Research Article

The Hilbert-Kunz Function for Binomial Hypersurfaces

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I give an iterative closed form formula for the Hilbert-Kunz function for any binomial hypersurface in general, over any field of arbitrary positive characteristic. I prove that the Hilbert-Kunz multiplicity associated with any binomial hypersurface over any field of arbitrary positive characteristic is rational. As an example, I also prove the well known fact that for 1-dimensional binomial hypersurfaces the Hilbert-Kunz multiplicity is a positive integer and give a precise account of the integer.

1. Introduction

Let $(A, \mathfrak{n})$ be a Noetherian local ring of dimension d and of prime characteristic $p > 0$. Let I be an $\mathfrak{n}$ -primary ideal and $q = p^n$ where $n > 0$ is an integer. The “Hilbert-Kunz function” of A with respect to I is defined as

$$HK_{I,A}(q) = l\left(\frac{A}{I^{[q]}}\right), \quad (1)$$

where $I^{[q]} = \text{nth Frobenius power of } I$, that is, the ideal generated by x^q , $x \in I$.

The associated Hilbert-Kunz multiplicity is defined to be

$$e_{HK}(I) = \lim_{q \rightarrow \infty} \frac{HK_{I,A}(q)}{q^d}. \quad (2)$$

Monsky showed in his paper [1] that the limit exists and is a real constant. Further he showed that

$$HK_{I,A}(q) = e_{HK}(I) q^d + O(q^{d-1}). \quad (3)$$

Several authors have investigated $e_{HK}(I)$. They showed that $e_{HK}(I)$ is a rational number for certain special kind of binomial hypersurfaces [2], cubic curves and surfaces [3] and full flag varieties, and elliptic curves [4].

A binomial hypersurface is defined in the beginning of Section 4. In this paper, we are interested in giving an iterative closed form formula for the Hilbert-Kunz function for any

binomial hypersurface in general. Our methods work for any positive characteristic. Our work generalizes the work of Conca [2]. In [2], Conca computes the Hilbert-Kunz function of monomial ideals and also of those Binomial hypersurfaces whose terms defining the hypersurface are relatively prime. In [2], Conca also proves that the Hilbert-Kunz multiplicity associated with these special Binomial Hypersurfaces is always rational. In this paper, we prove that the Hilbert-Kunz multiplicity associated with any binomial hypersurface over any field of arbitrary positive characteristic is rational. The rationality result appears in Eto's work (Theorem 2.2, [5]), where the multiplicity is interpreted as a volume of a polytope. Here we give a different proof for the rationality result. Our work also generalizes the work [6] of Watanabe, where he deals only with normal toric varieties.

The organization of this work is more or less clear from the table of contents. But to be precise, until Section 4 begins, the matter of this work holds true for any hypersurface over any field of positive characteristic and need not have to be a binomial hypersurface! Also, the filtration introduced in Section 2.1 is effective for any general ideal J of a polynomial ring $S = k[x_1, \dots, x_m]$ where k is a field of arbitrary prime characteristic $p > 0$, not just when the ideal J is generated by a single polynomial. The iterative closed form formula for the Hilbert-Kunz function for binomial hypersurfaces appears in Theorem 26 in Section 4.2.5. After that, in Section 4.4, we discuss the example of the 1-dimensional case. In the example of the 1-dimensional case, I prove the well known

fact that for any 1-dimensional binomial hypersurface, the associated Hilbert-Kunz multiplicity is a positive integer and I give a precise account of the integer. In Section 4.5, I give an expression for the Hilbert-Kunz multiplicity for a general binomial hypersurface (see (67)).

2. Stating the Problem

For the rest of this paper we will assume that $S = k[x_1, \dots, x_m]$ where k is a field of arbitrary prime characteristic $p > 0$ and J is an arbitrary ideal in S . Let $R = S/J$. Put $\widehat{\mathfrak{m}} := \mathfrak{m} + J$ where $\mathfrak{m} = (x_1, \dots, x_m)$. We will assume that $J \subseteq \mathfrak{m}$; otherwise, $\widehat{\mathfrak{m}} = R$. Consider

$$HK_{\widehat{\mathfrak{m}}R_{\widehat{\mathfrak{m}}}, R_{\widehat{\mathfrak{m}}}}(p^n) = l\left(\frac{R_{\widehat{\mathfrak{m}}}}{(\widehat{\mathfrak{m}}R_{\widehat{\mathfrak{m}}})^{[p^n]}}\right). \quad (4)$$

Note that the rings $R_{\widehat{\mathfrak{m}}}/(\widehat{\mathfrak{m}}R_{\widehat{\mathfrak{m}}})^{[p^n]}$ and $R/(\mathfrak{m}^{[p^n]} + J)$ are isomorphic, where $\mathfrak{m}^{[p^n]} = (x_1^{p^n}, \dots, x_m^{p^n})$. Hence the Hilbert-Kunz function $HK_{\widehat{\mathfrak{m}}R_{\widehat{\mathfrak{m}}}, R_{\widehat{\mathfrak{m}}}}(p^n)$ is given by

$$HK_{\widehat{\mathfrak{m}}R_{\widehat{\mathfrak{m}}}, R_{\widehat{\mathfrak{m}}}}(p^n) = l\left(\frac{R}{\mathfrak{m}^{[p^n]} + J}\right). \quad (5)$$

We are presently interested in the case when R is a hypersurface, that is, when the ideal J is generated by a single polynomial.

2.1. A Filtration for Computing the Length $l(R/(\mathfrak{m}^{[p^n]} + J))$. Let $\mathfrak{S}$ denote the set of all m -tuples $(u_1, \dots, u_m)$ such that for each $i \in \{1, \dots, m\}$, $u_i \in \{0, 1, \dots, p^n - 1\}$. Let us arrange the elements of the set $\mathfrak{S}$ in the lexicographic order; that is,

we say that $(u_1, \dots, u_m) <_{\text{lex}} (u'_1, \dots, u'_m)$ if in the vector difference $(u'_1, \dots, u'_m) - (u_1, \dots, u_m)$, the leftmost nonzero entry is > 0 .

It is easy to check that the set $\mathfrak{S}$ equipped with the lexicographic order $<_{\text{lex}}$ is in fact a totally ordered set, and this set contains p^{mn} many elements. Let us denote these p^{mn} many elements of the set $\mathfrak{S}$ arranged in the $<_{\text{lex}}$ order by $\mathbb{U}_0 <_{\text{lex}} \mathbb{U}_1 <_{\text{lex}} \mathbb{U}_2 <_{\text{lex}} \dots <_{\text{lex}} \mathbb{U}_{p^{mn}-1}$.

Let $\mathfrak{M} := \{x_1^{j_1} \dots x_m^{j_m} \mid 0 \leq j_i < p^n \forall 1 \leq i \leq m\}$. There are p^{mn} many elements in the set $\mathfrak{M}$. For any $\mathbb{U} = (u_1, u_2, \dots, u_m) \in \mathfrak{S}$, let us denote by $\mathbb{X}^{p^n-1-\mathbb{U}}$ the element of $\mathfrak{M}$ given by $x_1^{p^n-1-u_1} x_2^{p^n-1-u_2} \dots x_m^{p^n-1-u_m}$. The set $\mathfrak{S}$ is in one-to-one correspondence with the set $\mathfrak{M}$ via the map $\mathbb{U} \mapsto \mathbb{X}^{p^n-1-\mathbb{U}}$. Let us denote by $\mathbb{X}^{p^n-1-\mathbb{U}_j}$ the image of the element $\mathbb{U}_j$ under this correspondence.

For any $i \in \{1, \dots, m\}$ and any $\mathbb{U}_j = (u_{j,1}, \dots, u_{j,m}) \in \mathfrak{S}$, let us denote by $\mathbb{X}_i^{p^n-1-\mathbb{U}_j}$ the product $x_1^{p^n-1-u_{j,1}} \dots x_i^{p^n-1-u_{j,i}}$, that is, the product in $\mathbb{X}^{p^n-1-\mathbb{U}_j}$ up to the i th variable.

Given any $j \in \{0, \dots, p^{mn} - 1\}$ and any $i \in \{1, \dots, m\}$, define

$$\mathcal{B}_{j,i} := \begin{cases} \mathbb{X}_i^{p^n-1-\mathbb{U}_j} x_i & \text{if } u_{j,i} > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Definition 1. The ideals I_j for any $j \in \{0, \dots, p^{mn} - 1\}$, let $I_j := \langle \mathcal{B}_{j,i} \mid 1 \leq i \leq m \rangle + L$ where $L = \langle x_1^{p^n}, x_2^{p^n}, \dots, x_m^{p^n} \rangle$.

Declare $I_{p^{mn}}$ to be the polynomial ring S itself.

Remark 2. From the definition of the lexicographic order $<_{\text{lex}}$ on $\mathfrak{S}$, it follows easily that for any $j \in \{0, \dots, p^{mn} - 2\}$, either $\mathbb{X}^{p^n-1-\mathbb{U}_j} = \mathbb{X}^{p^n-1-\mathbb{U}_{j+1}} x_m$ or there exists $k \in \{1, \dots, m-1\}$ such that $\mathbb{X}_k^{p^n-1-\mathbb{U}_j} = \mathbb{X}_k^{p^n-1-\mathbb{U}_{j+1}} x_k$, $\mathbb{X}^{p^n-1-\mathbb{U}_j} = \mathbb{X}_k^{p^n-1-\mathbb{U}_j}$, and $\mathbb{X}^{p^n-1-\mathbb{U}_{j+1}} = \mathbb{X}_k^{p^n-1-\mathbb{U}_{j+1}} (x_{k+1} \dots x_m)^{p^n-1}$.

Notation 1. For any $j \in \{0, \dots, p^{mn} - 1\}$, let $i(j) := \max\{i \mid 1 \leq i \leq m, \mathcal{B}_{j,i} \neq 0\}$.

Lemma 3. (i) For any $j \in \{0, \dots, p^{mn} - 1\}$, $\mathbb{X}^{p^n-1-\mathbb{U}_j} \in I_{j+1}$;
(ii) $\mathcal{B}_{j,k} \in \mathcal{B}_{j+1,k}$ for all $j \in \{0, \dots, p^{mn} - 2\}$ and for all $k < i(j+1)$.

Proof. (i) It follows from Remark 2 that $\mathbb{X}^{p^n-1-\mathbb{U}_j} = \mathcal{B}_{j+1,i(j+1)}$. Hence $\mathbb{X}^{p^n-1-\mathbb{U}_j} \in I_{j+1}$;
(ii) the proof of this follows from Remark 2. $\square$

Lemma 4. For any $j \in \{0, \dots, p^{mn} - 1\}$, $I_j \subseteq I_{j+1}$.

Proof. As $I_j := \langle \mathcal{B}_{j,i} \mid 1 \leq i \leq m \rangle + L$ (where $L = \langle x_1^{p^n}, x_2^{p^n}, \dots, x_m^{p^n} \rangle$), it is enough to show that $\mathcal{B}_{j,i} \in I_{j+1}$ for all $j \in \{0, \dots, p^{mn} - 1\}$ and for all $i \in \{1, \dots, m\}$.

Case 1 (when $\mathbb{X}^{p^n-1-\mathbb{U}_j} = \mathbb{X}^{p^n-1-\mathbb{U}_{j+1}} x_m$). In this case, clearly $\mathcal{B}_{j,i} = \mathcal{B}_{j+1,i}$ for all $1 \leq i \leq m-1$. And $\mathcal{B}_{j,m}$ is either equal to 0 or equal to $\mathcal{B}_{j+1,m} x_m$. Hence in either case, $\mathcal{B}_{j,m} \in I_{j+1}$. Therefore, $I_j \subseteq I_{j+1}$.

Case 2 (when $\mathbb{X}^{p^n-1-\mathbb{U}_j} \neq \mathbb{X}^{p^n-1-\mathbb{U}_{j+1}} x_m$). In this case, there exists $k \in \{1, \dots, m-1\}$ such that $\mathbb{X}_k^{p^n-1-\mathbb{U}_j} = \mathbb{X}_k^{p^n-1-\mathbb{U}_{j+1}} x_k$, $\mathbb{X}^{p^n-1-\mathbb{U}_j} = \mathbb{X}_k^{p^n-1-\mathbb{U}_j}$, and $\mathbb{X}^{p^n-1-\mathbb{U}_{j+1}} = \mathbb{X}_k^{p^n-1-\mathbb{U}_{j+1}} (x_{k+1} \dots x_m)^{p^n-1}$.

Clearly then, $\mathcal{B}_{j,i} = \mathcal{B}_{j+1,i}$ for all $i < k$. Also, $\mathcal{B}_{j,k}$ is either equal to 0 or equal to $\mathcal{B}_{j+1,k} x_k$. Hence in either case, $\mathcal{B}_{j,k} \in I_{j+1}$. For $i > k$, $\mathcal{B}_{j,i} = \mathcal{B}_{j+1,k} x_i$. Hence $\mathcal{B}_{j,i} \in I_{j+1}$ for all $i > k$. Therefore, $I_j \subseteq I_{j+1}$. $\square$

Notation 2. Let M be any S -module. Let $\lambda(M)$ denote the length of M with respect to S .

Lemma 5. Let $j \in \{0, \dots, p^{mn} - 1\}$. Put $M_{j+1} := I_{j+1}/I_j$. Then $\lambda(M_{j+1}) = 1$.

Proof. Fix j . We will show that for each $i \in \{1, \dots, m\}$, $x_i I_{j+1} \subseteq I_j$. We claim that for any $i, k \in \{1, \dots, m\}$, $x_i \mathcal{B}_{j+1,k} \in I_j$.

Case 1 (when $k > i(j+1)$). In this case, we have $\mathcal{B}_{j+1,k} = 0$. Hence $x_i \mathcal{B}_{j+1,k} \in I_j$ for any $k > i(j+1)$ and for any $i \in \{1, \dots, m\}$.

Case 2 (when $k \leq i(j+1)$). It follows from Lemma 3 that $\mathcal{B}_{j,k} = \mathcal{B}_{j+1,k}$ for all $k < i(j+1)$. Therefore $\mathcal{B}_{j+1,k} \in I_j$ for all

$k < i(j+1)$. This implies that $x_i \mathcal{B}_{j+1,k} \in I_j$ for all $k < i(j+1)$ and for all $i \in \{1, \dots, m\}$.

It remains to show that $x_i \mathcal{B}_{j+1,i(j+1)} \in I_j$ for all $i \in \{1, \dots, m\}$. It follows from the proof of Lemma 3(i) that $\mathbb{X}^{p^n-1-\mathbb{U}_j} = \mathcal{B}_{j+1,i(j+1)}$. Hence for any $i \in \{1, \dots, m\}$, $x_i \mathcal{B}_{j+1,i(j+1)} = x_i \mathbb{X}^{p^n-1-\mathbb{U}_j}$ is either a multiple of $\mathcal{B}_{j,i}$ or a multiple of $x_i^{p^n}$. Therefore $x_i \mathcal{B}_{j+1,i(j+1)} \in I_j$ for all $i \in \{1, \dots, m\}$. $\square$

Lemma 6. For all $j \in \{0, \dots, p^{mn}-1\}$, $I_{j+1} = \langle \mathbb{X}^{p^n-1-\mathbb{U}_j} \rangle + I_j$.

Proof. It follows from Lemma 3 that $\mathbb{X}^{p^n-1-\mathbb{U}_j} \in I_{j+1}$ for all $j \in \{0, \dots, p^{mn}-1\}$. It also follows from the same lemma that $\mathcal{B}_{j,k} = \mathcal{B}_{j+1,k}$ for all $j \in \{0, \dots, p^{mn}-2\}$ and for all $k < i(j+1)$. Therefore the generators $\mathcal{B}_{j+1,k}$ (where $k \neq i(j+1)$) of the ideal I_{j+1} belong to I_j . Due to degree reasons, $\mathbb{X}^{p^n-1-\mathbb{U}_j} = \mathcal{B}_{j+1,i(j+1)}$ does not belong to I_j . Therefore $\mathbb{X}^{p^n-1-\mathbb{U}_j}$ is the only generator of the ideal I_{j+1} which does not belong to I_j . Hence $I_{j+1} = \langle \mathbb{X}^{p^n-1-\mathbb{U}_j} \rangle + I_j$. $\square$

It follows from Lemmas 4, 5, and 6 above that the following chain

$$I_0 \subsetneq I_1 \subsetneq \dots \subsetneq I_{p^{mn}-1} \subsetneq I_{p^{mn}} \quad (7)$$

is a filtration of ideals such that $\lambda(I_{j+1}/I_j) = 1$.

Therefore reducing modulo the ideal J , it follows that the following chain

$$\begin{aligned} (0) = \frac{I_0 + J}{I_0 + J} &\subseteq \frac{I_1 + J}{I_0 + J} \subseteq \frac{I_2 + J}{I_0 + J} \subseteq \dots \subseteq \frac{I_{p^{mn}} + J}{I_0 + J} \\ &= \frac{R}{\mathfrak{m}^{[p^n]} + J} \end{aligned} \quad (8)$$

is a filtration of ideals, which is having the property that *each successive quotient either is zero or is generated by a single element*. The total number of successive quotients of the above mentioned filtration which are generated by a single element equals the length $l(R/(\mathfrak{m}^{[p^n]} + J))$. The following corollary to Lemma 6 above gives a precise account of which successive quotient of the above mentioned filtration is generated by a single element and which successive quotient is zero.

Corollary 7. For any $j \in \{0, \dots, p^{mn}-1\}$, $\lambda((I_{j+1} + J)/(I_j + J)) \leq 1$, and $\lambda((I_{j+1} + J)/(I_j + J)) = 0$ if and only if $\mathbb{X}^{p^n-1-\mathbb{U}_j} \in I_j + J$.

Proof. Fix any $j \in \{0, \dots, p^{mn}-1\}$, it follows from Lemma 5 that $\lambda(I_{j+1}/I_j) = 1$. Therefore, reducing modulo the ideal J , we have $\lambda((I_{j+1} + J)/(I_j + J)) \leq 1$. Let M_{j+1} be as in Lemma 5. It follows from Lemma 6 that $\mathbb{X}^{p^n-1-\mathbb{U}_j}$ generates M_{j+1} . Therefore reducing modulo the ideal J , it follows that the quotient $(I_{j+1} + J)/(I_j + J)$ is zero if and only if $\mathbb{X}^{p^n-1-\mathbb{U}_j} \in I_j + J$ and is generated by the single element $\mathbb{X}^{p^n-1-\mathbb{U}_j}$ otherwise. $\square$

Hence for knowing that which successive quotient of the above mentioned filtration is generated by single element and which one is zero, the only condition that needs to be verified is the following:

$$\mathbb{X}^{p^n-1-\mathbb{U}_j} \in I_j + J \text{ or not for every } j \in \{0, \dots, p^{mn}-1\}?$$

This is the *key checking condition* for computing the length $l(R/(\mathfrak{m}^{[p^n]} + J))$.

3. A Procedure for Doing the Key Check

We are interested in the case when the ideal J is generated by a single polynomial, say $J = (f)$. In this section, we will first define a term order $\triangleright$ on the set of all monomials in the variables $x_1, \dots, x_m$, and then with respect to $\triangleright$, we will arrange the terms of the polynomial f , and with the help of all this notation, we will describe a way of doing the key checking.

3.1. The Term Order $\triangleright$. To order the terms of the polynomial f , let us put an order (denote it by $\triangleright$) on the set of all monomials in the variables $x_1, \dots, x_m$ as follows.

- (i) Set $x_1 \triangleright \dots \triangleright x_m$.
- (ii) On the set of all monomials in the variables $x_1, \dots, x_m$, $\triangleright$ is the degree lexicographic order with respect to the order $\triangleright$ defined on the variables $x_1, \dots, x_m$.

Say, the polynomial f has F many terms. Let us denote the most initial (with respect to $\triangleright$) term of f by $[F]$, the next most initial term of f by $[F-1]$, $\dots$, and so on till the least initial term $[1]$. Hence we have

$$J = (f) = ([F] + [F-1] + \dots + [1]). \quad (9)$$

For example, when $m = 5$ and $f = x_1^3 + x_2^3 + x_1 x_2 x_3 + x_4 x_5$, then $F = 4$, $[4] = x_1^3$, $[3] = x_1 x_2 x_3$, $[2] = x_2^3$, and $[1] = x_4 x_5$.

Remark 8. Note here that the terms $[F]$, $[F-1]$, $\dots$, $[1]$, of f are assumed to be containing scalar coefficients.

3.2. The Main Theorem. Given any term $[\tau]$ of the polynomial f , define $[\tau]^{-1} := 1/[\tau]$. Recall the set $\{\mathfrak{M} := \mathbb{X}^{p^n-1-\mathbb{U}_j} \mid j \in \{0, \dots, p^{mn}-1\}\}$. Let A be an arbitrary element of the set $\mathfrak{M}$. Say the monomial A chosen above from the set $\mathfrak{M}$ equals $\mathbb{X}^{p^n-1-\mathbb{U}_j}$ for some $j \in \{0, \dots, p^{mn}-1\}$. The key checking condition for the monomial A says that “ $A \in I_j + J$ or not.” Let us denote by A_c the ideal I_j . We call A_c the *ideal of convergence corresponding to A* . Given any $A \in \mathfrak{M}$, we need to check whether or not $A \in A_c + J$. Theorem 10 below provides a way for doing this checking.

Definition 9. A monomial in the variables $x_1, \dots, x_m$ of the type

$$A [\tau_1]^{-1} \dots [\tau_r]^{-1} [\alpha_1] \dots [\alpha_r], \quad (10)$$

where $[\tau_i]$ and $[\alpha_i]$ are terms of the polynomial f and r is any nonnegative integer, is called a *combination monomial* in A and f if $A[\tau_1]^{-1} \cdots [\tau_r]^{-1} [\alpha_1] \cdots [\alpha_r]$ contains no negative powers of any of the underlying variables $x_1, \dots, x_m$. In particular, the monomial A itself is a combination monomial in A and f .

Theorem 10. *Let $f = [F] + [F - 1] + \cdots + [1]$. $A \in A_c + J$ if and only if one of the following mutually exclusive conditions holds.*

- (i) *The term $[1]$ of f divides the monomial A .*
- (ii) *The term $[1]$ of f does not divide the monomial A , but there exists a polynomial h whose terms are of the type $cB[\tau]^{-1}$ where c is a nonzero scalar, B is a combination monomial in A and f such that $B \notin A_c$, $[\tau]$ is a term of f which divides B , such that the product $f \cdot h$ equals $aA + \text{finitely many terms all belonging to } A_c$ where a is a nonzero scalar.*

Proof. If condition (i) of the theorem holds, the proof follows easily from the construction of the ideal A_c . If condition (ii) of the theorem holds, then it is easy to see that $A \in A_c + J$.

Conversely, suppose $A \in A_c + J$. Note that due to degree reasons, the monomial A can never belong to the ideal A_c (the proof of this follows easily from the construction of the ideal A_c). Therefore the fact that $A \in A_c + J$ implies that there exists a polynomial (say g) such that the product $f \cdot g$ equals $aA + \text{finitely many terms all belonging to } A_c$ where a is a nonzero scalar in the ground field k .

Since the product $f \cdot g$ contains a term of the type aA (for some nonzero scalar a), therefore the polynomial g contains finitely many terms of the form $cA[\tau]^{-1}$ where c is a nonzero scalar and $[\tau]$ is a term of f such that $[\tau]$ divides A . Say $c_1A[\tau_1]^{-1} + \cdots + c_lA[\tau_l]^{-1}$ is the part of the polynomial g which has all its terms of the form $cA[\tau]^{-1}$ where c is some nonzero scalar and $[\tau]$ is some term of f which divides A . If at least one of the $[\tau_i]$'s appearing in the expression $c_1A[\tau_1]^{-1} + \cdots + c_lA[\tau_l]^{-1}$ is equal to $[1]$, then we can conclude that the term $[1]$ divides A and we are done.

If not, then look at the portion (say, g_1) of the polynomial g which consists of sum of all terms of the type $cB[\tau]^{-1}$ where c is a nonzero scalar, B is a combination monomial in A and f such that $B \notin A_c$, and $[\tau]$ is a term of f which divides B . Clearly then g_1 contains the sum $c_1A[\tau_1]^{-1} + \cdots + c_lA[\tau_l]^{-1}$. If $f \cdot g_1$ equals $aA + \text{finitely many terms all belonging to } A_c$ for some nonzero scalar a , then condition (ii) of the theorem holds and with $h = g_1$ and we are done.

So assume that the product $f \cdot g_1$ contains at least one term (say, D_1) which is a combination monomial in A and f not belonging to A_c . Since the product $f \cdot g$ equals $aA + \text{finitely many terms all belonging to } A_c$ for some nonzero scalar a , therefore there exists a nonzero term E in g such that E is not a term of the polynomial g_1 and $E[\tau] = bD_1$ for some nonzero scalar b and some term $[\tau]$ of f . Hence $E = bD_1[\tau]^{-1}$ for some nonzero scalar b and some term $[\tau]$ of f which divides D_1 . But since D_1 is a combination monomial in A and f not belonging to A_c , it follows from the equation

$E = bD_1[\tau]^{-1}$ that E is a term of the polynomial g_1 , hence a contradiction. $\square$

4. The Case of Binomial Hypersurfaces

In this section, we will study the case where the polynomial f contains only 2 terms; that is, $f = [2] + [1]$. The affine variety defined by the ideal J where $J = (f) = ([2] + [1])$ is called a binomial hypersurface.

4.1. The Main Theorem for Binomial Hypersurfaces. In this subsection, we will prove a theorem which will reduce the “key checking condition” mentioned above to checking of a combinatorial condition. The theorem which does this job is the following.

Theorem 11. *$A \in A_c + J$ if and only if either $[1]$ divides A or there exists a positive integer M for which $A[2]^{-M}[1]^{M-1}$ contains no negative powers of any of the underlying variables and $A[2]^{-M}[1]^M$ belongs to A_c . (Note here that by the notation $[2]^{-M}$, we mean $[2]^{-1}$ multiplied M times and similarly for the notation $[1]^M$.)*

Proof. If $[1]$ divides A , then it follows from Theorem 10 above that $A \in A_c + J$.

Suppose there exists a positive integer M for which $A[2]^{-M}[1]^{M-1}$ has no negative powers and $A[2]^{-M}[1]^M$ belongs to A_c . Let M_0 denote the least positive integer M for which this happens. Then the polynomial g given by

$$a_1 A [2]^{-1} + a_2 A [2]^{-2} [1]^1 + \cdots + a_{M_0} A [2]^{-M_0} [1]^{M_0-1}, \quad (11)$$

where a_1 is a nonzero scalar in the ground field k and $a_i + a_{i+1} = 0 \pmod{p}$ for each $i \in \{1, \dots, M_0 - 1\}$, is having the property that the product $f \cdot g$ equals $aA + \text{finitely many terms all belonging to } A_c$ where a is a nonzero scalar. Hence $A \in A_c + J$. We now need to prove the other way round.

Suppose $A \in A_c + J$ and $[1]$ does not divide A . Then by Theorem 10, there exists a polynomial h whose terms are of the type $cB[\tau]^{-1}$ where c is a nonzero scalar, B is a combination monomial in A and f not belonging to A_c , and $[\tau]$ is a term of f which divides B , such that the product $f \cdot h$ equals $aA + \text{finitely many terms all belonging to } A_c$ where a is a nonzero scalar.

It is easy to see that for the case of a binomial hypersurface, any combination monomial in A and f is either equal to A itself or it is of the type $A[2]^{-N}[1]^N$ for some positive integer N or it is of the type $A[1]^{-N}[2]^N$ for some positive integer N . It is also elementary to observe that any combination monomial in A and f of the type $A[1]^{-N}[2]^N$ always belongs to A_c . So any combination monomial not belonging to A_c is either equal to A itself or of the type $A[2]^{-N}[1]^N$ for some positive integer N . It hence follows that the terms of the polynomial h are either of the type $A[2]^{-N}[1]^N$ for some positive integer N or of the type $A[1]^{-1}$. But we have assumed that the term $[1]$ does not divide A , so $A[1]^{-1}$ cannot be a term of the polynomial h . Hence the

terms of the polynomial h are only of the type $A[2]^{-N}[1]^{N-1}$ for some positive integer N .

Let N_0 denote the largest positive integer N for which $A[2]^{-N}[1]^{N-1}$ is a term of the polynomial h . Clearly then $A[2]^{-N_0}[1]^{N_0-1}$ has no negative powers of any of the underlying variables $x_1, \dots, x_m$. Moreover, since the product $f \cdot h$ equals $aA + \text{finitely many terms all belonging to } A_c$ (where a is a nonzero scalar), it follows that $A[2]^{-N_0}[1]^{N_0}$ belongs to A_c . $\square$

4.2. A Formula for the Hilbert-Kunz Function for Binomial Hypersurfaces. In this subsection, we will give a closed form iterative formula for computing the Hilbert-Kunz function for any binomial hypersurface in general, over any field of positive characteristic. The notation and terminology remain the same as in the previous part of this paper.

4.2.1. Notation. Recall from Section 3.1 the variables $x_1, \dots, x_m$ and the term order $\triangleright$ on the set of all monomials in these variables. Let $f = [2] + [1]$ be the polynomial defining the binomial hypersurface where $[2]$ and $[1]$ are defined as in Section 3.1.

For any $i \in \{1, \dots, m\}$, let

$$\begin{aligned} \Delta_i &:= \text{the power of } x_i \text{ in the term } [1] \\ &\quad - \text{the power of } x_i \text{ in the term } [2]. \end{aligned} \quad (12)$$

Note that some of the Δ_i 's can be negative, some can be positive, and some can be 0. Without loss of generality, we can assume that $\Delta_1 \leq \dots \leq \Delta_m$.

Let r be the integer such that $0 \leq r \leq m$ and, the ordered set $\{x_1 \triangleright \dots \triangleright x_r\}$ equals the set $\{x_i \mid 1 \leq i \leq m, \Delta_i < 0\}$. For $i \in \{1, \dots, r\}$, let $N_i := x_{r+1-i}$ and $\Delta_{N_i} := \Delta_{r+1-i}$. Similarly, let s denote the number of x_i 's for which $\Delta_i = 0$ (note that s can also be 0). Let $x_{r+1} \triangleright \dots \triangleright x_{r+s}$ be the elements of the set $\{x_i \mid 1 \leq i \leq m, \Delta_i = 0\}$. For $i = \{1, \dots, s\}$, let $Z_i := x_{r+s+1-i}$ and $\Delta_{Z_i} := \Delta_{r+s+1-i}$. Finally, let $t := m - (r + s)$ and let $x_{r+s+1} \triangleright \dots \triangleright x_m$ be the elements of the set $\{x_i \mid 1 \leq i \leq m, \Delta_i > 0\}$. Let us denote by $P_1 \triangleright \dots \triangleright P_t$ the ordered set $x_{r+s+1} \triangleright \dots \triangleright x_m$ and for $i = \{1, \dots, t\}$, let $\Delta_{P_i} := \Delta_{r+s+i}$. In other words, we can say that the ordered set $x_1 \triangleright \dots \triangleright x_m$ is the same as the ordered set $N_r \triangleright \dots \triangleright N_1 \triangleright Z_s \triangleright \dots \triangleright Z_1 \triangleright P_1 \triangleright \dots \triangleright P_t$. We call N_i 's the *negative difference variables*, Z_i 's the *zero difference variables*, and P_i 's the *positive difference variables*. For any $i \in \{1, \dots, r\}$, let $N_{i,\min}$ and $N_{i,\max}$ denote the minimum and the maximum powers, respectively, of the variable N_i that appears in the expression of the polynomial f . We can similarly define $P_{q,\min}$, $P_{q,\max}$, $Z_{j,\min}$, and $Z_{j,\max}$ for all $q \in \{1, \dots, t\}$ and for all $j \in \{1, \dots, s\}$. Observe that since each Z_j is a zero difference variable, we have $Z_{j,\min} = Z_{j,\max}$ for all $j \in \{1, \dots, s\}$.

For any $i \in \{1, \dots, r\}$, let $a_i := -(\Delta_{N_i})$ and $b_i := N_{i,\min}$. For any $i \in \{1, \dots, r\}$ and any positive integer n , let $M_{a_i,n}$ and $r_{a_i,n}$ be defined by the equation $p^n - 1 = a_i M_{a_i,n} + r_{a_i,n}$.

4.2.2. The Number $M_{\max,A}$

Definition 12 (of $M_{\max,A}$). Let A be an arbitrary element of the set $\mathfrak{M}$ (this set has been defined earlier) such that $[7]$ divides A . It follows from Theorem 11 that for those monomials A which are divisible by $[7]$, we need to keep an account of the largest positive integer M for which the monomial $A[2]^{-M}[1]^{M-1}$ has no negative power for any of the variables $x_1, \dots, x_m$. Let $M_{\max,A}$ denote this largest positive integer.

4.2.3. An Explicit Account of $M_{\max,A}$. Corollary 15 below gives an explicit account of the number $M_{\max,A}$ for any monomial A which is divisible by $[7]$. The number $M_{\max,A}$ is important to us because of Theorem 11 which provides us with a way to count the required length $l(R/(\mathfrak{m}^{[p^n]} + J))$.

Remark 13. Let A be any monomial in $\mathfrak{M}$ which $[2]$ divides; then it is of the form

$$N_r^{p^n - k_{N_r}} \dots N_1^{p^n - k_{N_1}} Z_s^{p^n - k_{Z_s}} \dots Z_1^{p^n - k_{Z_1}} P_1^{j_1} \dots P_t^{j_t}, \quad (13)$$

where $p^n - 1 \geq j_q \geq P_{q,\min}$ for any $q \in \{1, \dots, t\}$, $p^n - 1 \geq p^n - k_{N_i} \geq N_{i,\max}$ for every $1 \leq i \leq r$, and $p^n - 1 \geq p^n - k_{Z_i} \geq Z_{i,\max}$ for every $1 \leq i \leq s$.

Hence for any positive integer M , $A[2]^{-M}[1]^{M-1}$ is of the form

$$\prod_{i=1}^r N_i^{p^n - k_{N_i} - a_i M - b_i} \prod_{i=1}^s Z_i^{p^n - k_{Z_i} - Z_{i,\min}} \prod_{q=1}^t P_q^{j_q + M \Delta_{P_q} - P_{q,\max}}, \quad (14)$$

where $p^n - 1 \geq j_q \geq P_{q,\min}$ for any $q \in \{1, \dots, t\}$, $p^n - 1 \geq p^n - k_{N_i} \geq N_{i,\max}$ for every $1 \leq i \leq r$, and $p^n - 1 \geq p^n - k_{Z_i} \geq Z_{i,\max}$ for every $1 \leq i \leq s$.

It is now easy to see that for any $M \geq 1$, the power of any zero difference variable or any positive difference variable in $A[2]^{-M}[1]^{M-1}$ is ≥ 0 . But the power of any negative difference variable in $A[2]^{-M}[1]^{M-1}$ may be negative for some values of M .

The following lemma and its corollary give an explicit account of $M_{\max,A}$ for any monomial A in $\mathfrak{M}$ which $[2]$ divides. But for stating the lemma, we need some notation. Let $i \in \{1, \dots, r\}$ and let n be any positive integer. If $b_i \geq a_i$, then let y_{a_i} denote the least positive integer $\geq (b_i - r_{a_i,n})$ which is divisible by a_i . Let q_{a_i} be defined by the equation $y_{a_i} = a_i q_{a_i}$. Let

$$p_{a_i} := \begin{cases} 0 & \text{if } b_i < a_i \text{ and } r_{a_i,n} \geq b_i, \\ 1 & \text{if } b_i < a_i \text{ and } r_{a_i,n} < b_i, \\ q_{a_i} & \text{if } b_i \geq a_i \text{ [clearly here } r_{a_i,n} < b_i]. \end{cases} \quad (15)$$

And let

$$E_{a_i,b_i} := \begin{cases} r_{a_i,n} - b_i & \text{if } p_{a_i} = 0, \\ (r_{a_i,n} - b_i) + a_i & \text{if } p_{a_i} = 1, \\ y_{a_i} - (b_i - r_{a_i,n}) & \text{if } p_{a_i} = q_{a_i}. \end{cases} \quad (16)$$

For any real number β , let $\langle \beta \rangle$ denote the smallest positive integer $\geq \beta$. For any $1 \leq i \leq r$, let

$$\tilde{k}_{N_i, \text{last}} := \left\langle \frac{p^n - N_{i, \text{max}} - 1 - E_{a_i, b_i}}{a_i} \right\rangle. \quad (17)$$

Lemma 14. For any $1 \leq i \leq r$, if k_{N_i} is such that $1 \leq k_{N_i} \leq p^n - N_{i, \text{max}}$, then the maximum value of the positive integer M for which $p^n - k_{N_i} - a_i M - b_i \geq 0$ equals

$$\begin{array}{lll} M_{a_i, n} - p_{a_i} & \text{if} & 1 \leq k_{N_i} \leq 1 + E_{a_i, b_i} \\ M_{a_i, n} - p_{a_i} - 1 & \text{if} & 1 + E_{a_i, b_i} < k_{N_i} \leq 1 + E_{a_i, b_i} + a_i \\ M_{a_i, n} - p_{a_i} - 2 & \text{if} & 1 + E_{a_i, b_i} + a_i < k_{N_i} \leq 1 + E_{a_i, b_i} + 2a_i \\ \vdots & \vdots & \vdots \\ M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i} & \text{if} & 1 + E_{a_i, b_i} + a_i (\tilde{k}_{N_i} - 1) < k_{N_i} \leq 1 + E_{a_i, b_i} + a_i \tilde{k}_{N_i} \\ \vdots & \vdots & \vdots \\ M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i, \text{last}} & \text{if} & 1 + E_{a_i, b_i} + a_i (\tilde{k}_{N_i, \text{last}} - 1) < k_{N_i} \leq p^n - N_{i, \text{max}}. \end{array} \quad (18)$$

Proof. Observe that since $p^n - 1 = a_i M_{a_i, n} + r_{a_i, n}$, we have $p^n - k_{N_i} - a_i M - b_i \geq 0$ if and only if $M \leq M_{a_i, n} - ((b_i - r_{a_i, n})/a_i) - ((k_{N_i} - 1)/a_i)$. So we need to compute the greatest positive integer M , that is, $\leq M_{a_i, n} - ((b_i - r_{a_i, n})/a_i) - ((k_{N_i} - 1)/a_i)$.

Case 1 (when $b_i \geq a_i$). In this case, we have $0 \leq r_{a_i, n} < a_i \leq b_i$. The least positive integer $\geq b_i - r_{a_i, n}$ which is divisible by a_i is y_{a_i} . Therefore the least positive integer $\geq (b_i - r_{a_i, n})/a_i$ is $q_{a_i} = p_{a_i}$. Therefore the greatest negative integer $\leq -(b_i - r_{a_i, n})/a_i$ is $-p_{a_i}$. And hence the greatest positive integer $\leq M_{a_i, n} - (b_i - r_{a_i, n})/a_i$ is $M_{a_i, n} - p_{a_i}$.

Since $k_{N_i} \leq p^n - N_{i, \text{max}}$, it follows that $M_{a_i, n} - ((b_i - r_{a_i, n})/a_i) - ((k_{N_i} - 1)/a_i) \geq M_{a_i, n} - p_{a_i} - ((p^n - N_{i, \text{max}} - 1 - E_{a_i, b_i})/a_i)$. The smallest positive integer $\geq (p^n - N_{i, \text{max}} - 1 - E_{a_i, b_i})/a_i$ is $\tilde{k}_{N_i, \text{last}}$. Therefore the greatest positive integer $\leq M_{a_i, n} - p_{a_i} - ((p^n - N_{i, \text{max}} - 1 - E_{a_i, b_i})/a_i)$ is $M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i, \text{last}}$. As k_{N_i} increases, $M_{a_i, n} - ((b_i - r_{a_i, n})/a_i) - ((k_{N_i} - 1)/a_i)$ decreases, the lower bound being equal to $M_{a_i, n} - p_{a_i} - ((p^n - N_{i, \text{max}} - 1 - E_{a_i, b_i})/a_i)$. Therefore the greatest positive integer $\leq M_{a_i, n} - ((b_i - r_{a_i, n})/a_i) - ((k_{N_i} - 1)/a_i)$ equals $M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i, \text{last}}$. Hence the greatest positive integer $\leq C_{i, n} := M_{a_i, n} - ((b_i - r_{a_i, n})/a_i) - ((k_{N_i} - 1)/a_i)$ is equal to

$$\begin{array}{lll} M_{a_i, n} - p_{a_i} & \text{if} & M_{a_i, n} - p_{a_i} \leq C_{i, n} \\ M_{a_i, n} - p_{a_i} - 1 & \text{if} & M_{a_i, n} - p_{a_i} - 1 \leq C_{i, n} < M_{a_i, n} - p_{a_i} \\ M_{a_i, n} - p_{a_i} - 2 & \text{if} & M_{a_i, n} - p_{a_i} - 2 \leq C_{i, n} < M_{a_i, n} - p_{a_i} - 1 \\ \vdots & \vdots & \vdots \\ M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i, \text{last}} & \text{if} & M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i, \text{last}} \leq C_{i, n} < M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i, \text{last}} + 1. \end{array} \quad (19)$$

This is equivalent to saying that the maximum value of the positive integer M for which $p^n - k_{N_i} - a_i M - b_i \geq 0$ equals

$$\begin{array}{lll} M_{a_i, n} - p_{a_i} & \text{if} & 1 \leq k_{N_i} \leq 1 + E_{a_i, b_i} \\ M_{a_i, n} - p_{a_i} - 1 & \text{if} & 1 + E_{a_i, b_i} < k_{N_i} \leq 1 + E_{a_i, b_i} + a_i \\ M_{a_i, n} - p_{a_i} - 2 & \text{if} & 1 + E_{a_i, b_i} + a_i < k_{N_i} \leq 1 + E_{a_i, b_i} + 2a_i \\ \vdots & \vdots & \vdots \\ M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i} & \text{if} & 1 + E_{a_i, b_i} + a_i (\tilde{k}_{N_i} - 1) < k_{N_i} \leq 1 + E_{a_i, b_i} + a_i \tilde{k}_{N_i} \\ \vdots & \vdots & \vdots \\ M_{a_i, n} - p_{a_i} - \tilde{k}_{N_i, \text{last}} & \text{if} & 1 + E_{a_i, b_i} + a_i (\tilde{k}_{N_i, \text{last}} - 1) < k_{N_i} \leq p^n - N_{i, \text{max}}. \end{array} \quad (20)$$

Case 2 (when $b_i < a_i$). In this case, we have 2 subcases.

Subcase 2.1 (when $b_i \leq r_{a_i,n}$). Here we have $p_{a_i} = 0$ and $0 \leq b_i \leq r_{a_i,n} < a_i$. Therefore $0 \leq (r_{a_i,n} - b_i)/a_i < 1$ and hence $-1 < (b_i - r_{a_i,n})/a_i \leq 0$. Therefore the least nonnegative integer $\geq (b_i - r_{a_i,n})/a_i$ is $p_{a_i} = 0$. The rest of the proof is similar to the proof of Case 1.

Subcase 2.2 (when $b_i > r_{a_i,n}$). Here we have, $p_{a_i} = 1$ and $0 \leq r_{a_i,n} < b_i < a_i$. Therefore $0 < (b_i - r_{a_i,n})/a_i < 1$. Hence the least positive integer $\geq (b_i - r_{a_i,n})/a_i$ is $p_{a_i} = 1$. The rest of the proof is similar to the proof of Case 1. $\square$

The following corollary is immediate.

Corollary 15. *Let A be a monomial in $\mathfrak{M}$ which [2] divides and let M be a positive integer. Let A and $A[2]^{-M}[1]^{M-1}$ be of the form as given in Remark 13. For $1 \leq i \leq r$, let $M_{i,A}$ be the maximum value of the positive integer M for which $p^n - k_{N_i} - a_i M - b_i$ is ≥ 0 .*

Then $M_{\max,A}$ equals the minimum of all $M_{i,A}$ s as i ranges from 1 to r .

Let $\mathfrak{M}[2] :=$ the set of all monomials in $\mathfrak{M}$ which are divisible by [2]. Let $M_{\max}[2] := \{M_{\max,A} \mid A \in \mathfrak{M}[2]\}$.

Remark 16. It follows from Corollary 15 above that $M_{\max}[2] = \{M_{a_r,n} - p_{a_r} - j \mid 0 \leq j \leq \tilde{k}_{N_r, \text{last}}\}$.

4.2.4. The Length Count for a Fixed $M_{\max,A}$

Remark 17. It is easy to verify that if M_1 is a positive integer such that $A[2]^{-M_1}[1]^{M_1-1}$ contains no negative powers and M_2 is a positive integer such that $M_2 \leq M_1$, then $A[2]^{-M_2}[1]^{M_2-1}$ contains no negative powers. It is also easy to verify that if N_1 is a positive integer such that $A[2]^{-N_1}[1]^{N_1}$ belongs to A_c and N_2 is a positive integer such that $N_1 \leq N_2$, then $A[2]^{-N_2}[1]^{N_2}$ belongs to A_c . Hence for any $A \in \mathfrak{M}[2]$, $A \in A_c + J$ if and only if either [1] divides A or $A[2]^{-M_{\max,A}}[1]^{M_{\max,A}}$ belongs to A_c .

Remark 18. Recall from Remark 13 as well as from Corollary 15 that given any monomial A in $\mathfrak{M}[2]$, the value of $M_{\max,A}$ depends only on the powers of the negative difference variables appearing in A . It now follows that for each $i \in \{1, \dots, r\}$, if we fix numbers $k_{N_i}^0$ such that $p^n - 1 \geq p^n - k_{N_i}^0 \geq N_{i,\max}$, then for any A in the set

$$\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0) := \left\{ A \mid A = N_r^{p^n - k_{N_r}^0} \dots N_1^{p^n - k_{N_1}^0} Z_s^{l_s} \dots Z_1^{l_1} P_1^{j_1} \dots P_t^{j_t} \right. \\ \left. \text{where } p^n - 1 \geq l_j \geq 0 \right. \quad (21)$$

for every $j \in \{1, \dots, s\}$ and,

$$p^n - 1 \geq j_q \geq 0 \text{ for every } q \in \{1, \dots, t\} \Big\},$$

the value of $M_{\max,A}$ is the same (call it $M_{\max}^0$).

The main result of this subsection is Lemma 24 below which provides an account of the length count for a fixed value of $M_{\max,A}$. But for stating the lemma, we need to introduce some notation first.

For an arbitrary element $M_{\max}$ in $M_{\max}[2]$ and for every $q \in \{1, \dots, t\}$, let

$$\text{Min}_{P_q, M_{\max}}^c := \text{Min} \left\{ p^n - P_{q,\min}, M_{\max} \Delta_{P_q} \right\}. \quad (22)$$

The symbol Min^c in the above definition of $\text{Min}_{P_q, M_{\max}}^c$ stands for “minimum convergent.” The reason behind this notation is explained in Remark 19 below.

Remark 19. Observe that if the power of the variable P_q in the monomial A is j_q , then the power of P_q in the monomial $A[2]^{-M_{\max}}[1]^{M_{\max}}$ is $j_q + M_{\max} \Delta_{P_q}$. If A is divisible by [2], then $p^n - 1 \geq j_q \geq P_{q,\min}$. Now if $j_q + M_{\max} \Delta_{P_q} \geq p^n$, then $A[2]^{-M_{\max}}[1]^{M_{\max}}$ belongs to A_c . That is, if $p^n - j_q \leq \text{Min}_{P_q, M_{\max}}^c$, then $A[2]^{-M_{\max}}[1]^{M_{\max}}$ belongs to A_c . That is, if $p^n - \text{Min}_{P_q, M_{\max}}^c \leq j_q \leq p^n - 1$, then $A[2]^{-M_{\max}}[1]^{M_{\max}}$ belongs to A_c . So $p^n - \text{Min}_{P_q, M_{\max}}^c$ is the minimum possible power of the variable P_q in the monomial A for which $A[2]^{-M_{\max}}[1]^{M_{\max}}$ belongs to A_c .

Recall the ordered set $P_1 \triangleright \dots \triangleright P_t$ of positive difference variables. Let $A_{t+1} := 0$ and $C_t := 1$. For any $q \in \{1, \dots, t-1\}$, let

$$C_q := p^{(t-q)n}. \quad (23)$$

And for any $q \in \{2, \dots, t\}$, let

$$A_q := (p^n - P_{q,\min}) A_{q+1} + P_{q,\min} C_q. \quad (24)$$

For any $M_{\max}$ in $M_{\max}[2]$, let $D_{t+1, M_{\max}} := 1$ and $\bar{D}_{t+1, M_{\max}} := 0$. For any $M_{\max}$ in $M_{\max}[2]$ and any $q \in \{1, \dots, t\}$, let

$$D_{q, M_{\max}} := \left(\text{Min}_{P_q, M_{\max}}^c \right) A_{q+1} \\ + \left(p^n - P_{q,\min} - \text{Min}_{P_q, M_{\max}}^c \right) D_{q+1, M_{\max}} \\ + \left(P_{q,\min} \right) C_q, \quad (25)$$

$$\bar{D}_{q, M_{\max}} := \begin{cases} \left(\text{Min}_{P_q, M_{\max}}^c \right) A_{q+1} \\ + \left(p^n - \text{Min}_{P_q, M_{\max}}^c - P_{q,\max} \right) \bar{D}_{q+1, M_{\max}} \\ + \left(P_{q,\max} - P_{q,\min} \right) D_{q+1, M_{\max}} + \left(P_{q,\min} \right) C_q \\ \quad \text{if } \text{Min}_{P_q, M_{\max}}^c < p^n - P_{q,\min} \\ \quad \text{and } P_{q,\max} < p^n - \text{Min}_{P_q, M_{\max}}^c \\ D_{q, M_{\max}} \\ \quad \text{if } \text{Min}_{P_q, M_{\max}}^c < p^n - P_{q,\min} \\ \quad \text{and } P_{q,\max} \geq p^n - \text{Min}_{P_q, M_{\max}}^c \\ \left(\text{Min}_{P_q, M_{\max}}^c \right) A_{q+1} + \left(P_{q,\min} \right) C_q \\ \quad \text{if } \text{Min}_{P_q, M_{\max}}^c = p^n - P_{q,\min}. \end{cases} \quad (26)$$

Recall the set $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$ from Remark 18. For any A in this set $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$, the value of $M_{\max, A}$ is the same, namely, $M_{\max}^0$. Lemma 24 below gives an account of the length count for the set $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$. But we will break up the proof of Lemma 24 in a few short steps as follows.

Lemma 20. *For every $q \in \{2, \dots, t\}$, let*

$$\begin{aligned} & \mathfrak{F}_{q, A, J_1, \dots, J_{q-1}} \\ & := \left\{ A = N_r^{p^n - k_{N_r}^0} \dots N_1^{p^n - k_{N_1}^0} Z_s^{l_s} \dots Z_1^{l_1} P_1^{j_1} \dots P_t^{j_t} \right. \\ & \quad \in \mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0) \mid l_s = l_{s-1} = \dots \\ & \quad \left. = l_1 = p^n - 1 \text{ and } j_1 = J_1, \dots, j_{q-1} = J_{q-1} \right\}, \end{aligned} \quad (27)$$

where for each $k \in \{1, \dots, q-1\}$, J_k is a fixed integer such that $P_{k, \min} \leq J_k \leq p^n - 1$ and at least one J_k is $\geq p^n - \text{Min}_{P_k, M_{\max}^0}^c$.

Then the total number of monomials A in the set $\mathfrak{F}_{q, A, J_1, \dots, J_{q-1}}$ for which $A \notin A_c + J$ equals A_q .

Proof. The proof is by reverse induction on q , the base case of induction being $q = t$. Observe that if A is any monomial in $\mathfrak{F}_{t, A, J_1, \dots, J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $P_{t, \min} \leq j_t \leq p^n - 1$, then [2] divides A and $A[2]^{-M_{\max}^0} [1]^{M_{\max}^0} \in A_c$. Hence such a monomial A satisfies $A \in A_c + J$. The total number of such monomials in $\mathfrak{F}_{t, A, J_1, \dots, J_{t-1}}$ is $p^n - P_{t, \min}$. And if A is any monomial in $\mathfrak{F}_{t, A, J_1, \dots, J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $0 \leq j_t < P_{t, \min}$, then neither [2] nor [1] divides A and hence $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t, A, J_1, \dots, J_{t-1}}$ is $P_{t, \min}$. Hence the total number of monomials A in the set $\mathfrak{F}_{t, A, J_1, \dots, J_{t-1}}$ for which $A \notin A_c + J$ equals

$$\begin{aligned} (p^n - P_{t, \min}) 0 + (P_{t, \min}) 1 &= (p^n - P_{t, \min}) A_{t+1} \\ &+ (P_{t, \min}) C_t = A_t. \end{aligned} \quad (28)$$

This proves the base case of induction. The rest of the proof follows by reverse induction and is left to the reader. $\square$

Lemma 21. *For every $q \in \{2, \dots, t\}$, let*

$$\begin{aligned} & \mathfrak{F}_{q, C, J_1, \dots, J_{q-1}} \\ & := \left\{ A = N_r^{p^n - k_{N_r}^0} \dots N_1^{p^n - k_{N_1}^0} Z_s^{l_s} \dots Z_1^{l_1} P_1^{j_1} \dots P_t^{j_t} \right. \\ & \quad \in \mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0) \mid l_s = l_{s-1} = \dots \\ & \quad \left. = l_1 = p^n - 1, j_1 = J_1, \dots, j_{q-1} = J_{q-1} \right\}, \end{aligned} \quad (29)$$

where for each $k \in \{1, \dots, q-1\}$, J_k is a fixed integer such that $0 \leq J_k \leq p^n - 1$ and at least one J_k is $< P_{k, \min}$.

Then the total number of monomials A in the set $\mathfrak{F}_{q, C, J_1, \dots, J_{q-1}}$ for which $A \notin A_c + J$ equals C_{q-1} .

Proof. The proof is by reverse induction on q , the base case of induction being $q = t$. Observe that if A is any monomial in $\mathfrak{F}_{t, C, J_1, \dots, J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $0 \leq j_t \leq p^n - 1$, then neither [2] nor [1] divides A and hence $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t, C, J_1, \dots, J_{t-1}}$ is p^n . Hence the total number of monomials A in the set $\mathfrak{F}_{t, C, J_1, \dots, J_{t-1}}$ for which $A \notin A_c + J$ equals $p^n = p^{(t-(t-1))n} = C_{t-1}$. This proves the base case of induction. The rest of the proof follows by reverse induction and is left to the reader. $\square$

Lemma 22. *For every $q \in \{2, \dots, t\}$, let*

$$\begin{aligned} & \mathfrak{F}_{q, D, J_1, \dots, J_{q-1}} \\ & := \left\{ A = N_r^{p^n - k_{N_r}^0} \dots N_1^{p^n - k_{N_1}^0} Z_s^{l_s} \dots Z_1^{l_1} P_1^{j_1} \dots P_t^{j_t} \right. \\ & \quad \in \mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0) \mid l_s = l_{s-1} = \dots \\ & \quad \left. = l_1 = p^n - 1 \text{ and } j_1 = J_1, \dots, j_{q-1} = J_{q-1} \right\}, \end{aligned} \quad (30)$$

where for each $k \in \{1, \dots, q-1\}$, J_k is a fixed integer such that $P_{k, \min} \leq J_k < p^n - \text{Min}_{P_k, M_{\max}^0}^c$.

Then the total number of monomials A in the set $\mathfrak{F}_{q, D, J_1, \dots, J_{q-1}}$ for which $A \notin A_c + J$ equals

- (i) $D_{q, M_{\max}^0}$ if there exists at least one $k \in \{1, \dots, q-1\}$ such that $P_{k, \max} \geq p^n - \text{Min}_{P_k, M_{\max}^0}^c$;
- (ii) $\bar{D}_{q, M_{\max}^0}$ if for every $k \in \{1, \dots, q-1\}$, we have $P_{k, \max} < p^n - \text{Min}_{P_k, M_{\max}^0}^c$ and $P_{k, \max} \leq J_k < p^n - \text{Min}_{P_k, M_{\max}^0}^c$;
- (iii) $D_{q, M_{\max}^0}$ if for every $k \in \{1, \dots, q-1\}$, we have $P_{k, \max} < p^n - \text{Min}_{P_k, M_{\max}^0}^c$ and there exists at least one $k \in \{1, \dots, q-1\}$ such that $P_{k, \min} \leq J_k < P_{k, \max}$.

Proof. The proof is by reverse induction on q , the base case of induction being $q = t$. Let us first prove the base case of induction.

Case 1. When there exists at least one $k \in \{1, \dots, t-1\}$ such that $P_{k, \max} \geq p^n - \text{Min}_{P_k, M_{\max}^0}^c$.

Then since we have that for each $k \in \{1, \dots, t-1\}$, J_k is a fixed integer such that $P_{k, \min} \leq J_k < p^n - \text{Min}_{P_k, M_{\max}^0}^c$, it follows that monomials in the set $\mathfrak{F}_{t, D, J_1, \dots, J_{t-1}}$ are not divisible by [1].

Observe now that if A is any monomial in $\mathfrak{F}_{t, D, J_1, \dots, J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $p^n - \text{Min}_{P_t, M_{\max}^0}^c \leq j_t \leq p^n - 1$, then [2] divides A and $A[2]^{-M_{\max}^0} [1]^{M_{\max}^0} \in A_c$. Hence such a monomial A satisfies $A \in A_c + J$. The total number of such monomials in $\mathfrak{F}_{t, D, J_1, \dots, J_{t-1}}$ is $\text{Min}_{P_t, M_{\max}^0}^c$. If A is any monomial in $\mathfrak{F}_{t, D, J_1, \dots, J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $P_{t, \min} \leq j_t < p^n - \text{Min}_{P_t, M_{\max}^0}^c$, then [2] divides A but [1] does not divide A and $A[2]^{-M_{\max}^0} [1]^{M_{\max}^0} \notin A_c$. Hence

such a monomial A satisfies $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\min}$. And if A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $0 \leq j_t < P_{t,\min}$, then neither [2] nor [1] divides A and hence $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $P_{t,\min}$. Hence the total number of monomials A in the set $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ for which $A \notin A_c + J$ equals

$$\begin{aligned} & (\text{Min}_{P_t, M_{\max}^0}^c) 0 + (p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\min}) 1 + (P_{t,\min}) 1 \\ &= (\text{Min}_{P_t, M_{\max}^0}^c) A_{t+1} + (p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\min}) D_{t+1, M_{\max}^0} \\ &+ (P_{t,\min}) C_t = D_{t, M_{\max}^0}. \end{aligned} \quad (31)$$

This proves the base case of induction in Case 1.

Case 2. When for every $k \in \{1, \dots, t-1\}$, we have $P_{k,\max} < p^n - \text{Min}_{P_k, M_{\max}^0}^c$ and $P_{k,\max} \leq J_k < p^n - \text{Min}_{P_k, M_{\max}^0}^c$. In this case, there are 3 subcases.

Subcase 2.1 ($\text{Min}_{P_t, M_{\max}^0}^c = p^n - P_{t,\min}$). Observe that if A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $p^n - \text{Min}_{P_t, M_{\max}^0}^c \leq j_t \leq p^n - 1$, then [2] divides A and $A[2]^{-M_{\max}^0}[1]^{M_{\max}^0} \in A_c$. Hence such a monomial A satisfies $A \in A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $\text{Min}_{P_t, M_{\max}^0}^c$. And if A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $0 \leq j_t < P_{t,\min} = p^n - \text{Min}_{P_t, M_{\max}^0}^c$, then neither [2] nor [1] divides A and hence $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $P_{t,\min}$. Hence the total number of monomials A in the set $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ for which $A \notin A_c + J$ equals

$$\begin{aligned} & (\text{Min}_{P_t, M_{\max}^0}^c) 0 + (P_{t,\min}) 1 \\ &= (\text{Min}_{P_t, M_{\max}^0}^c) A_{t+1} + (P_{t,\min}) C_t = \widetilde{D}_{t, M_{\max}^0}. \end{aligned} \quad (32)$$

This proves the base case of induction in this subcase.

Subcase 2.2 ($\text{Min}_{P_t, M_{\max}^0}^c < p^n - P_{t,\min}$ and $P_{t,\max} \geq p^n - \text{Min}_{P_t, M_{\max}^0}^c$). Observe that if A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $p^n - \text{Min}_{P_t, M_{\max}^0}^c \leq j_t \leq p^n - 1$, then [2] divides A and $A[2]^{-M_{\max}^0}[1]^{M_{\max}^0} \in A_c$. Hence such a monomial A satisfies $A \in A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $\text{Min}_{P_t, M_{\max}^0}^c$. If A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $P_{t,\min} \leq j_t < p^n - \text{Min}_{P_t, M_{\max}^0}^c$, then [2] divides A but [1] does not divide A (since $P_{t,\max} \geq p^n - \text{Min}_{P_t, M_{\max}^0}^c$) and $A[2]^{-M_{\max}^0}[1]^{M_{\max}^0} \notin A_c$. Hence such a monomial A satisfies $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\min}$. And if A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $0 \leq j_t < P_{t,\min}$, then neither [2] nor [1] divides A and hence $A \notin A_c + J$. The total number of such

monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $P_{t,\min}$. Hence the total number of monomials A in the set $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ for which $A \notin A_c + J$ equals

$$\begin{aligned} & (\text{Min}_{P_t, M_{\max}^0}^c) 0 + (p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\min}) 1 + (P_{t,\min}) 1 \\ &= (\text{Min}_{P_t, M_{\max}^0}^c) A_{t+1} + (p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\min}) D_{t+1, M_{\max}^0} \\ &+ (P_{t,\min}) C_t = D_{t, M_{\max}^0} = \widetilde{D}_{t, M_{\max}^0}. \end{aligned} \quad (33)$$

This proves the base case of induction in this subcase.

Subcase 2.3 ($\text{Min}_{P_t, M_{\max}^0}^c < p^n - P_{t,\min}$ and $P_{t,\max} < p^n - \text{Min}_{P_t, M_{\max}^0}^c$). Observe that if A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $p^n - \text{Min}_{P_t, M_{\max}^0}^c \leq j_t \leq p^n - 1$, then [2] divides A and $A[2]^{-M_{\max}^0}[1]^{M_{\max}^0} \in A_c$. Hence such a monomial A satisfies $A \in A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $\text{Min}_{P_t, M_{\max}^0}^c$. If A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $P_{t,\max} \leq j_t < p^n - \text{Min}_{P_t, M_{\max}^0}^c$, then [1] divides A (since $P_{t,\max} \leq j_t$ and $P_{k,\max} \leq J_k < p^n - \text{Min}_{P_k, M_{\max}^0}^c$ for every $k \in \{1, \dots, t-1\}$). Hence such a monomial A satisfies $A \in A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\max}$. If A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $P_{t,\min} \leq j_t < P_{t,\max}$, then [2] divides A but [1] does not divide A (since $j_t < P_{t,\max}$) and $A[2]^{-M_{\max}^0}[1]^{M_{\max}^0} \notin A_c$ (since $j_t < P_{t,\max} < p^n - \text{Min}_{P_t, M_{\max}^0}^c$). Hence such a monomial A satisfies $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $P_{t,\max} - P_{t,\min}$. And if A is any monomial in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ such that the power j_t of the variable P_t in it satisfies the inequality $0 \leq j_t < P_{t,\min}$, then neither [2] nor [1] divides A and hence $A \notin A_c + J$. The total number of such monomials in $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ is $P_{t,\min}$. Hence the total number of monomials A in the set $\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ for which $A \notin A_c + J$ equals

$$\begin{aligned} & (\text{Min}_{P_t, M_{\max}^0}^c) 0 + (p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\max}) 0 \\ &+ (P_{t,\max} - P_{t,\min}) 1 + (P_{t,\min}) 1 \\ &= (\text{Min}_{P_t, M_{\max}^0}^c) A_{t+1} + (p^n - \text{Min}_{P_t, M_{\max}^0}^c - P_{t,\max}) \\ &\times \widetilde{D}_{t+1, M_{\max}^0} + (P_{t,\max} - P_{t,\min}) D_{t+1, M_{\max}^0} + (P_{t,\min}) C_t \\ &= \widetilde{D}_{t, M_{\max}^0}. \end{aligned} \quad (34)$$

This proves the base case of induction in this subcase. Hence the base case of induction is proved in Case 2.

Case 3. When for every $k \in \{1, \dots, t-1\}$, we have $P_{k,\max} < p^n - \text{Min}_{P_k, M_{\max}^0}^c$ and there exists at least one $k \in \{1, \dots, t-1\}$ such that $P_{k,\min} \leq J_k < P_{k,\max}$.

Since there exists at least one $k \in \{1, \dots, t-1\}$ such that $P_{k,\min} \leq J_k < P_{k,\max}$, it follows that monomials in the set

$\mathfrak{F}_{t,D,J_1,\dots,J_{t-1}}$ are not divisible by [1]. The rest of the proof of this case is similar to the proof of Case 1. This proves the base case of induction in Case 3.

This proves the base case of induction. The rest of the proof follows by reverse induction and is left to the reader. $\square$

Let

$$\begin{aligned} \mathfrak{F}_1 &:= \left\{ A = N_r^{p^n - k_{N_r}^0} \dots N_1^{p^n - k_{N_1}^0} Z_s^{l_s} \dots Z_1^{l_1} P_1^{j_1} \dots P_t^{j_t} \right. \\ &\quad \left. \in \mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0) \mid l_s = l_{s-1} = \dots \right. \\ &\quad \left. = l_1 = p^n - 1 \right\}. \end{aligned} \quad (35)$$

Lemma 23. *The total number of monomials A in the set $\mathfrak{F}_1$ for which $A \notin A_c + J$ equals $\tilde{D}_{1,M_{\max}^0}$.*

Proof. Let J_1 be an arbitrarily fixed integer such that $p^n - \text{Min}_{P_1, M_{\max}^0}^c \leq J_1 \leq p^n - 1$. It then follows from Lemma 20 that for any such fixed J_1 , the total number of monomials A in $\mathfrak{F}_{2,A,J_1}$ such that $A \notin A_c + J$ equals A_2 . Also if J_1 is an arbitrarily fixed integer such that $0 \leq J_1 < P_{1,\min}$, then it follows from Lemma 21 that for any such fixed J_1 , the total number of monomials A in $\mathfrak{F}_{2,C,J_1}$ such that $A \notin A_c + J$ equals C_1 . Now there are 3 cases.

Case 1 (if $\text{Min}_{P_1, M_{\max}^0}^c = p^n - P_{1,\min}$). In this case, the total number of monomials A in the set $\mathfrak{F}_1$ for which $A \notin A_c + J$ equals

$$\begin{aligned} &(p^n - 1 - p^n + \text{Min}_{P_1, M_{\max}^0}^c + 1) A_2 + (P_{1,\min}) C_1 \\ &= (\text{Min}_{P_1, M_{\max}^0}^c) A_2 + (P_{1,\min}) C_1 = \tilde{D}_{1,M_{\max}^0}. \end{aligned} \quad (36)$$

Case 2 (if $\text{Min}_{P_1, M_{\max}^0}^c < p^n - P_{1,\min}$ and $P_{1,\max} \geq p^n - \text{Min}_{P_1, M_{\max}^0}^c$). It follows from Lemma 22 that for any fixed integer J_1 such that $P_{1,\min} \leq J_1 < p^n - \text{Min}_{P_1, M_{\max}^0}^c$, the total number of monomials A in $\mathfrak{F}_{2,D,J_1}$ for which $A \notin A_c + J$ equals $D_{2,M_{\max}^0}$. Hence in this case, the total number of monomials A in the set $\mathfrak{F}_1$ for which $A \notin A_c + J$ equals

$$\begin{aligned} &(p^n - 1 - p^n + \text{Min}_{P_1, M_{\max}^0}^c + 1) A_2 \\ &\quad + (p^n - \text{Min}_{P_1, M_{\max}^0}^c - P_{1,\min}) D_{2,M_{\max}^0} + (P_{1,\min}) C_1 \\ &= (\text{Min}_{P_1, M_{\max}^0}^c) A_2 + (p^n - \text{Min}_{P_1, M_{\max}^0}^c - P_{1,\min}) D_{2,M_{\max}^0} \\ &\quad + (P_{1,\min}) C_1 = D_{1,M_{\max}^0} = \tilde{D}_{1,M_{\max}^0}. \end{aligned} \quad (37)$$

Case 3 (if $\text{Min}_{P_1, M_{\max}^0}^c < p^n - P_{1,\min}$ and $P_{1,\max} < p^n - \text{Min}_{P_1, M_{\max}^0}^c$). It follows from Lemma 22 that for any fixed integer J_1 such that $P_{1,\max} \leq J_1 < p^n - \text{Min}_{P_1, M_{\max}^0}^c$, the total number of monomials A in $\mathfrak{F}_{2,D,J_1}$ for which $A \notin A_c + J$ equals

$\tilde{D}_{2,M_{\max}^0}$. It again follows from Lemma 22 that for any fixed integer J_1 such that $P_{1,\min} \leq J_1 < P_{1,\max}$, the total number of monomials A in $\mathfrak{F}_{2,D,J_1}$ for which $A \notin A_c + J$ equals $D_{2,M_{\max}^0}$. Hence in this case, the total number of monomials A in the set $\mathfrak{F}_1$ for which $A \notin A_c + J$ equals

$$\begin{aligned} &(p^n - 1 - p^n + \text{Min}_{P_1, M_{\max}^0}^c + 1) A_2 \\ &\quad + (p^n - \text{Min}_{P_1, M_{\max}^0}^c - P_{1,\max}) \tilde{D}_{2,M_{\max}^0} \\ &\quad + (P_{1,\max} - P_{1,\min}) D_{2,M_{\max}^0} + (P_{1,\min}) C_1 \\ &= (\text{Min}_{P_1, M_{\max}^0}^c) A_2 + (p^n - \text{Min}_{P_1, M_{\max}^0}^c - P_{1,\max}) \tilde{D}_{2,M_{\max}^0} \\ &\quad + (P_{1,\max} - P_{1,\min}) D_{2,M_{\max}^0} + (P_{1,\min}) C_1 = \tilde{D}_{1,M_{\max}^0}. \end{aligned} \quad (38)$$

$\square$

Lemma 24. *The total number of monomials A in the set $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$ for which $A \notin A_c + J$ equals $\mathfrak{S}_{M_{\max}^0, s}$ where $\mathfrak{S}_{M_{\max}^0, s}$ is defined inductively as follows:*

$$\begin{aligned} \mathfrak{S}_{M_{\max}^0, 0} &:= \tilde{D}_{1,M_{\max}^0}, \quad \text{for every } j \in \{1, \dots, s\}, \\ \mathfrak{S}_{M_{\max}^0, j} &:= (\mathfrak{S}_{M_{\max}^0, j-1}) (p^n - Z_{j,\min}) + (p^{(t+j-1)n}) (Z_{j,\min}). \end{aligned} \quad (39)$$

Proof. Fix an arbitrary integer L_1 such that $p^n - 1 \geq L_1 \geq Z_{1,\min}$. Let

$$\begin{aligned} \mathfrak{P}_{L_1} &:= \left\{ A = N_r^{p^n - k_{N_r}^0} \dots N_1^{p^n - k_{N_1}^0} Z_s^{l_s} \dots Z_1^{l_1} P_1^{j_1} \dots P_t^{j_t} \right. \\ &\quad \left. \in \mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0) \mid l_s = l_{s-1} = \dots \right. \\ &\quad \left. = l_2 = p^n - 1, l_1 = L_1 \right\}. \end{aligned} \quad (40)$$

It is now easy to see that the total number of monomials A in $\mathfrak{P}_{L_1}$ for which $A \notin A_c + J$ equals $\tilde{D}_{1,M_{\max}^0} = \mathfrak{S}_{M_{\max}^0, 0}$ (the proof of this is similar to the proof of Lemma 23).

Fix an arbitrary integer M_1 such that $0 \leq M_1 < Z_{1,\min}$. Let

$$\begin{aligned} \mathfrak{P}_{M_1} &:= \left\{ A = N_r^{p^n - k_{N_r}^0} \dots N_1^{p^n - k_{N_1}^0} Z_s^{l_s} \dots Z_1^{l_1} P_1^{j_1} \dots P_t^{j_t} \right. \\ &\quad \left. \in \mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0) \mid l_s = l_{s-1} = \dots \right. \\ &\quad \left. = l_2 = p^n - 1, l_1 = M_1 \right\}. \end{aligned} \quad (41)$$

Since $0 \leq M_1 < Z_{1,\min}$, therefore any monomial $A \in \mathfrak{P}_{M_1}$ is divisible neither by [1] nor by [2]. Hence the total number of monomials A in $\mathfrak{P}_{M_1}$ for which $A \notin A_c + J$ equals p^{tn} .

So if we define $\mathfrak{P}_1$ as the subset of $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$ consisting of all those monomials in $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$ for which $l_s = l_{s-1} = \dots = l_2 = p^n - 1$, then the total number of monomials A in the set $\mathfrak{P}_1$ for which $A \notin A_c + J$

equals $\mathfrak{S}_{M_{\max}^0,1}$. Similarly, if we define $\mathfrak{P}_2$ as the subset of $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$ consisting of all those monomials in it for which $l_s = l_{s-1} = \dots = l_3 = p^n - 1$, then the total number of monomials A in the set $\mathfrak{P}_2$ for which $A \notin A_c + J$ equals $\mathfrak{S}_{M_{\max}^0,2}$. Proceeding inductively, it is now easy to see that the total number of monomials A in the set $\mathfrak{P}(k_{N_1}^0, k_{N_2}^0, \dots, k_{N_r}^0)$ for which $A \notin A_c + J$ equals $\mathfrak{S}_{M_{\max}^0}$. $\square$

Definition 25. Given any $M_{\max}$ in $M_{\max}[2]$, let $\Phi_{M_{\max}} := \mathfrak{S}_{M_{\max},s}$ where $\mathfrak{S}_{M_{\max},s}$ is defined in the same way as $\mathfrak{S}_{M_{\max}^0,s}$ is defined in Lemma 24 (just replace $M_{\max}^0$ in Lemma 24 by $M_{\max}$).

4.2.5. The Formula for the Hilbert-Kunz Function. We will now state the formula for the Hilbert-Kunz function for the general binomial hypersurface that we have considered above. But for stating the formula, we need some more notation, which we provide first.

Set $G_{P_{(t-1)}} := P_{t,\max}$ and for every $q \in \{1, \dots, (t-2)\}$, let $G_{P_q} := (p^n - P_{q+1,\max})G_{P_{q+1}} + (P_{q+1,\max})p^{n(t-q-1)}$;

Set $G_{Z_l} := (p^n - P_{1,\max})G_{P_1} + (P_{1,\max})p^{n(t-1)}$, and for every $l \in \{2, \dots, s\}$, let $G_{Z_l} := (p^n - Z_{l-1,\max})G_{Z_{l-1}} + (Z_{l-1,\max})p^{n(t+l-2)}$;

Set $G_{N_i} := (p^n - Z_{s,\max})G_{Z_s} + (Z_{s,\max})p^{n(t+s-1)}$, and for every $i \in \{2, \dots, r\}$, let $G_{N_i} := (p^n - N_{i-1,\min})G_{N_{i-1}} + (N_{i-1,\min})p^{n(t+s+i-2)}$.

Recall that for any $i \in \{1, \dots, r\}$, let $a_i := -(\Delta_{N_i})$ and for any $i \in \{1, \dots, r\}$ and any positive integer n , $M_{a_i,n}$ and $r_{a_i,n}$ are defined by the equation $p^n - 1 = a_i M_{a_i,n} + r_{a_i,n}$. Also recall the definitions of E_{a_i,b_i} and p_{a_i} from Section 4.2.3.

For every $i \in \{1, \dots, r\}$, let

$$\begin{aligned} D_{a_i,a_r} &:= (1 + E_{a_i,b_i}) + a_i (M_{a_i,n} - p_{a_i} - M_{a_r,n} + p_{a_r}) \\ a'_i &:= p^n - N_{i,\max} - [1 + E_{a_i,b_i} + a_i (\tilde{k}_{N_i,\text{last}} - 1)], \\ \tilde{X}_{N_i,\text{last}} &:= M_{a_r,n} - p_{a_r} - M_{a_i,n} + p_{a_i} + \tilde{k}_{N_i,\text{last}}. \end{aligned} \quad (42)$$

Clearly $a'_i \leq a_i$ for every $i \in \{1, \dots, r\}$.

For any nonnegative integer j such that $M_{a_r,n} - p_{a_r} - j$ belongs to the set $M_{\max}[2]$, let

$$\int_j^{(N_i)} := \Phi_{M_{a_r,n} - p_{a_r} - j}, \quad \text{for any } i \in \{2, \dots, r\}, \quad (43)$$

and let

$$\int_j^{(N_i)} := \begin{cases} (D_{a_{i-1},a_r} + j a_{i-1}) \int_j^{(N_{i-1})} + a_{i-1} \int_{j+1}^{(N_{i-1})} + \dots \\ + a_{i-1} \int_{(\tilde{X}_{N_{i-1},\text{last}})-1}^{(N_{i-1})} + a'_{i-1} \int_{\tilde{X}_{N_{i-1},\text{last}}}^{(N_{i-1})} \\ + (N_{i-1,\max} - N_{i-1,\min}) G_{N_{i-1}} \\ + (N_{i-1,\min}) p^{n(s+t+i-2)} \quad \text{if } j < \tilde{X}_{N_{i-1},\text{last}} \\ (p^n - N_{i-1,\max}) \int_j^{(N_{i-1})} \\ + (N_{i-1,\max} - N_{i-1,\min}) G_{N_{i-1}} \\ + (N_{i-1,\min}) p^{n(s+t+i-2)} \quad \text{otherwise.} \end{cases} \quad (44)$$

Theorem 26. The formula for the Hilbert-Kunz function of the general binomial hypersurface under consideration is

$$\begin{aligned} D_{a_r,a_r} \int_0^{(N_r)} + a_r \int_1^{(N_r)} + \dots + a_r \int_{(\tilde{X}_{N_r,\text{last}})-1}^{(N_r)} + a'_r \int_{\tilde{X}_{N_r,\text{last}}}^{(N_r)} \\ + (N_{r,\max} - N_{r,\min}) G_{N_r} + (N_{r,\min}) p^{n(s+t+r-1)}. \end{aligned} \quad (45)$$

Proof. Observe that for any fixed $i \in \{2, \dots, r\}$ and any $j \in \mathbb{Z}$ such that $0 \leq j \leq \tilde{k}_{N_r,\text{last}}$, the number $\int_j^{(N_i)}$ provides an account of the number of monomials A in $\mathfrak{M}$ which satisfy the following 3 properties simultaneously.

- (i) $A \notin A_c + J$.
- (ii) The power of each of the variables $N_{i+1}, \dots, N_r$ is $p^n - 1$.
- (iii) The power of the variable N_i is $p^n - k_{N_i}$ where

$$1 + E_{a_i,b_i} + a_i (\tilde{k}_{N_i} - 1) < k_{N_i} \leq 1 + E_{a_i,b_i} + a_i \tilde{k}_{N_i}, \quad (46)$$

$$\text{and } \tilde{k}_{N_i} = M_{a_i,n} - p_{a_i} - M_{a_r,n} + p_{a_r} + j.$$

Therefore for any $j \in \mathbb{Z}$ such that $0 \leq j \leq \tilde{X}_{N_r,\text{last}}$, $\int_j^{(N_r)}$ denotes the number of monomials A in $\mathfrak{M}$ for which $A \notin A_c + J$ and the power of the variable N_r is $p^n - k_{N_r}$ where $1 + E_{a_r,b_r} + a_r (j-1) < k_{N_r} \leq 1 + E_{a_r,b_r} + a_r j$.

Clearly then the number of monomials A in $\mathfrak{M}$ for which $A \notin A_c + J$ and the power of the variable N_r is $p^n - k_{N_r}$ where $1 \leq k_{N_r} \leq 1 + E_{a_r,b_r}$ equals $D_{a_r,a_r} \int_0^{(N_r)}$. Similarly, the number of monomials A in $\mathfrak{M}$ for which $A \notin A_c + J$ and the power of the variable N_r is $p^n - k_{N_r}$ where $1 + E_{a_r,b_r} + a_r (j-1) < k_{N_r} \leq 1 + E_{a_r,b_r} + a_r j$ and $1 \leq j \leq (\tilde{X}_{N_r,\text{last}}) - 1$ equals $a_r \int_j^{(N_r)}$.

We can similarly account for the terms $a'_r \int_{\tilde{X}_{N_r,\text{last}}}^{(N_r)}$, $(N_{r,\max} - N_{r,\min}) G_{N_r}$, and $(N_{r,\min}) p^{n(s+t+r-1)}$ in the formula (45) of the Hilbert-Kunz function. $\square$

4.3. An Illustrative Example. For this example as well as for the example given in Section 4.4 below, we will need the following notation.

Given any real number β , let $\langle \beta \rangle$ denote the smallest integer $\geq \beta$ and let $[\beta]$ denote the largest integer $\leq \beta$. We call $\langle \beta \rangle$ and $[\beta]$ the upper and the lower integral parts of β .

Let $S = k[N_2, N_1, Z_2, Z_1, P_1, P_2]$ where k is a field of arbitrary prime characteristic $p > 0$, and let J be the ideal in S defined by $J = (f)$ where $f = N_2^{10} N_1^6 Z_2^5 Z_1^2 P_1^3 P_2^4 + N_2^4 N_1^2 Z_2^5 Z_1^2 P_1^6 P_2^9$. The differences $\Delta_{N_2}, \Delta_{N_1}, \Delta_{Z_2}, \Delta_{Z_1}, \Delta_{P_1}$, and Δ_{P_2} are therefore given by $-6, -4, 0, 0, 3, 5$, respectively. Hence the N_i 's are the negative difference variables, the Z_i 's are the zero difference variables, and the P_i 's are the positive difference variables. Here $a_1 = -\Delta_{N_1} = 4$, $a_2 = -\Delta_{N_2} = 6$, $b_1 = N_{1,\min} = 2$, and $b_2 = N_{2,\min} = 4$. Let $E_{a_1, b_1}, E_{a_2, b_2}$ be as defined in Section 4.2.3 above. Here, $E_{a_1, b_1} = E_{4,2}$ and $E_{a_2, b_2} = E_{6,4}$. For any $i \in \{1, 2\}$, let $M_{a_i, n}$ and $r_{a_i, n}$ be defined by the equation(s) $p^n - 1 = a_i M_{a_i, n} + r_{a_i, n}$. That is,

$$M_{a_1, n} = M_{4, n} = \frac{p^n - 1 - r_{4, n}}{4},$$

$$M_{a_2, n} = M_{6, n} = \frac{p^n - 1 - r_{6, n}}{6},$$

$$\text{Also } \tilde{k}_{N_1, \text{last}} = \left\langle \frac{p^n - 6 - 1 - E_{a_1, b_1}}{4} \right\rangle,$$

$$\tilde{k}_{N_2, \text{last}} = \left\langle \frac{p^n - 10 - 1 - E_{a_2, b_2}}{6} \right\rangle,$$

$$\text{the set } M_{\max}[2] = \{M_{6, n} - p_6 - j \mid j \in \mathbb{Z}, 0 \leq j \leq \tilde{k}_{N_2, \text{last}}\}. \quad (47)$$

For every $j \in \mathbb{Z}$ such that $0 \leq j \leq \tilde{k}_{N_2, \text{last}}$, we have

$$\begin{aligned} \text{Min}_{P_1, M_{6, n} - p_6 - j}^c &:= \text{Min} \{p^n - 3, (M_{6, n} - p_6 - j) \Delta_{P_1}\} \\ &= 3(M_{6, n} - p_6 - j), \\ \text{Min}_{P_2, M_{6, n} - p_6 - j}^c &:= \text{Min} \{p^n - 4, (M_{6, n} - p_6 - j) \Delta_{P_2}\} \\ &= 5(M_{6, n} - p_6 - j). \end{aligned} \quad (48)$$

Using the formulae given in Lemma 24, it follows that

$$\begin{aligned} \mathfrak{S}_{M_{6, n} - p_6 - j, 0} &= 15p^n - 30(M_{6, n} - p_6 - j) - 54, \\ \mathfrak{S}_{M_{6, n} - p_6 - j, 1} &= [15p^n - 30(M_{6, n} - p_6 - j) - 54] \\ &\quad \times (p^n - 2) + 2p^{2n}, \\ \mathfrak{S}_{M_{6, n} - p_6 - j, 2} &= [15p^n - 30(M_{6, n} - p_6 - j) - 54] \\ &\quad \times (p^n - 2)(p^n - 5) + 7p^{3n} - 10p^{2n}. \end{aligned} \quad (49)$$

Since for any $j \in \mathbb{Z}$ such that $0 \leq j \leq \tilde{k}_{N_2, \text{last}}$, we know that $\int_j^{(N_1)} := \mathfrak{S}_{M_{6, n} - p_6 - j, s}$ and s = the total number of zero difference variables = 2 here, therefore we have

$$\begin{aligned} \int_j^{(N_1)} &= [15p^n - 30(M_{6, n} - p_6 - j) - 54] \\ &\quad \times (p^n - 2)(p^n - 5) + 7p^{3n} - 10p^{2n}. \end{aligned} \quad (50)$$

For getting the formula for the Hilbert-Kunz function in this case, we need to compute $\int_j^{(N_2)}$ for the various values of j under consideration. A straight forward computation tells us that for any $j \in \mathbb{Z}$ such that $0 \leq j < \tilde{X}_{N_1, \text{last}}$, we have

$$\begin{aligned} \int_j^{(N_2)} &= (D_{a_1, a_2} + 4j) \int_j^{(N_1)} + 4 \sum_{k=j+1}^{\tilde{X}_{N_1, \text{last}}-1} \int_k^{(N_1)} \\ &\quad + a'_1 \int_{\tilde{X}_{N_1, \text{last}}}^{(N_1)} + (6-2) G_{N_1} + 2p^{n(2+2+2-2)}, \end{aligned} \quad (51)$$

where

$$\begin{aligned} D_{a_1, a_2} &= (1 + E_{a_1, b_1}) + 4(M_{4, n} - p_4 - M_{6, n} + p_6) \\ &= (1 + E_{4,2}) + 4(M_{4, n} - p_4 - M_{6, n} + p_6) \\ a'_1 &= p^n - 6 - 1 - E_{a_1, b_1} - 4(\tilde{k}_{N_1, \text{last}} - 1), \\ \tilde{X}_{N_1, \text{last}} &= M_{6, n} - p_6 - M_{4, n} + p_4 + \tilde{k}_{N_1, \text{last}}, \\ G_{N_1} &= 19p^{3n} - 130p^{2n} + 372p^n - 360. \end{aligned} \quad (52)$$

Similarly, for any $j \in \mathbb{Z}$ such that $j \geq \tilde{X}_{N_1, \text{last}}$, we have

$$\begin{aligned} \int_j^{(N_2)} &= (p^n - 6) \int_j^{(N_1)} + (6-2) G_{N_1} + 2p^{n(2+2+2-2)} \\ &= (p^n - 6) [(15p^n - 30(M_{6, n} - p_6 - j) - 54) \\ &\quad \times (p^n - 2)(p^n - 5) + 7p^{3n} - 10p^{2n}] \\ &\quad + 4[19p^{3n} - 130p^{2n} + 372p^n - 360] + 2p^{4n}. \end{aligned} \quad (53)$$

Clearly then $\int_j^{(N_2)}$ is a rational function in p and n for all values of j . Therefore, the formula for the corresponding Hilbert-Kunz function is

$$\begin{aligned} D_{a_2, a_2} \int_0^{(N_2)} &+ 6 \sum_{j=1}^{\tilde{X}_{N_2, \text{last}}-1} \int_j^{(N_2)} + a'_2 \int_{\tilde{X}_{N_2, \text{last}}}^{(N_2)} \\ &+ (10-4) G_{N_2} + 4p^{n(2+2+2-1)}, \end{aligned} \quad (54)$$

where

$$\begin{aligned} D_{a_2, a_2} &= 1 + E_{a_2, b_2} = 1 + E_{6,4}, \\ a'_2 &= p^n - 10 - 1 - E_{a_2, b_2} - 6(\tilde{k}_{N_2, \text{last}} - 1), \\ \tilde{X}_{N_2, \text{last}} &= M_{6, n} - p_6 - M_{4, n} + p_4 + \tilde{k}_{N_2, \text{last}}, \\ G_{N_2} &= 21p^{4n} - 168p^{3n} + 632p^{2n} - 1104p^n + 720, \end{aligned} \quad (55)$$

the formulae of $\int_j^{(N_2)}$ are given above.

Since all the quantities involved in the above formula for the Hilbert-Kunz function are rational functions in the variables p and n , therefore the Hilbert-Kunz function is a rational function in p and n .

4.4. Example of the One-Dimensional Case. In this subsection, we will discuss the case of 1-dimensional binomial hypersurfaces and will observe that using the above formula for the Hilbert-Kunz function, we do get that in this case of 1-dimension, the associated Hilbert-Kunz multiplicity is an integer (e.g., see Chapter 6, Corollary 6.2 of [7]). In this case, this integer happens to be equal to the ordinary multiplicity because the Hilbert-Kunz multiplicity is equal to the ordinary multiplicity for 1-dimensional rings.

In the case of 1-dimensional binomial hypersurfaces, we have $S = k[x_1, x_2]$, $J = (f)$ where $f = [2] + [1]$, $R = S/J$ and the terms $[2]$ and $[1]$ of f are monomials in the 2 variables x_1 and x_2 .

There are 3 possible cases:

- (1) $x_1 \triangleright x_2$ where x_1 is a negative difference variable and x_2 is a positive difference variable;
- (2) $x_1 \triangleright x_2$ where x_1 is a negative difference variable and x_2 is a zero difference variable;
- (3) $x_1 \triangleright x_2$ where x_1 is a zero difference variable and x_2 is a positive difference variable.

We will first discuss Cases (2) and (3).

In Cases (2) and (3), it is easy to check (without using the formula above) following.

The Hilbert-Kunz function evaluated at $p^n = (x_{1,\min} + x_{2,\min})p^n - x_{1,\min}x_{2,\min}$ and The associated Hilbert-Kunz multiplicity = $x_{1,\min} + x_{2,\min}$ which is an integer.

We will now discuss Case (1).

Since x_1 is a negative difference variable and x_2 is a positive difference variable, then using our notation in Section 4.2.1, we can denote x_1 by N_1 and x_2 by P_1 . Then writing $x_1 \triangleright x_2$ is the same as writing $N_1 \triangleright P_1$. In this case, there are 3 subcases.

Subcase 1.1 (when $a_1 < \Delta_{P_1}$). It is an exercise to check that using the above formula for the Hilbert-Kunz function, we get that for n large enough.

The Hilbert-Kunz function evaluated at p^n equals

$$\begin{aligned}
 & (1 + E_{a_1, b_1}) P_{1,\min} + a_1 P_{1,\min} \\
 & \times \left[p^n \frac{(\Delta_{P_1} - a_1)}{a_1 \Delta_{P_1}} - \frac{1}{a_1} - \frac{r_{a_1, n}}{a_1} - p_{a_1} + \frac{P_{1,\min}}{\Delta_{P_1}} \right] \\
 & - a_1 P_{1,\max} + a_1 P_{1,\max} \left\langle \frac{p^n - N_{1,\max} - 1 - E_{a_1, b_1}}{a_1} \right\rangle \\
 & - a_1 P_{1,\max} \left[p^n \frac{(\Delta_{P_1} - a_1)}{a_1 \Delta_{P_1}} - \frac{1}{a_1} - \frac{r_{a_1, n}}{a_1} - p_{a_1} + \frac{P_{1,\min}}{\Delta_{P_1}} \right] \\
 & + a'_1 P_{1,\max} + (N_{1,\max} - N_{1,\min}) P_{1,\max} + (N_{1,\min}) p^n, \quad (56)
 \end{aligned}$$

where for any real number β , $\langle \beta \rangle$ and $[\beta]$ denote the upper and lower integral parts of β , respectively.

Hence the associated Hilbert-Kunz multiplicity equals

$$a_1 P_{1,\min} \frac{(\Delta_{P_1} - a_1)}{a_1 \Delta_{P_1}} + \frac{a_1 P_{1,\max}}{\Delta_{P_1}} + N_{1,\min} \quad (57)$$

which is nothing but $P_{1,\min} + a_1 + N_{1,\min}$, which is an integer.

Subcase 1.2 (when $a_1 > \Delta_{P_1}$). It is an exercise to check that using the above formula for the Hilbert-Kunz function, we get that for n large enough

the Hilbert-Kunz function evaluated at p^n equals

$$(p^n - N_{1,\max}) P_{1,\max} + (N_{1,\max} - N_{1,\min}) P_{1,\max} + (N_{1,\min}) p^n. \quad (58)$$

Hence the associated Hilbert-Kunz multiplicity equals $P_{1,\max} + N_{1,\min}$, which is an integer.

Subcase 1.3 (when $a_1 = \Delta_{P_1}$). Let $\tilde{N}_0$ be the smallest positive integer such that if $\tilde{k} > \tilde{N}_0$, then $P_{1,\min} - 1 - r_{a_1, n} - a_1 p_{a_1} < a_1 \tilde{k}$. Similarly, let $\tilde{N}'_0$ be the smallest positive integer such that if $\tilde{k} > \tilde{N}'_0$, then $P_{1,\max} - 1 - r_{a_1, n} - a_1 p_{a_1} < a_1 \tilde{k}$. Clearly then $\tilde{N}'_0 \geq \tilde{N}_0$.

It is an exercise to check that using the above formula for the Hilbert-Kunz function, we get that for n large enough,

the Hilbert-Kunz function evaluated at p^n equals

$$\begin{aligned}
 & (1 + E_{a_1, b_1}) P_{1,\min} + a_1 P_{1,\min} \tilde{N}_0 \\
 & + a_1 \sum_{\tilde{k}=\tilde{N}_0+1}^{\tilde{N}'_0} (1 + r_{a_1, n} + a_1 p_{a_1} + a_1 \tilde{k}) \\
 & + a_1 P_{1,\max} \left(\left\langle \frac{p^n - N_{1,\max} - 1 - E_{a_1, b_1}}{a_1} \right\rangle - 1 - \tilde{N}'_0 \right) \\
 & + a'_1 P_{1,\max} + (N_{1,\max} - N_{1,\min}) P_{1,\max} + (N_{1,\min}) p^n, \quad (59)
 \end{aligned}$$

where for any real number β , $\langle \beta \rangle$ denotes the upper integral part of β .

Hence the associated Hilbert-Kunz multiplicity equals $P_{1,\max} + N_{1,\min}$, which is an integer.

4.5. Rationality of the Hilbert-Kunz Multiplicity. The Hilbert-Kunz multiplicity for the general Binomial hypersurface (whose Hilbert-Kunz function is given by (45) above) is by definition equal to the coefficient of $p^{n(s+t+r-1)}$ in the formula given by (45). We can easily see from the above discussion that for any j such that $0 \leq j \leq \tilde{X}_{N_r, \text{last}}$, $\int_j^{(N_r)}$ is obtained iteratively by first computing $\int_j^{(N_1)}$ for all j , then $\int_j^{(N_2)}$ (for all j), ..., and so on till $\int_j^{(N_{r-1})}$ for all j . Moreover, for any nonnegative integer j such that $0 \leq j \leq \tilde{k}_{N_r, \text{last}}$, we know that $M_{a_r, n} - p_{a_r} - j$ belongs to the set $M_{\max}[2]$ and

$$\int_j^{(N_1)} := \Phi_{M_{a_r, n} - p_{a_r} - j} = \mathfrak{S}_{M_{a_r, n} - p_{a_r} - j, s}. \quad (60)$$

So let us first give a formula for $\mathfrak{S}_{M_{a_r,n}-p_{a_r}-j,s}$. I will rather give a formula for $\mathfrak{S}_{M_{\max},s}$ for any $M_{\max}$ in $M_{\max}[2]$ in general and then one can put $M_{\max} = M_{a_r,n} - p_{a_r} - j$ in it to get the required formula for $\mathfrak{S}_{M_{a_r,n}-p_{a_r}-j,s}$.

Let $M_{\max} \in M_{\max}[2]$ be arbitrary. To compute $\mathfrak{S}_{M_{\max},s}$, we first need to compute $\mathfrak{S}_{M_{\max},0}$. I will compute $\mathfrak{S}_{M_{\max},0}$ only in the most general case, that is, when

$$\begin{aligned} \text{Min}_{P_q, M_{\max}}^c &< p^n - P_{q,\max} < p^n - P_{q,\min} \\ \forall q \in \{1, \dots, t\}, \forall M_{\max} \in M_{\max}[2]. \end{aligned} \quad (61)$$

In this case, we have

$$\text{Min}_{P_q, M_{\max}}^c = M_{\max} \Delta_{P_q}. \quad (62)$$

In all other cases, it is a tedious job to compute $\mathfrak{S}_{M_{\max},0}$ because we will have to take care of various complicated possibilities like

for some $q \in \{1, \dots, t\}$, we may have $\text{Min}_{P_q, M_{\max}}^c < p^n - P_{q,\min}$ and for some other $q' \in \{1, \dots, t\}$, we may have $\text{Min}_{P_{q'}, M_{\max}}^c = p^n - P_{q',\min}$.

This makes the computation of $\mathfrak{S}_{M_{\max},0}$ highly tedious and complicated. Let us now compute $\mathfrak{S}_{M_{\max},0}$ for the most general case.

For $2 \leq l \leq t$, let $E_{l,\min}$ denote the sum of all possible terms which are products of l -many items either of the type $M_{\max} \Delta_{P_q}$ or of the type $P_{q,\min}$ where $q \in \{1, \dots, t\}$ such that the following two properties hold.

- (i) At least one of the l items in each such product is of the type $M_{\max} \Delta_{P_q}$;
- (ii) not all of the l items in any such product are of the type $M_{\max} \Delta_{P_q}$.

Similarly, for any $l \in \{2, \dots, t\}$, let $E_{l,\max}$ denote the sum of all possible terms which are products of l -many items either of the type $M_{\max} \Delta_{P_q}$ or of the type $P_{q,\max}$ where $q \in \{1, \dots, t\}$ such that the following two properties hold.

- (i) At least one of the l items in each such product is of the type $M_{\max} \Delta_{P_q}$;
- (ii) not all of the l items in any such product are of the type $M_{\max} \Delta_{P_q}$.

For $2 \leq l \leq t$, let T_l denote the sum of all possible terms which are products of the type $\prod_{\substack{i_1, i_2, \dots, i_l \in \{1, \dots, t\} \\ i_1 \neq i_2 \neq \dots \neq i_l}} P_{i_1, \max} P_{i_2, \max} \dots P_{i_l, \max}$.

Then

$$\begin{aligned} \mathfrak{S}_{M_{\max},0} &= p^{(t-1)n} \left[\sum_{i=1}^t P_{i,\max} \right] \\ &+ \sum_{l=2}^t (-1)^l p^{(t-l)n} \{E_{l,\min} - E_{l,\max} - T_l\}. \end{aligned} \quad (63)$$

Hence the formula for $\Phi_{M_{\max}} (:= \mathfrak{S}_{M_{\max},s})$ is given by

$$\begin{aligned} &\left[p^{(t-1)n} \left[\sum_{i=1}^t P_{i,\max} \right] + \sum_{l=2}^t (-1)^l p^{(t-l)n} \right. \\ &\quad \times \{E_{l,\min} - E_{l,\max} - T_l\} \left. \right] \left[\prod_{i=1}^s (p^n - Z_{i,\min}) \right] \\ &+ \sum_{k=1}^s \left[(-1)^{k+1} p^{(t+s-k)n} \right. \\ &\quad \times \left(\sum_{\substack{i_1, i_2, \dots, i_k \in \{1, \dots, s\} \\ i_1 \neq i_2 \neq \dots \neq i_k}} Z_{i_1, \min} Z_{i_2, \min} \dots Z_{i_k, \min} \right) \left. \right]. \end{aligned} \quad (64)$$

Since $\int_j^{(N_1)} = \Phi_{M_{a_r,n}-p_{a_r}-j}$, we get the formula for $\int_j^{(N_1)}$ from (64) above by putting $M_{\max} = M_{a_r,n} - p_{a_r} - j$ in it.

Let us now write down the formula (45) for the Hilbert-Kunz function, in a different way which will help us see how the formula for the Hilbert-Kunz multiplicity will look like. But for that, we first need some notation.

For each $j \in \mathbb{N}$, let $F_j^{(1)} := 1, F_j^{(2)} := 2j - 1$ and for any positive integer $k \geq 3$, let $F_j^{(k)} := \sum_{l=1}^{j-1} F_l^{(k-1)} + jF_j^{(k-1)}$.

For each $0 \leq j \leq \tilde{X}_{N_r, \text{last}}$, let $C_j^{(N_r)}$ denote the coefficient of $\int_j^{(N_r)}$ in the formula (45) for the Hilbert-Kunz function.

For any $m \in \{1, \dots, r\}$, let $V^{(m)} := \{D_{a_r, a_r}, D_{a_{r-1}, a_r}, \dots, D_{a_m, a_r}\}$, $U^{(m)} := \{a_r, a_{r-1}, \dots, a_m\}$, and $C_0^{(N_m)} := \prod_{k=m}^r D_{a_k, a_r}$.

For any integer j such that $1 \leq j < \tilde{X}_{N_m, \text{last}}$ and any integer l such that $1 \leq l \leq r - m + 1$, let $H_l^{(m)}$:= sum of all possible terms which are products of $r - l - m + 1$ many D_{a_k, a_r} s and l many a_k s where the D_{a_k, a_r} s belong to the set $V^{(m)}$ and the a_k s belong to the set $U^{(m)}$.

Let $m \in \{1, \dots, r - 1\}$ be arbitrary. Then for any integer j such that $1 \leq j < \tilde{X}_{N_m, \text{last}}$, let

$$C_j^{(N_m)} := \sum_{l=1}^{r-m+1} F_j^{(l)} H_l^{(m)}. \quad (65)$$

And for any integer j such that $\tilde{X}_{N_m, \text{last}} \leq j \leq \tilde{X}_{N_r, \text{last}}$, let us define $C_j^{(N_m)}$ inductively as follows.

Let $C_{\tilde{X}_{N_m, \text{last}}}^{(N_m)} := a'_m [\sum_{j=0}^{\tilde{X}_{N_m, \text{last}}-1} C_j^{(N_{m+1})}] + (p^n - N_{m, \max}) C_{\tilde{X}_{N_m, \text{last}}}^{(N_{m+1})}$ and for any $\tilde{X}_{N_m, \text{last}} < j \leq \tilde{X}_{N_r, \text{last}}$, let $C_j^{(N_m)} := (p^n - N_{m, \max}) C_j^{(N_{m+1})}$.

Now if we expand the formula ((45) above) of the Hilbert-Kunz function, assuming that $\bar{X}_{N_1, \text{last}} \leq \bar{X}_{N_2, \text{last}} \leq \dots \leq \bar{X}_{N_r, \text{last}}$, we get that the expression in (45) equals

$$\begin{aligned}
 & \sum_{j=0}^{(\bar{X}_{N_1, \text{last}})-1} C_j^{(N_1)} \int_j^{(N_1)} \\
 & + \left[a'_1 \left(\sum_{j=0}^{(\bar{X}_{N_1, \text{last}})-1} C_j^{(N_2)} \right) + (p^n - N_{1, \text{max}}) C_{\bar{X}_{N_1, \text{last}}}^{(N_2)} \right] \int_{\bar{X}_{N_1, \text{last}}}^{(N_1)} \\
 & + (p^n - N_{1, \text{max}}) \sum_{j=(\bar{X}_{N_1, \text{last}})+1}^{\bar{X}_{N_r, \text{last}}} C_j^{(N_2)} \int_j^{(N_1)} \\
 & + (N_{r, \text{max}} - N_{r, \text{min}}) G_{N_r} + N_{r, \text{min}} p^{n(s+t+r-1)} \\
 & + \sum_{k=1}^{r-1} \left[(N_{r-k, \text{max}} - N_{r-k, \text{min}}) G_{N_{r-k}} \right. \\
 & \quad \left. + N_{r-k, \text{min}} p^{n(s+t+r-k-1)} \right] \\
 & \times \left[C_0^{(N_{r-k+1})} + \sum_{l=1}^k \left(H_l^{(r-k+1)} \sum_{j=1}^{(\bar{X}_{N_{r-k+1}, \text{last}})-1} F_j^{(l)} \right) \right. \\
 & \quad \left. + \sum_{j=\bar{X}_{N_{r-k+1}, \text{last}}}^{\bar{X}_{N_r, \text{last}}} C_j^{(N_{r-k+1})} \right], \tag{66}
 \end{aligned}$$

where the formulae for $\int_j^{(N_1)}$ can be obtained from (64). The Hilbert-Kunz multiplicity is by definition the coefficient of $p^{n(s+t+r-1)}$ in the expression (66) above. Computing the coefficient of $p^{n(s+t+r-1)}$ in the expression (66) above, we get that the Hilbert-Kunz multiplicity equals

$$\begin{aligned}
 & N_{r, \text{min}} + \sum_{k=1}^{r-1} N_{r-k, \text{min}} \left[\text{Coefficient of } p^{kn} \text{ in} \right. \\
 & \quad \left. \sum_{l=1}^k H_l^{(r-k+1)} \left(\sum_{j=1}^{\bar{X}_{N_{r-k+1}, \text{last}}-1} F_j^{(l)} \right) \right] \\
 & + \sum_{q=0}^{t-1} \left\{ \left[\text{coefficient of } j^q p^{n(t+s-1-q)} \text{ in } \int_j^{(N_1)} \right] \right. \\
 & \quad \times \left\{ \left[\text{terms of } H_1^{(1)} \text{ of degree } p^{n(r-1)} \right] \right. \\
 & \quad \times \sum_{j=1}^{\bar{X}_{N_1, \text{last}}-1} j^q F_j^{(1)}
 \end{aligned}$$

$$\begin{aligned}
 & + \left[\text{terms of } H_2^{(1)} \text{ of degree } p^{n(r-2)} \right] \\
 & \times \sum_{j=1}^{\bar{X}_{N_1, \text{last}}-1} j^q F_j^{(2)} + \dots \\
 & + \left[\text{terms of } H_r^{(1)} \text{ of degree } p^{n(r-r)} \right] \\
 & \times \sum_{j=1}^{\bar{X}_{N_1, \text{last}}-1} j^q F_j^{(r)} \left. \right\} \left. \right\}. \tag{67}
 \end{aligned}$$

If we expand the above formula (expression (67)) of the Hilbert-Kunz multiplicity, then inside the expansion, we will get terms which are of the form:

- (i) finite (by “finite”, I mean depending upon n) sums of the form $\sum_{j=\epsilon_n}^{\theta_n} j^v$ where v is some positive integer and ϵ_n and θ_n are upper or lower integral parts of some real numbers which are of the form “polynomials in p^n with rational coefficients”;
- (ii) polynomials in p^n with rational coefficients (including constant terms which are rational numbers);
- (iii) products of the above mentioned 2 types of terms.

It hence follows from the above discussion that the Hilbert-Kunz multiplicity associated with any general binomial hypersurface is always rational.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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