $a_1(x, t) > 0, a_2(x, t) > 0$ in Q_{T_0} . In fact, otherwise, by Theorem 9 there exists $\bar{t} \in (0, T_0)$ such that $a_1(x, t) = a_2(x, t) \equiv 0$ in $Q_{\bar{t}}$. Suppose that $c_1(x_0, 0) > 0$ for some $x_0 \in \Omega$. Then we get

$$\begin{aligned} c_1(h_2 + v)^p &= c_1(w_2 + v)^p = c_1 v_{\max}^p \\ &= (u_{\max})_t - \Delta u_{\max} = (w_1 + u)_t - \Delta(w_1 + u) \\ &= (h_1 + u)_t - \Delta(h_1 + u) = c_1(h_2^p + v^p) \end{aligned} \quad (52)$$

for $x \in \Omega$ and $0 < t < \bar{t}$. This is a contradiction since $h_2(x, t) > 0, v(x, t) > 0$ in $Q_{T_0}, c_1(x_0, t_0) > 0$ for some $t_0 \in (0, \bar{t})$, and $0 < p < 1$.

If $k_1(x, y_0, 0) > 0$ for any $x \in \partial\Omega$ and some $y_0 \in \partial\Omega$ we can obtain a contradiction by another way. Indeed,

$$\begin{aligned} & \int_{\Omega} k_1(x, y, t) (h_1^m(y, t) + u^m(y, t)) dy \\ &= h_1(x, t) + u(x, t) = w_1(x, t) + u(x, t) \\ &= u_{\max}(x, t) = \int_{\Omega} k_1(x, y, t) u_{\max}^m(y, t) dy \\ &= \int_{\Omega} k_1(x, y, t) (w_1(y, t) + u(y, t))^m dy \\ &= \int_{\Omega} k_1(x, y, t) (h_1(y, t) + u(y, t))^m dy \end{aligned} \quad (53)$$

for $x \in \partial\Omega$ and $0 < t < \bar{t}$ and we get again a contradiction since $h_1(x, t) > 0$, $u(x, t) > 0$ in Q_{T_0} , and $0 < m < 1$.

Since $a_1 > 0$, $a_2 > 0$ in Q_{T_0} by comparison principle with arguments of Theorems 13 and 4 we conclude that $a_1 \geq h_1, a_2 \geq h_2$ in Q_{T_0} . This implies (46) and completes the proof for the first case.

Now suppose that $\max(p, q, m, n) > 1$. Assume, for example, that $0 < p \leq 1$, $q > 1$, $0 < m < 1$, $n > 1$. Then, as in the first case, we introduce the functions $w_1(x, t), w_2(x, t)$. We use the following relations:

$$\begin{aligned} u_{\max}^q(x, t) - u^q(x, t) &= q\theta_1^{q-1}(u_{\max}(x, t) - u(x, t)) \\ &\leq M_1(u_{\max}(x, t) - u(x, t)), \\ v_{\max}^n(x, t) - v^n(x, t) &= n\theta_2^{n-1}(v_{\max}(x, t) - v(x, t)) \\ &\leq M_2(v_{\max}(x, t) - v(x, t)), \end{aligned} \quad (54)$$

where θ_1, θ_2 are nonnegative continuous functions which are between $u_{\max}(x, t)$, $u(x, t)$ and $v_{\max}(x, t)$, $v(x, t)$, respectively, $q\theta_1^{q-1} \leq M_1$, $n\theta_2^{n-1} \leq M_2$ in Q_T . Then we have that a pair of functions $(w_1(x, t), w_2(x, t))$ satisfies relations

$$\begin{aligned} w_{1t} &\leq \Delta w_1 + c_1(x, t) w_2^p, \\ w_{2t} &\leq \Delta w_2 + c_2(x, t) M_1 w_1, \\ x &\in \Omega, \quad 0 < t < T_1, \\ w_1(x, t) &\leq \int_{\Omega} k_1(x, y, t) w_1^m(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_1, \\ w_2(x, t) &\leq \int_{\Omega} k_2(x, y, t) M_2 w_2(y, t) dy, \\ x &\in \partial\Omega, \quad 0 < t < T_1, \\ w_1(x, 0) &\equiv 0, \quad w_2(x, 0) \equiv 0, \quad x \in \Omega. \end{aligned} \quad (55)$$

Further develop the arguments, as in the first case, only for $q = 1$, $n = 1$. Using the linearization of terms with powers greater than 1 in the equations and boundary conditions of (1) as above we can prove the theorem for the remaining cases in a similar way. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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