Research Article

A Regularization Method for the Elliptic Equation with Inhomogeneous Source

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We consider the following Cauchy problem for the elliptic equation with inhomogeneous source in a rectangular domain with Dirichlet boundary conditions at $x = 0$ and $x = \pi$. The problem is ill-posed. The main aim of this paper is to introduce a regularization method and use it to solve the problem. Some sharp error estimates between the exact solution and its regularization approximation are given and a numerical example shows that the method works effectively.

1. Introduction

The Cauchy problem for the elliptic equation has been extensively investigated in many practical areas. For example, some problems relating to geophysics [1], plasma physics [2], and bioelectric field problems [3] are equivalent to solving the Cauchy problem for the elliptic equation. In this paper, we consider the following Cauchy problem for elliptic equation with nonhomogeneous source:

$$u_{xx} + u_{yy} = f(x, y), \quad (x, y) \in (0, \pi) \times (0, 1), \quad (1)$$

$$u(0, y) = u(\pi, y) = 0, \quad (2)$$

$$u_y(x, 0) = 0, \quad (3)$$

$$u(x, 0) = g(x), \quad (4)$$

where $g \in L^2(0, \pi)$, $f \in L^2(0, 1; L^2(0, \pi))$ are given.

Problem (1)–(4) is well known to be ill-posed in the sense of Hadamard: a small perturbation in the data g may cause dramatically large errors in the solution $u(x, y)$ for $0 < y \leq 1$. An explicit example to emphasize this fact is given in [4]. In the past, there were many studies on the homogeneous problem, that is $f = 0$ in (1). Using the boundary element method,

the homogeneous problems were considered in [5–7] and the references therein. Similarly, many authors have investigated the Cauchy problem for linear homogeneous elliptic equation, for example, the quasireversibility method [4, 8–10], fourth-order modified method [11, 12], Fourier truncation regularized method (or spectral regularized method) [13–15], the Backus-Gilbert algorithm [16] and so forth. Some other authors also considered the homogeneous problem such as Beskos [5], Eldén et al. [17, 18], Marin and Lesnic [19], Qin and Wei [20], Regińska and Tautenhahn [21], Tautenhahn [22].

Very recently, in 2009, Hào and his group [23] applied the nonlocal-boundary value method to regularize the abstract homogeneous elliptic problem. This method is also given in [24] for solving an elliptic problem with homogeneous source in a cylindrical domain. A mollification regularization method for the Cauchy problem in a multidimensional case has been considered in the recent paper of Cheng and his group [25].

Although there are many papers on the homogeneous elliptic equation, the result on the inhomogeneous case is very scarce, while the inhomogeneous case is, of course, more general and nearer to practical application than the homogeneous one. Shortly, it allows the occurrence of some elliptic source which is inevitable in nature. The main aim of this

paper is to present a simple and effective regularization method and investigate the error estimate between the regularization solution and the exact solution. In a sense, this paper may be an extension of many previous results.

The remainder of the paper is divided into two sections. In Section 2, we will study the regularization of problem (1)–(4) and obtain convergence estimates. In Section 3, a numerical test case for inhomogeneous problems is given to describe the effectiveness of our method.

2. Regularization and Error Estimate

By the method of separation of variables, the solution of problem (1)–(4) is given by

$$u(x, y) = \sum_{n=1}^{\infty} \left[\left(\frac{e^{ny} + e^{-ny}}{2} \right) g_n + \int_0^y \left(\frac{e^{n(y-s)} - e^{n(s-y)}}{2n} \right) f_n(s) ds \right] \sin nx, \quad (5)$$

where

$$g_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin nx dx, \quad (6)$$

$$f_n(s) = \frac{2}{\pi} \int_0^{\pi} f(x, s) \sin nx dx.$$

We can see that the instability is caused by the fast growth of e^{ny} , $y > 0$ as n tends to infinity. Even though these exact Fourier coefficients $g_n, f_n(s)$ may tend to zero rapidly, in practice, performing classical calculation is impossible because the given data is usually diffused by a variety of reasons such as round-off error and measurement error. A small perturbation in the data can arbitrarily deduce a large error in the solution. Therefore, some special regularization methods are required. From (5), we replace the term e^{ny} that causes dramatically the increasing of the right side by several bounded approximations. We assume that the exact data $g(x)$ and the measured data $g^\epsilon(x)$ both belong to $L^2(0, \pi)$ and satisfy $\|g^\epsilon - g\|_2 \leq \epsilon$ where $\|\cdot\|_2$ is the norm on $L^2(0, \pi)$ and ϵ denotes the noise level, respectively.

In the paper, we will use a modification method to regularize our problem. The regularized solution is given as follows:

$$u^\epsilon(x, y) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2} \left(\frac{1}{\alpha + e^{-ny}} + e^{-ny} \right) g_n + \int_0^y \frac{1}{2n} \left(\frac{e^{-ns}}{\alpha + e^{-ny}} - e^{n(s-y)} \right) f_n(s) ds \right] \sin nx. \quad (7)$$

Here $\alpha \in (0, 1)$ is a parameter regularization which depends on ϵ . The explicit error estimates including error estimates

have been given according to some priori assumptions on the regularity of the exact solution.

Let v^ϵ be the solution of problem (7) corresponding to the measured data g^ϵ . Then, it is given by

$$v^\epsilon(x, y) = \sum_{n=1}^{\infty} \left[\frac{1}{2} \left(\frac{1}{\alpha + e^{-ny}} + e^{-ny} \right) g_n^\epsilon + \int_0^y \frac{1}{2n} \left(\frac{e^{-ns}}{\alpha + e^{-ny}} - e^{n(s-y)} \right) f_n(s) ds \right] \sin nx. \quad (8)$$

We first have the following theorem.

Theorem 1. Let $g, g^\epsilon \in L^2(0, \pi)$ such that $\|g^\epsilon - g\|_2 \leq \epsilon$. Then one has

$$\|v^\epsilon(\cdot, y) - u^\epsilon(\cdot, y)\|_2 \leq \alpha^{-1} \epsilon, \quad (9)$$

for all $y \in [0, 1]$.

Proof. It follows from (7) and (8) that

$$v^\epsilon(\cdot, y) - u^\epsilon(\cdot, y) = \sum_{n=1}^{\infty} \left[\frac{1}{2} \left(\frac{1}{\alpha + e^{-ny}} + e^{-ny} \right) (g_n^\epsilon - g_n) \right] \sin nx. \quad (10)$$

We have

$$\begin{aligned} \|v^\epsilon(\cdot, y) - u^\epsilon(\cdot, y)\|_2^2 &= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{1}{\alpha + e^{-ny}} + e^{-ny} \right)^2 |g_n^\epsilon - g_n|^2 \\ &\leq \frac{\pi}{8} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha} + 1 \right)^2 |g_n^\epsilon - g_n|^2 \\ &\leq \alpha^{-2} \|g^\epsilon - g\|_2^2 \leq \alpha^{-2} \epsilon^2. \end{aligned} \quad (11)$$

Therefore, we get

$$\|v^\epsilon(\cdot, y) - u^\epsilon(\cdot, y)\|_2 \leq \alpha^{-1} \epsilon. \quad (12)$$

This completes the proof of Theorem 1. $\square$

Theorem 2. Let g, g^ϵ be as in Theorem 1. Assume that $\int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds < \infty$. If we select $\alpha = \epsilon^{1/2}$, then for every $y \in [0, 1]$ one has

$$\|v^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \leq M \epsilon^{(1-y)/2}, \quad (13)$$

where

$$M = 1 + \frac{\sqrt{3}}{2} \times \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds}. \quad (14)$$

Proof. First, we have

$$u_n(y) = \left(\frac{e^{ny} + e^{-ny}}{2} \right) g_n + \int_0^y \left(\frac{e^{n(y-s)} - e^{n(s-y)}}{2n} \right) f_n(s) ds, \quad (15)$$

$$u_n^\epsilon(y) = \frac{1}{2} \left(\frac{1}{\alpha + e^{-ny}} + e^{-ny} \right) g_n + \int_0^y \frac{1}{2n} \left(\frac{e^{-ns}}{\alpha + e^{-ny}} - e^{n(s-y)} \right) f_n(s) ds. \quad (16)$$

Subtracting (16) to (15), we have

$$\begin{aligned} u_n^\epsilon(y) - u_n(y) &= \frac{1}{2} \left(\frac{1}{\alpha + e^{-ny}} - e^{ny} \right) g_n \\ &\quad + \frac{1}{2n} \int_0^y \left(\frac{e^{-ns}}{\alpha + e^{-ny}} - e^{n(y-s)} \right) f_n(s) ds \\ &= \frac{-1}{2} \left(\frac{\alpha}{\alpha + e^{-ny}} \right) \left[e^{ny} g_n + \frac{1}{n} \int_0^y e^{n(y-s)} f_n(s) ds \right]. \end{aligned} \quad (17)$$

We have

$$\begin{aligned} u_y(x, y) &= \sum_{n=1}^{\infty} \left[n \left(\frac{e^{ny} - e^{-ny}}{2} \right) g_n \right. \\ &\quad \left. + \int_0^y n \left(\frac{e^{n(y-s)} + e^{n(s-y)}}{2n} \right) f_n(s) ds \right] \sin nx \\ &= \sum_{n=1}^{\infty} n \left[\left(\frac{e^{ny} - e^{-ny}}{2} \right) g_n \right. \\ &\quad \left. + \int_0^y \left(\frac{e^{n(y-s)} + e^{n(s-y)}}{2n} \right) f_n(s) ds \right] \sin nx. \end{aligned} \quad (18)$$

From (15), we have

$$\begin{aligned} \frac{1}{n} \frac{d}{dy} u_n(y) &= \left(\frac{e^{ny} - e^{-ny}}{2} \right) g_n \\ &\quad + \int_0^y \left(\frac{e^{n(y-s)} + e^{n(s-y)}}{2n} \right) f_n(s) ds. \end{aligned} \quad (19)$$

Combining (15) and (19), we get

$$u_n(y) + \frac{1}{n} \frac{d}{dy} u_n(y) = e^{ny} g_n + \frac{1}{n} \int_0^y e^{n(y-s)} f_n(s) ds. \quad (20)$$

Let $y = 1$; we have

$$\begin{aligned} u_n(1) + \frac{1}{n} \frac{d}{dy} u_n(1) &= e^n g_n + \frac{1}{n} \int_0^1 e^{n(1-s)} f_n(s) ds \\ &= e^n \left[g_n + \frac{1}{n} \int_0^1 e^{-ns} f_n(s) ds \right]. \end{aligned} \quad (21)$$

Therefore, we get

$$g_n = e^{-n} \left[u_n(1) + \frac{1}{n} \frac{d}{dy} u_n(1) \right] - \frac{1}{n} \int_0^1 e^{-ns} f_n(s) ds. \quad (22)$$

From (17) and (22), we have

$$\begin{aligned} |u_n^\epsilon(y) - u_n(y)| &= \frac{1}{2} \left(\frac{\alpha}{\alpha + e^{-ny}} \right) \left| e^{ny} \left[e^{-n} u_n(1) + \frac{e^{-n}}{n} \frac{d}{dy} u_n(1) \right. \right. \\ &\quad \left. \left. - \frac{1}{n} \int_0^1 e^{-ns} f_n(s) ds \right] \right. \\ &\quad \left. + \frac{1}{n} \int_0^y e^{n(y-s)} f_n(s) ds \right| \\ &= \frac{1}{2} \left(\frac{\alpha}{\alpha + e^{-ny}} \right) \left| e^{n(y-1)} \left[u_n(1) + \frac{1}{n} \frac{d}{dy} u_n(1) \right] \right. \\ &\quad \left. - \frac{1}{n} \int_y^1 e^{n(y-s)} f_n(s) ds \right| \\ &\leq \frac{1}{2} \alpha \left(\frac{e^{-n(1-y)}}{\alpha + e^{-n}} \right) \left[|u_n(1)| + \left| \frac{d}{dy} u_n(1) \right| \right. \\ &\quad \left. + \int_0^1 |e^n f_n(s)| ds \right]. \end{aligned} \quad (23)$$

Moreover, one has, for $\tau > t > 0$ and $\alpha > 0$,

$$\frac{e^{-t\tau}}{\alpha + e^{-t\tau}} = \frac{1}{(\alpha e^{t\tau} + 1)^{t/\tau} (\alpha + e^{-t\tau})^{1-t/\tau}} \leq \alpha^{t/\tau-1}. \quad (24)$$

Letting $\tau = 1$, $t = 1 - y$, we get

$$\frac{e^{-n(1-y)}}{\alpha + e^{-n}} \leq \alpha^{-y}. \quad (25)$$

From (23), (25), we have

$$\begin{aligned} |u_n^\epsilon(y) - u_n(y)| &\leq \frac{1}{2} \alpha^{1-y} \left[|u_n(1)| + \left| \frac{d}{dy} u_n(1) \right| + \int_0^1 |e^n f_n(s)| ds \right]. \end{aligned} \quad (26)$$

Applying the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, we have

$$\begin{aligned} |u_n^\epsilon(y) - u_n(y)|^2 &\leq \frac{3}{4} \alpha^{2(1-y)} \left[|u_n(1)|^2 \right. \\ &\quad \left. + \left| \frac{d}{dy} u_n(1) \right|^2 + \left(\int_0^1 |e^n f_n(s)| ds \right)^2 \right] \\ &\leq \frac{3}{4} \alpha^{2(1-y)} \left[|u_n(1)|^2 \right. \\ &\quad \left. + \left| \frac{d}{dy} u_n(1) \right|^2 + \int_0^1 e^{2n} f_n^2(s) ds \right]. \end{aligned} \quad (27)$$

Thus

$$\begin{aligned}
& \|u^\epsilon(\cdot, y) - u(\cdot, y)\|_2^2 \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} |u_n^\epsilon(y) - u_n(y)|^2 \\
&\leq \frac{3}{4} \alpha^{2(1-y)} \left(\frac{\pi}{2} \sum_{n=1}^{\infty} |u_n(1)|^2 \right. \\
&\quad \left. + \frac{\pi}{2} \sum_{n=1}^{\infty} \left| \frac{d}{dy} u_n(1) \right|^2 \right. \\
&\quad \left. + \frac{\pi}{2} \sum_{n=1}^{\infty} \int_0^1 e^{2n} f_n^2(s) ds \right) \\
&= \frac{3}{4} \alpha^{2(1-y)} \left(\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 \right. \\
&\quad \left. + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds \right),
\end{aligned} \tag{28}$$

or we get

$$\begin{aligned}
& \|u^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \\
&\leq \frac{\sqrt{3}}{2} \alpha^{1-y} \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds}.
\end{aligned} \tag{29}$$

Using Theorem 1 and (29), we get

$$\begin{aligned}
& \|v^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \\
&\leq \|v^\epsilon(\cdot, y) - u^\epsilon(\cdot, y)\|_2 + \|u^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \\
&\leq \alpha^{-1} \epsilon \\
&\quad + \frac{\sqrt{3}}{2} \alpha^{1-y} \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds} \\
&= \epsilon^{1/2} \\
&\quad + \frac{\sqrt{3}}{2} \epsilon^{(1-y)/2} \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds} \\
&\leq M \epsilon^{(1-y)/2},
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
M &= 1 + \frac{\sqrt{3}}{2} \\
&\quad \times \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds}.
\end{aligned} \tag{31}$$

This completes the proof of Theorem 2. $\square$

Remark 3. From (13), as $y \rightarrow 1$, the accuracy of regularized solution becomes progressively lower. To retain the continuous dependence of the solution at $y = 1$, we introduce the following theorem.

Theorem 4. Assume that problem (1)–(4) has a solution u such that $u_y \in L^2((0, 1); L^2(0, \pi))$ and

$$\int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds < \infty. \tag{32}$$

Then for all $\epsilon \in (0, 1)$ there exists a $y_\epsilon > 0$ such that

$$\|u^\epsilon(\cdot, y_\epsilon) - u(\cdot, 1)\|_2 \leq 2C_1 \left(\ln \left(\frac{1}{\epsilon} \right) \right)^{-1/4}, \tag{33}$$

where

$$N = \sqrt{\int_0^1 \|u_y(\cdot, s)\|_2^2 ds},$$

C_1

$$\begin{aligned}
&= \max \left\{ N, \frac{\sqrt{3}}{2} \right. \\
&\quad \left. \times \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds} \right\}.
\end{aligned} \tag{34}$$

Proof. We have

$$u(x, 1) - u(x, y) = \int_y^1 u_y(x, s) ds. \tag{35}$$

It follows that

$$\|u(\cdot, y) - u(\cdot, 1)\|_2^2 \leq (1-y) \int_y^1 \|u_y(\cdot, s)\|_2^2 ds = N^2 (1-y). \tag{36}$$

Using (29), noticing that $\alpha = \epsilon^{1/2}$, and (34), we have

$$\begin{aligned}
\|u^\epsilon(\cdot, y) - u(\cdot, 1)\|_2 &\leq \|u^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \\
&\quad + \|u(\cdot, y) - u(\cdot, 1)\|_2 \\
&\leq C_1 \left(\sqrt{1-y} + \epsilon^{(1-y)/2} \right).
\end{aligned} \tag{37}$$

For every $\epsilon \in (0, 1)$, there exists uniquely a positive number y_ϵ such that $\sqrt{1-y_\epsilon} = \epsilon^{(1-y_\epsilon)/2}$; that is

$$\frac{\ln(1-y_\epsilon)}{1-y_\epsilon} = \ln \epsilon. \tag{38}$$

Using inequality $\ln(1-y) > -1/(1-y)$ for every $0 < y < 1$, we get

$$\|u^\epsilon(\cdot, y_\epsilon) - u(\cdot, 1)\|_2 \leq 2C_1 \left(\ln \left(\frac{1}{\epsilon} \right) \right)^{-1/4}. \tag{39}$$

This completes the proof of Theorem 4. $\square$

Theorem 5. Let g, g^ϵ, u be as in Theorem 4 and (32) holds. Then one can construct a function w^ϵ satisfying

$$\|w^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \leq M\epsilon^{(1-y)/2} \quad (40)$$

for every $y \in (0, 1)$ and

$$\|w^\epsilon(\cdot, 1) - u(\cdot, 1)\|_2 \leq C \left(\ln \left(\frac{1}{\epsilon} \right) \right)^{-1/4}, \quad (41)$$

where

$$\begin{aligned} M &= 2 + \frac{\sqrt{3}}{2} \\ &\times \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds}, \\ C_1 &= \max \left\{ \sqrt{\int_0^1 \|u_y(\cdot, s)\|_2^2 ds}, \frac{\sqrt{3}}{2} \right. \\ &\times \left. \sqrt{\|u(\cdot, 1)\|_2^2 + \|u_y(\cdot, 1)\|_2^2 + \frac{\pi}{2} \int_0^1 \sum_{n=1}^{\infty} e^{2n} f_n^2(s) ds} \right\}, \\ C &= 2 + 2C_1. \end{aligned} \quad (42)$$

Proof. Let y_ϵ be the unique solution of

$$\sqrt{1 - y_\epsilon} = \epsilon^{(1-y_\epsilon)/2}. \quad (43)$$

We define a function w^ϵ as follows:

$$w^\epsilon(\cdot, y) = \begin{cases} v^\epsilon(\cdot, y), & 0 < y < 1, \\ v^\epsilon(\cdot, y_\epsilon), & y = 1. \end{cases} \quad (44)$$

From Theorem 1, we have

$$\begin{aligned} \|w^\epsilon(\cdot, y) - u(\cdot, y)\|_2 &= \|v^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \\ &\leq M\epsilon^{(1-y)/2} \end{aligned} \quad (45)$$

for every $y \in (0, 1)$. From Theorem 2, we have

$$\|u^\epsilon(\cdot, y_\epsilon) - u(\cdot, 1)\|_2 \leq 2C_1 \left(\ln \left(\frac{1}{\epsilon} \right) \right)^{-1/4}. \quad (46)$$

Using Theorem 1, noticing that $\alpha = \epsilon^{1/2}$, (43), and (46), we get

$$\begin{aligned} \|w^\epsilon(\cdot, 1) - u(\cdot, 1)\|_2 &= \|v^\epsilon(\cdot, y_\epsilon) - u(\cdot, 1)\|_2 \\ &\leq \|v^\epsilon(\cdot, y_\epsilon) - u^\epsilon(\cdot, y_\epsilon)\|_2 \\ &\quad + \|u^\epsilon(\cdot, y_\epsilon) - u(\cdot, 1)\|_2 \\ &\leq \epsilon^{1/2} + \epsilon + 2C_1 \left(\ln \left(\frac{1}{\epsilon} \right) \right)^{-1/4} \\ &\leq C \left(\ln \left(\frac{1}{\epsilon} \right) \right)^{-1/4}, \end{aligned} \quad (47)$$

where

$$C = 2 + 2C_1. \quad (48)$$

This completes the proof of Theorem 5. $\square$

Remark 6. In this theorem, we require a condition on the Fourier expansion coefficient f_n in (32). This condition is very difficult to check. To improve this, in the next theorem, we only require the assumption of u , not to depend on the function f .

Theorem 7. Let g, g^ϵ be as in Theorem 1. Assume that problem (1)–(4) has a solution u such that

$$\|u_x(\cdot, y)\|_2^2 + \|u_y(\cdot, y)\|_2^2 < \infty. \quad (49)$$

If we select $\alpha = \epsilon^k$ for $0 < k < 1$, then

$$\|v^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \leq \epsilon^{1-k} + \frac{P}{2k \ln(1/\epsilon)} \quad (50)$$

for every $y \in [0, 1]$ where

$$P = \sqrt{\|u_x(\cdot, y)\|_2^2 + \|u_y(\cdot, y)\|_2^2}. \quad (51)$$

Proof. Combining (15) and (20), we obtain

$$\begin{aligned} u_n^\epsilon(y) - u_n(y) &= \frac{-1}{2} \left(\frac{\alpha}{\alpha + e^{-ny}} \right) \left[e^{ny} g_n + \frac{1}{n} \int_0^y e^{n(y-s)} f_n(s) ds \right] \\ &= \frac{-1}{2} \left(\frac{\alpha}{\alpha + e^{-ny}} \right) \left[u_n(y) + \frac{1}{n} \frac{d}{dy} u_n(y) \right] \\ &= \frac{-1}{2} \left(\frac{\alpha}{\alpha n + n e^{-ny}} \right) \left[n u_n(y) + \frac{d}{dy} u_n(y) \right]. \end{aligned} \quad (52)$$

For $z > 0$, we consider the function $\varphi(z) = 1/(\alpha z + e^{-z})$. By taking the derivative of φ , we get

$$\varphi'(z) = -\frac{\alpha - e^{-z}}{(\alpha z + e^{-z})^2}. \quad (53)$$

The function $\varphi(z)$ attains maximum value at the z_0 such that $\varphi'(z_0) = 0$. It follows that $e^{z_0} = 1/\alpha$ or $z_0 = \ln(1/\alpha)$. Hence

$$\begin{aligned} \frac{1}{\alpha z + e^{-z}} &\leq \frac{1}{\alpha z_0 + e^{-z_0}} \\ &= \frac{1}{\alpha \ln(1/\alpha) + \alpha} \\ &\leq \frac{1}{\alpha \ln(1/\alpha)}. \end{aligned} \quad (54)$$

TABLE 1

y	$\epsilon_1 = 10^{-1}$	$\ v^\epsilon - u\ _2$
0		0.0053
0.1		0.0285
0.2		0.0019
0.3		0.0309
0.4		0.0722
0.5		0.1245
0.6		0.1914
0.7		0.2769
0.8		0.3865
0.9		0.5268

TABLE 2

y	$\epsilon_2 = 10^{-5}$	$\ v^\epsilon - u\ _2$
0		4.8387×10^{-4}
0.1		7.6623×10^{-4}
0.2		0.0011
0.3		0.0016
0.4		0.0023
0.5		0.0032
0.6		0.0043
0.7		0.0059
0.8		0.0080
0.9		0.0109

TABLE 3

y	$\epsilon_3 = 10^{-10}$	$\ v^\epsilon - u\ _2$
0		1.5665×10^{-6}
0.1		2.4632×10^{-6}
0.2		3.6404×10^{-6}
0.3		5.1995×10^{-6}
0.4		7.2769×10^{-6}
0.5		1.0056×10^{-5}
0.6		1.3786×10^{-5}
0.7		1.8801×10^{-5}
0.8		2.5551×10^{-5}
0.9		3.4647×10^{-5}

Using this inequality, we obtain

$$\begin{aligned}
& \|u^\epsilon(\cdot, y) - u(\cdot, y)\|_2^2 \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} |u_n^\epsilon(y) - u_n(y)|^2 \\
&\leq \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{\alpha}{\alpha n + e^{-n}} \right)^2 \left[nu_n(y) + \frac{d}{dy} u_n(y) \right]^2
\end{aligned}$$

FIGURE 1: The exact solution in the case $0 < y < 1$.

$$\begin{aligned}
&\leq \frac{1}{2\ln^2(1/\alpha)} \frac{\pi}{2} \sum_{n=1}^{\infty} \left[n^2 |u_n(y)|^2 + \left| \frac{d}{dy} u_n(y) \right|^2 \right] \\
&\leq \frac{1}{2\ln^2(1/\alpha)} \left(\|u_x(\cdot, y)\|_2^2 + \|u_y(\cdot, y)\|_2^2 \right).
\end{aligned} \tag{55}$$

By Theorem 1 and $\alpha = \epsilon^k$ ($0 < k < 1$) and using the triangle inequality, we get

$$\begin{aligned}
\|v^\epsilon(\cdot, y) - u(\cdot, y)\|_2 &\leq \|v^\epsilon(\cdot, y) - u^\epsilon(\cdot, y)\|_2 \\
&\quad + \|u^\epsilon(\cdot, y) - u(\cdot, y)\|_2 \\
&\leq \epsilon^{1-k} + \frac{P}{2k \ln(1/\epsilon)}.
\end{aligned} \tag{56}$$

□

Remark 8. Condition (49) is natural and reasonable.

3. A Numerical Experiment

To illustrate the theoretical results obtained before, we will discuss the corresponding numerical aspects in this section. We consider a simple problem as follows:

$$\begin{aligned}
\Delta u &= \frac{3}{8} (e^{2y} + e^{-2y}) \sin x, \quad (x, y) \in (0, \pi) \times (0, 1), \\
u(0, y) &= u(\pi, y) = 0, \\
u_y(x, 0) &= 0, \\
u(x, 0) &= g(x).
\end{aligned} \tag{57}$$

Consider the exact data $g(x) = \sin x/4$; then the exact solution to this problem is

$$u(x, y) = \frac{e^{2y} + e^{-2y}}{8} \sin x. \tag{58}$$

FIGURE 2: The regularized solution with $\epsilon_1 = 10^{-1}$. FIGURE 3: The regularized solution with $\epsilon_2 = 10^{-5}$. FIGURE 4: The regularized solution with $\epsilon_3 = 10^{-10}$.

Considering the measured data $g^\epsilon(x) = (\sqrt{32/\pi\epsilon} + 1)g(x)$, we have

$$\begin{aligned} \|g^\epsilon - g\|_2 &= \left(\int_0^\pi \frac{32}{\pi} \epsilon^2 (g(x))^2 dx \right)^{1/2} \\ &= \left(\frac{2}{\pi} \epsilon^2 \int_0^\pi \sin^2 x dx \right) = \epsilon. \end{aligned} \quad (59)$$

Let ϵ be $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-5}$, $\epsilon_3 = 10^{-10}$, respectively. If we put

$$y = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} \quad (60)$$

we get Tables 1, 2, and 3 for the case $0 < y < 1$ and we have the graphic that is displayed in Figures 1, 2, 3, and 4 on the interval $[0, \pi] \times [0, 0.9]$.

FIGURE 5: The exact solution and the regularized solution in the case $y = 1$.

TABLE 4

		$\ v^\epsilon(\cdot, y_\epsilon) - u(\cdot, 1)\ _2$
$\epsilon_1 = 10^{-1}$	$y_{\epsilon_1} = 0.4$	0.8319
$\epsilon_2 = 10^{-5}$	$y_{\epsilon_2} = 0.8$	0.3792
$\epsilon_3 = 10^{-10}$	$y_{\epsilon_3} = 0.99$	0.0225

For each figure, we can find that the smaller the ϵ is, the better the computed approximation is. And the bigger the y is, the worse the computed approximation is. Figure 5 shows the comparisons of the exact solution $u(x, y)$ and the approximation $v^\epsilon(x, y)$ at the point $y = 1$. In the case $y = 1$, from (43) and using inequality $\ln(1-y) > -1/(1-y)$ for every $0 < y < 1$, we get

$$y_\epsilon > 1 - \frac{1}{\sqrt{2 \ln(1/\epsilon)}}. \quad (61)$$

Therefore, we will choose $y_{\epsilon_1} = 0.4$, $y_{\epsilon_2} = 0.8$, and $y_{\epsilon_3} = 0.99$, with $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-5}$, and $\epsilon_3 = 10^{-10}$, respectively. Numerical results are given in Table 4.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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