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Research Article

On Henstock Method to Stratonovich Integral with respect to Continuous Semimartingale

Haifeng Yang and Tin Lam Toh

National Institute of Education, Nanyang Technological University, Singapore 637616

Correspondence should be addressed to Tin Lam Toh; tinlam.toh@nie.edu.sg

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We will use the Henstock (or generalized Riemann) approach to define the Stratonovich integral with respect to continuous semimartingale in L^2 space. Our definition of Stratonovich integral encompasses the classical definition of Stratonovich integral.

1. Introduction

Classically, it has been emphasized that defining stochastic integrals using Riemann approach is impossible [1, p. 40], since the integrators have paths of unbounded variation. Moreover, the integrands are usually highly oscillatory [1–3]. The uniform mesh in classical Riemann setting is unable to handle highly oscillatory integrators and integrands. Kurzweil and Henstock [4–6] independently modified the Riemann approach by using nonuniform mesh. This modification leads to a larger class of integrals being studied (see [4, 7–12]).

The generalized Riemann approach, using nonuniform meshes, has been successful in giving an alternative definition to the Stratonovich integral with respect to Brownian motion (see [13]). This paper attempts to further generalize the result of [13] to define the Stratonovich integral with respect to continuous semimartingale.

The difficulty to extend from Brownian motion to semimartingale is that the quadratic variation of the continuous martingale may not be absolutely continuous. Hence, the definition of the Stratonovich-Henstock belated integral in [13] may not be valid. In [12, 14, 15], the definition of weakly Henstock variation belated integral was used to address this difficulty. Similarly, we tap on this approach to define Stratonovich integral with respect to continuous L^2-martingale.

Further, to consider continuous local martingale as an integrator, we will show that a (δ, η)-fine belated partial stochastic division $D = \{(I_k, \xi_k)\}_{k=1}^{n}$ of $[0, t]$ in [12, 16] is also a (δ, η)-fine belated partial stochastic division of $[0, t \wedge S]$, where S is a stopping time.

Lastly, the definition of the Stratonovich integral with respect to continuous semimartingale will be further refined. We divide the continuous semimartingale into two parts: a continuous local martingale and the difference of continuous, nondecreasing, and adapted processes. The former is discussed in Section 3.2. The latter has finite total variation on each bounded interval. Hence, integration with respect to the latter process is classical Riemann-Stieltjes. This part has already been addressed in [17].

We give a new definition of the Stratonovich integral with continuous semimartingale in L^2 space. This integrand is weaker than the classical case.

2. Stratonovich Integral

In this section, we will generalize the definition of Stratonovich integral to include the case where the integrator is continuous L^2-semimartingale. We will develop the definition of Stratonovich-Henstock belated integral where the integrator is a Brownian motion in [13]. Let $(\Omega, \mathscr{F}, P)$ be a probability

space such that $(\Omega, \mathscr{F}, P)$ is complete. Also, let $C^n(\mathbb{R})$ denote the class of functions which have continuous n derivatives.

Definition 1. Let $X = \{X_t, \mathscr{F}_t; 0 \le t < \infty\}$ be a process. If there exists a nondecreasing sequence $\{T_n\}_{n=1}^{\infty}$ of the stopping times of the filtration $\{\mathscr{F}_t\}$, such that $\{X_t^{(n)} \triangleq X_{t\wedge T_n}, \mathscr{F}_t; 0 \le t < \infty\}$ is a martingale for each $n \ge 1$ and $P[\lim_{n\to\infty} T_n = \infty] = 1$, then one says that X is local martingale. If, in addition, $X_0 = 0$ a.s., one writes $X \in \mathscr{M}^{\text{loc}}$ (or $X \in \mathscr{M}^{\text{c,loc}}$, if X is continuous).

Definition 2. A continuous semimartingale $X = \{X_t, \mathscr{F}_t; 0 \le t < \infty\}$ is an adapted process which has the decomposition

$$X_t = X_0 + M_t + C_t, \quad 0 \le t < \infty \text{ a.s.}, \tag{1}$$

where $M = \{M_t, \mathscr{F}_t; 0 \le t < \infty\} \in \mathscr{M}^{\text{c,loc}}$ and C_t has finite variation.

Definition 3. Let $X = \{X_t, \mathscr{F}_t; 0 \le t < \infty\}$ and $Y = \{Y_t, \mathscr{F}_t; 0 \le t < \infty\}$ be two continuous semimartingales such that

$$X_t = X_0 + M_t + C_t; \qquad Y_t = Y_0 + N_t + D_t, \tag{2}$$

where M and N are in $\mathscr{M}^{\text{c,loc}}$ and C_t and D_t are adapted, continuous process of bounded variation with $C_0 = D_0 = 0$ a.s. The Stratonovich integral of Y_t with respect to X_t is

$$\int_0^t Y_s \circ dX_s \triangleq \int_0^t Y_s dM_s + \int_0^t Y_s dC_s + \frac{1}{2}\langle M, N\rangle_t, \tag{3}$$

where the first integral on the right-hand side of (3) is an Itô integral, the second one is pathwise integral of the Riemann-Stieltjes type, and $\langle M, N\rangle_t$ is the cross variation process of M and N. If $M = N$, then $\langle M, N\rangle_t = \langle M\rangle_t$ is the quadratic variation of M (see [18, p. 31]).

Proposition 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $C^3(\mathbb{R})$ and let $X = \{X_t, \mathscr{F}_t; 0 \le t < \infty\}$ be a continuous semimartingale with decomposition in (2). Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s \text{ a.s.} \tag{4}$$

Proof. By Itô formula [18, p. 149], we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle M\rangle_s. \tag{5}$$

We only have to prove that

$$\int_0^t f'(X_s) \circ dX_s = \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle M\rangle_s. \tag{6}$$

Since $f'(X_s) \in C^2(\mathbb{R})$, then, by Itô formula,

$$\begin{aligned} f'(X_s) = f'(X_0) &+ \int_0^t f''(X_s)\, dM_s + \int_0^t f''(X_s)\, dA_s \\ &+ \frac{1}{2}\int_0^t f'''(X_s)\, d\langle M\rangle_s. \end{aligned} \tag{7}$$

Then $f'(X_s)$ is a semimartingale, $\int_0^t f''(X_s)dM_s$ is a local martingale, and $\int_0^t f''(X_s)dA_s$ and $(1/2)\int_0^t f'''(X_s)d\langle M\rangle_s$ are bounded continuous processes. By definition of a semimartingale,

$$\begin{aligned} \int_0^t f'(X_s) \circ dX_s = &\int_0^t f'(X_s)\, dX_s \\ &+ \frac{1}{2}\left\langle M, \int_0^t f''(X_s)\, dM_s\right\rangle_t. \end{aligned} \tag{8}$$

By the property of cross variation (see [18, p. 143])

$$\left\langle M, \int_0^t f''(X_s)\, dM_s\right\rangle_t = \int_0^t f''(X_s)\, d\langle M\rangle_s. \tag{9}$$

Hence,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle M\rangle_s \\ &= f(X_0) + \int_0^t f'(X_s) \circ dX_s. \end{aligned} \tag{10}$$

□

3. Henstock Approach

3.1. Henstock Approach to Define Stratonovich Integral with respect to Bounded Continuous Martingale. In [13–16, 19], a positive function δ on $[0, t]$, $0 \le t < \infty$, where δ does not depend on $\omega \in \Omega$, was used. From the proof of the main results in [13, 14, 16, 20], $\delta(\xi)$ is deterministic, because the quadratic variation of a Brownian motion is deterministic and Fubini's theorem can be applied in switching the order of the two integrals. In this section, we consider martingales as integrators. To define stochastic integrals using Riemann sums, the δ-function needs to depend on Ω and stochastic intervals are needed. Therefore, we need to redefine δ as also dependent on Ω (see [12]).

Definition 5. Let $\delta : [0, t] \times \Omega \to (0, t)$, $0 \le t < \infty$, be a measurable function with respect to the two variables $\xi \in [0, t]$ and $\omega \in \Omega$. Then δ is called a *locally stopping process* [12] if, for each $\xi \in [0, t]$, $\xi + \delta$ is a stopping time.

For each $(c, d] \subset [0, t]$, we denote the measure [15] induced by the quadratic variation process of an adapted process $X = \{X_t, 0 \le t < \infty\}$ by

$$\mu_X(c, d] \triangleq E\left(\langle X\rangle_d - \langle X\rangle_c\right). \tag{11}$$

Definition 6. Let δ be a locally stopping process and $X = \{X_t, \mathscr{F}_t; 0 \le t < \infty\}$. A finite collection of stochastic interval-point pairs $(((S_i, T_i], \xi_i), i = 1, 2, \ldots, n)$, where S_i and T_i are stopping times of X for all $i = 1, 2, \ldots, n$, is said to be a (δ, η)-fine belated partial stochastic division of $[0, t]$ if

(1) for each $i(S_i, T_i]$ is a stochastic interval and, for each $\omega \in \Omega, (S_i, T_i]$, $i = 1, 2, \ldots, n$, are disjoint left-open subintervals of $[0, 1]$;

(2) each $((S_i, T_i], \xi_i)$ is a δ-fine belated division; that is, for each $\omega \in \Omega$, one has

$$(S_i(\omega), T_i(\omega)] \subset [\xi_i, \xi_i + \delta(\xi_i, \omega)]; \tag{12}$$

(3) for any $\eta > 0 |\mu_X(0,t] - \sum_{i=1}^{n} \mu_X(S_i, T_i]| < \eta$.

For the case $S_i(\omega) = T_i(\omega)$, $(S_i(\omega), T_i(\omega)]$ denotes $\{T_i(\omega)\}$. Note that, for each $\xi \in [0,t]$, $\xi + \delta$ is a stopping time. Thus, for each ξ_i, there exists a stochastic interval $(S_i(\omega), T_i(\omega)]$ with $S_i(\omega) < T_i(\omega)$ such that $(S_i(\omega), T_i(\omega)] \subset [\xi_i, \xi_i + \delta(\xi_i, \omega)]$. For each $\omega \in \Omega$, namely,

$$S_i(\omega) = \xi_i(\omega), \qquad T_i(\omega) = \inf\{\xi_i + \delta(\xi_i, \omega), t\}. \tag{13}$$

In addition, by Vitali's covering theorem, the (δ, η)-fine belated partial stochastic division could cover $[0,t]$ except for an arbitrarily small μ_X-measure [19, p. 44] [12, 16].

Definition 7. An adapted process $Y = \{Y_t, \mathcal{F}_t; 0 \le t < \infty\}$ in $(\Omega, \mathcal{F}, P)$ is said to be weakly Stratonovich-Henstock belated (denoted by WSHB) integrable to a process A in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ on $[0,t]$, $0 \le t < \infty$, with respect to a stochastic process X if, for every $\varepsilon > 0$, there exists a locally stopping process δ on $[0,t] \times \Omega$ for which

$$E\left|\sum_{i=1}^{n}\left\{\frac{Y(\xi_{i+1},\omega) + Y(\xi_i,\omega)}{2}\left(X(T_i(\omega),\omega) - X(S_i(\omega),\omega)\right) - \left[A(T_i(\omega),\omega) - A(S_i(\omega),\omega)\right]\right\}\right|^2 \le \varepsilon, \tag{14}$$

where $\xi_{n+1} = t$. For succinctness, one may write

$$E\left(\left|\sum_{i=1}^{n}\frac{Y(\xi_{i+1},\omega) + Y(\xi_i,\omega)}{2}\left(X_{T_i} - X_{S_i}\right) - \left(A_{T_i} - A_{S_i}\right)\right|^2\right) \le \varepsilon, \tag{15}$$

for every (δ, η)-fine belated partial stochastic division $D = \{((S_i, T_i], \xi_i), i = 1, 2, \ldots, n\}$ of $[0,t]$.

Proposition 8. *If an adapted process $Y = \{Y_t, \mathcal{F}_t; 0 \le t < \infty\}$ is WSHB integrable with respect to stochastic process X, then the weakly Henstock variational belated integral of Y is unique a.s.*

In light of Proposition 8, we will denote the integral of the process Y with respect to stochastic process X by the notation

$$(\mathrm{WSH}) \int_0^t Y_s \circ dW_s \triangleq A_t. \tag{16}$$

Proposition 9. *Let Y and $Y' = \{Y'_t, \mathcal{F}_t; 0 \le t < \infty\}$ be WSHB integrable with respect to stochastic process X and $\alpha \in \mathbb{R}$. Then $Y + Y'$ and αY are WSHB integrable with respect to stochastic process X. Furthermore,*

$$(WSH)\int_0^t \left(Y_s + Y'_s\right) \circ dX_s = (WSH)\int_0^t Y_s \circ dX_s + (WSH)\int_0^t Y'_s \circ dX_s; \tag{17}$$

$$(WSH)\int_0^t \alpha Y_s \circ dX_s = \alpha\,(WSH)\int_0^t Y_s \circ dX_s.$$

Proposition 10. *If stochastic process Y is WSHB integrable with respect to stochastic process X, then so is it on subinterval $[c,d] \subset [0,t]$.*

Proposition 11. *Let the adapted process Y be WSHB integrable on $[0,a]$ and $[a,t]$ with respect to stochastic process X, where $0 < a < t$. Then Y is WSHB integrable on $[0,t]$ and, furthermore,*

$$(WSH)\int_0^t Y_s \circ dX_s = (WSH)\int_0^a Y_s \circ dX_s + (WSH)\int_a^t Y_s \circ dX_s. \tag{18}$$

The proofs of Propositions 8 to 11 are similar to the results of classical Henstock integral [17]; hence we omit these proofs in this paper.

Next, we will consider the relationship between WSHB integral and Stratonovich integral with respect to continuous martingale. Here, we assume that the continuous martingale is bounded. In the next section we will address the continuous local martingale. Note that we can choose the stopping time such that a continuous local martingale becomes a bounded continuous martingale up to any finite stopping time [18, p. 44].

Theorem 12. *If $f(x) \in C^2(\mathbb{R})$ and $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is bounded continuous martingale, then $f(X_t)$ is WSHB integrable with respect to bounded continuous martingale X on $[0,t]$. Furthermore,*

$$(WSH)\int_0^t f(X_s) \circ dX_s = \int_0^t f(X_s) \circ dX_s. \tag{19}$$

Proof. Since X is a bounded continuous martingale and $f(x) \in C^2(\mathbb{R})$, then $f(X_t)$, $f'(X_t)$, and $f''(X_t)$ are bounded and continuous with respect to t. By Definition 1, the classical Stratonovich integral with respect to bounded continuous martingale X is

$$\int_0^t f(X_s) \circ dX_s = \int_0^t f(X_s)\,dX_s + \int_0^t f'(X_s)\,d\langle X\rangle_s. \tag{20}$$

Let $D_1 = \{([s_i, t_i], \xi_i)\}_{i=1}^n$ be a (δ, η)-fine belated partial stochastic division of $[0, t]$. Then,

$$\begin{aligned}
&E\left|\sum_{i=1}^{n-1}\frac{1}{2}\left(f\left(X_{\xi_i}\right)+f\left(X_{\xi_{i+1}}\right)\right)\left(X_{t_i}-X_{s_i}\right)\right.\\
&\quad\left.-\int_{s_i}^{t_i} f\left(X_s\right)\circ dX_s\right|^2\\
&\quad= E\left|\sum_{i=1}^{n-1} f\left(X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)\right.\\
&\qquad+\sum_{i=1}^{n-1}\frac{1}{2}\left(f\left(X_{\xi_{i+1}}\right)-f\left(X_{\xi_i}\right)\right)\left(X_{t_i}-X_{s_i}\right)\\
&\qquad\left.-\int_{s_i}^{t_i} f(X_s)\circ dX_s\right|^2\\
&\quad= E\left|\sum_{i=1}^{n-1} f\left(X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)\right.\\
&\qquad+\sum_{i=1}^{n-1}\frac{1}{2}\left(f\left(X_{\xi_{i+1}}\right)-f\left(X_{\xi_1}\right)\right)\left(X_{t_i}-X_{s_i}\right)\\
&\qquad\left.-\int_{s_i}^{t_i} f\left(X_s\right)dX_s-\int_{s_i}^{t_i}\frac{\partial f\left(X_s\right)}{\partial x}d\left\langle X\right\rangle_s\right|^2\\
&\quad\le 2E\left|\sum_{i=1}^{n-1} f\left(X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)-\int_{s_i}^{t_i} f\left(X_s\right)dX_s\right|^2\\
&\qquad+E\left|\sum_{i=1}^{n-1}\left(f\left(X_{\xi_{i+1}}\right)-f\left(X_{\xi_i}\right)\right)\left(X_{t_i}-X_{s_i}\right)\right.\\
&\qquad\left.-\int_{s_i}^{t_i}\frac{\partial f\left(X_s\right)}{\partial x}\left\langle X\right\rangle_s\right|^2 = 2R_1+R_2,
\end{aligned}\tag{21}$$

where

$$\begin{aligned}
R_1 &= E\left|\sum_{i=1}^{n-1} f\left(X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)-\int_{s_i}^{t_i} f\left(X_s\right)dX_s\right|^2,\\
R_2 &= E\left|\sum_{i=1}^{n-1} f\left(X_{\xi_{i+1}}\right)-f\left(X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)\right.\\
&\qquad\left.-\int_{s_i}^{t_i}\frac{\partial f\left(X_s\right)}{\partial x}d\left\langle X\right\rangle_s\right|^2.
\end{aligned}\tag{22}$$

From [12], given $\varepsilon_1 > 0$, there exists a (δ_1, η_1)-fine belated partial division of $[0, t]$, $D_1 = \{([s_i, t_i], \xi_i)\}_{i=1}^n$, such that $R_1 < \varepsilon_1$. We just need to consider R_2. By mean value theorem, $f(X_{\xi_{i+1}}) - f(X_{\xi_i}) = (\partial f/\partial x)(\widetilde{X_i})(X_{\xi_{i+1}} - X_{\xi_i})$, where $\widetilde{X_i} \in [X_{\xi_i}, X_{\xi_{i+1}}]$. Since X_t is continuous with respect to t, there exists a $\widehat{\xi_i} \in [\xi_i, \xi_{i+1}]$ such that $X_{\widehat{\xi_i}} = \widetilde{X_i}$ a.s. Similarly, there exists $\overline{\xi_i} \in [s_i, t_i]$ such that $\int_{s_i}^{t_i}(\partial f(X_t)/\partial x)d\langle X\rangle_t = f'(X_{\overline{\xi_i}})(X_{t_i} - X_{s_i})^2$ in $L^2(\Omega)$ space. Then,

$$\begin{aligned}
R_2 &= E\left|\sum_{i=1}^{n-1}\left(f\left(X_{\xi_{i+1}}\right)-f\left(X_{\xi_i}\right)\right)\left(X_{t_i}-X_{s_i}\right)\right.\\
&\qquad\left.-\int_{s_i}^{t_i}\frac{\partial f\left(X_t\right)}{\partial x}d\left\langle X\right\rangle_t\right|^2\\
&= E\left|\sum_{i=1}^{n-1}\left\{f'\left(X_{\widehat{\xi_i}}\right)\left(X_{\xi_{i+1}}-X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)\right.\right.\\
&\qquad\left.\left.-\frac{\partial f\left(X_{\overline{\xi_i}}\right)}{\partial x}\left(X_{t_i}-X_{s_i}\right)^2\right\}\right|^2\\
&\le 2E\left|\sum_{i=1}^{n-1}\left\{f'\left(X_{\widehat{\xi_i}}\right)\left(X_{\xi_{i+1}}-X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)\right.\right.\\
&\qquad\left.\left.-f'\left(X_{\widehat{\xi_i}}\right)\left(X_{t_i}-X_{s_i}\right)^2\right\}\right|^2\\
&\quad+2E\left|\sum_{i=1}^{n-1} f'\left(X_{\widehat{\xi_i}}\right)\left(X_{t_i}-X_{s_i}\right)^2\right.\\
&\qquad\left.-f'\left(X_{\overline{\xi_i}}\right)\left(X_{t_i}-X_{s_i}\right)^2\right|^2.
\end{aligned}\tag{23}$$

Let

$$\begin{aligned}
r_1 &= E\left|\sum_{i=1}^{n-1}\left\{f'\left(X_{\widehat{\xi_i}}\right)\left(X_{\xi_{i+1}}-X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)\right.\right.\\
&\qquad\left.\left.-f'\left(X_{\widehat{\xi_i}}\right)\left(X_{t_i}-X_{s_i}\right)^2\right\}\right|^2,\\
r_2 &= E\left|\sum_{i=1}^{n-1} f'\left(X_{\widehat{\xi_i}}\right)\left(X_{t_i}-X_{s_i}\right)^2-f'\left(X_{\overline{\xi_i}}\right)\left(X_{t_i}-X_{s_i}\right)^2\right|^2,\\
r_1 &\le KE\left|\sum_{i=1}^{n-1}\left(X_{\xi_{i+1}}-X_{\xi_i}\right)\left(X_{t_i}-X_{s_i}\right)-\left(X_{t_i}-X_{s_i}\right)^2\right|^2\\
&\le KE\left|\sum_{i=1}^{n-1}\left(X_{t_i}-X_{s_i}\right)\left[\left(X_{t_i}-X_{s_i}\right)-\left(X_{\xi_{i+1}}-X_{\xi_i}\right)\right]\right|^2\\
&\le KE\left|\sum_{i=1}^{n-1}\left(X_{t_i}-X_{s_i}\right)\left[\left(X_{\xi_{i+1}}-X_{t_i}\right)+\left(X_{s_i}-X_{\xi_i}\right)\right]\right|^2\\
&\le K\sum_{i=1}^{n-1} E\left|\left(X_{\xi_{i+1}}-X_{t_i}\right)+\left(X_{s_i}-X_{\xi_i}\right)\right|^2 E\left(X_{t_i}-X_{s_i}\right)^2,
\end{aligned}\tag{24}$$

where $K = \sup\{f'(W_t) : 0 \le t \le 1 \ \& \ \omega \in \Omega\}$. Given that X_t is a bounded martingale,

$$\begin{aligned} E\left|\left(X_{\xi_{i+1}} - X_{t_i}\right) + \left(X_{s_i} - X_{\xi_i}\right)\right|^2 &= E\left(X_{\xi_{i+1}} - X_{t_i}\right)^2 + E\left(X_{s_i} - X_{\xi_i}\right)^2 \\ &= E\langle X\rangle_{\xi_{i+1}} - E\langle X\rangle_{t_i} + E\langle X\rangle_{s_i} - E\langle X\rangle_{\xi_i}. \end{aligned} \tag{25}$$

However, there exists a (δ_2, η_2)-fine belated partial stochastic division $D_2 = \{[\xi_i, t_i], \xi_i) : i = 1, 2, \ldots, n\}$ of $[0, t]$ such that $|\mu_X(0, t] - \sum_{i=1}^{n} \mu_X(s_i, t_i]| < \eta_2$. Then we have

$$E\langle X\rangle_{\xi_{i+1}} - E\langle X\rangle_{t_i} + E\langle X\rangle_{s_i} - E\langle X\rangle_{\xi_i} < \eta_2. \tag{26}$$

Hence,

$$r_1 \le K\varepsilon_2 \sum_{i=1}^{n-1} E\left(X_{t_i} - X_{s_i}\right)^2 \le KE\langle X\rangle_t \eta_2. \tag{27}$$

Since X is bounded continuous martingale, by Lemma 5.10 in [18, p. 33], we have

$$E\left|\sum_{i=1}^{n-1}\left(X_{t_i} - X_{s_i}\right)^4\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{28}$$

Hence, given $\varepsilon_2 > 0$,

$$\begin{aligned} r_2 &\le \sup_{1\le i\le n \ \& \ \omega\in\Omega}\left|f'\left(X_{\widehat{\xi}_i}\right) - f'\left(X_{\overline{\xi}_i}\right)\right| E\left|\sum_{i=1}^{n-1}\left(X_{t_i} - X_{s_i}\right)^4\right| \\ &\le 2K\varepsilon_2. \end{aligned} \tag{29}$$

Now take $\eta = \min\{\eta_1, \eta_2\}$ and let $\delta(\xi_i) = \min\{\delta_1(\xi_i), \delta_2, \xi_{i+1} - \xi_i\}$; then a (δ, η)-fine belated partial division $D = \{[\xi_i, t_i], \xi_i) : i = 1, 2, \ldots, n\}$ of $[0, t]$ is both (δ_1, η_1)-fine belated partial stochastic division and (δ_2, η_2)-fine belated partial stochastic division. Hence, for $\varepsilon = \max\{\varepsilon_1, \eta_2, \varepsilon_2\}$, we have

$$\begin{aligned} E\left|\sum_{i=1}^{n-1}\frac{f_{\xi_i} + f_{\xi_{i+1}}}{2}\left(W_{t_i} - W_{s_i}\right) - \int_{s_i}^{t_i} X_t \circ dW_t\right|^2 \\ \le R_1 + r_1 + r_2 \le \overline{K}\varepsilon, \end{aligned} \tag{30}$$

where $\overline{K}$ is a constant. Hence, $f(X_t)$ is WSHB integrable on $[0, t]$ and, by the unique property of WSHB,

$$(\text{WSH}) \int_0^t f(X_s) \circ dX_s = \int_0^t f(X_s) \circ dX_s. \tag{31}$$

□

3.2. Henstock Approach to Define Stratonovich Integral with respect to Continuous Local Martingale

Lemma 13. *Let $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous local martingale. Suppose f is WSHB integrable to a process A_t with respect to local continuous martingale X. Let T be a stopping time; then f is WSHB integrable to a process A^T with respect to X^T, where $X^T = \{X_{t\wedge T}; 0 \le t < \infty\}$ and $A^T = \{A_{t\wedge T}; 0 \le t < \infty\}$.*

Proof. Let $\varepsilon > 0$ and $\delta(\xi, \omega) > 0$ be given as in Definition 6 for the WSHB integral of f with respect to continuous local martingale X. Define

$$\delta^T(\xi, \omega) \triangleq \begin{cases} T(\omega) - \xi, & \text{if } \xi < T(\omega) \le \xi + \delta(\xi, \omega) \\ \delta(\xi, \omega), & \text{otherwise.} \end{cases} \tag{32}$$

Then $\delta^T(\xi, \omega)$ is locally stopping process, since $(T(\omega) \wedge \xi) \wedge (\xi + \delta(\xi, \omega))$ is a stopping time and $\{\omega : T(\omega) < \xi\} \in \mathcal{F}_t$ if $\xi < t$. Let $((U, V], \xi)$ be a $(\delta^T(\xi, \omega), \eta)$-fine belated stochastic interval-point pair.

(i) If $T(w) \le \xi$, then

$$\begin{aligned} X^T(V(\omega), \omega) - X^T(U(\omega), \omega) \\ = X^T(T(\omega), \omega) - X^T(T(\omega), \omega) = 0. \end{aligned} \tag{33}$$

(ii) If $\xi < T(\omega) \le \xi + \delta(\xi, \omega)$, then $[U, V] \subset [\xi, T] \subset [\xi, \xi + \delta(\xi, \omega)]$. Consequently, $((U, V], \xi)$ will be a $(\delta(\xi, \omega), \eta)$-fine belated stochastic interval-point pair, $U \wedge T = U$ and $V \wedge T = V$.

(iii) If $T(\omega) > \xi + \delta(\xi, \omega)$, then

$$\begin{aligned} X^T(U(\omega), \omega) - X^T(V(\omega), \omega) \\ = X(V(\omega) \wedge T(\omega), \omega) - X(U(\omega) \wedge T(\omega), \omega) \\ = X(U(\omega), \omega) - X(V(\omega), \omega). \end{aligned} \tag{34}$$

It is easy to verify that $D = \{((U_i, V_i], \xi_i), i = 1, 2, \ldots, n\}$ is also a $(\delta^T(\xi, \omega), \eta)$-fine belated stochastic division of $[0, t \wedge T(w)]$ only if $D = \{((U_i, V_i], \xi_i), i = 1, 2, \ldots, n\}$ is a $(\delta(\xi, \omega), \eta)$-fine belated stochastic division of $[0, t]$.

Similar results hold for A^T. Since the $D = \{((U_i, V_i], \xi_i), i = 1, 2, \ldots, n\}$ is also a $(\delta^T(\xi, \omega), \eta)$-fine belated stochastic division of $[0, t \wedge T(w)]$, by the property of being WSHB integrable, we have that f is WSHB integrable to a process A^T with respect to X^T. □

Lemma 14. *Let $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous local martingale with the corresponding nondecreasing sequence of stopping times $\{T_n\}_{n=1}^{\infty}$. Suppose that f is WSHB integrable to a process A with respect to X; then f is WSHB integrable to a process A^{T_n} with respect to X^{T_n} for each $n = 1, 2, \ldots$. Furthermore, for each $t \in [0, t]$, $\lim_{n\to\infty} A^{T_n} = A$ a.s.*

Proof. This follows directly form Lemma 13 and the uniqueness of the WSHB integral. Then we have the last statement

$$\lim_{n\to\infty} A^{T_n} = A \text{ a.s.} \tag{35}$$

□

Theorem 15. *If $f(x) \in C^2(\mathbb{R})$ and $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous local martingale, then $f(X_t)$ is WSHB integrable with respect to X on $[0, t]$. Furthermore,*

$$(WSH) \int_0^t f(X_s) \circ dX_s = \int_0^t f(X_s) \circ dX_s. \tag{36}$$

Proof. First we introduce, for each $m \geq 1$, the stopping time

$$T_m = \inf\{t \geq 0; |X_t| \geq m \text{ or } \langle X\rangle_t \geq m\}. \tag{37}$$

The resulting sequence $\{T_m\}$ is nondecreasing with $P[\lim_{m\to\infty} T_m = \infty] = 1$. Thus, the stopping process $X_{t\wedge T_m}$ is bounded; that is, $X_{t\wedge T_m}$ is a bounded martingale for each $m = 1, 2, \ldots$. Under this situation, the values of f outside $[-m, m]$ are irrelevant. We may assume without loss of generality that f has compact support, so f, f', and f'' are bounded.

Let $D = \{((U_i, V_i], \xi_i), i = 1, 2, \ldots, n\}$ be a $(\delta(\xi, \omega), \eta)$-fine belated stochastic division of $[0, 1]$. From Lemma 13, we have that, for each $m \geq 1$, $D_m = \{((U_i, V_i], \xi_i), i = 1, 2, \ldots, n\}$ is a $(\delta^{T_m}(\xi, \omega), \eta)$-fine belated stochastic division of $[0, t\wedge T_m(\omega)]$. From Lemma 13, there is a process $A^{T_m} = \{A_{t\wedge T_m}; 0 \leq t < \infty\}$ such that

$$E\left(\left|\sum_{i=1}^{n-1} \frac{1}{2} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i\wedge T_m} - X_{U_i\wedge T_m}\right) - \left(A_{V_i\wedge T_m} - A_{U_i\wedge T_m}\right)\right|^2\right) < \varepsilon, \tag{38}$$

for every (δ, η)-fine belated partial stochastic division $D = \{((U_i, V_i], \xi_i) : i = 1, 2, \ldots, n\}$ of $[0, t]$. Since

$$\begin{aligned}
&\lim_{m\to\infty}\left|\sum_{i=1}^{n-1} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i\wedge T_m} - X_{U_i\wedge T_m}\right) - \left(A_{V_i\wedge T_m} - A_{U_i\wedge T_m}\right)\right| \\
&= \left|\sum_{i=1}^{n-1} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i} - X_{U_i}\right) - \left(A_{V_i} - A_{U_i}\right)\right| \text{ a.s.,}
\end{aligned} \tag{39}$$

we must have

$$\begin{aligned}
&E\left|\sum_{i=1}^{n-1} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i} - X_{U_i}\right) - \left(A_{V_i} - A_{U_i}\right)\right| \\
&= E\left(\lim_{m\to\infty}\left|\sum_{i=1}^{n-1} \frac{1}{2} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i\wedge T_m} - X_{U_i\wedge T_m}\right) - \left(A_{V_i\wedge T_m} - A_{U_i\wedge T_m}\right)\right|^2\right) \\
&= E\left(\liminf_{m\to\infty}\left|\sum_{i=1}^{n-1} \frac{1}{2} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i\wedge T_m} - X_{U_i\wedge T_m}\right) - \left(A_{V_i\wedge T_m} - A_{U_i\wedge T_m}\right)\right|^2\right) \\
&\leq \liminf_{m\to\infty} E\left(\left|\sum_{i=1}^{n-1} \frac{1}{2} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i\wedge T_m} - X_{U_i\wedge T_m}\right) - \left(A_{V_i\wedge T_m} - A_{U_i\wedge T_m}\right)\right|^2\right) \\
&< \liminf_{m\to\infty} \varepsilon \leq \varepsilon.
\end{aligned} \tag{40}$$

This shows that $f(X_t)$ is WSHB integrable on $[0, t]$ and

$$(\text{WSH})\int_0^t f(X_s) \circ dX_s = \int_0^t f(X_s) \circ dX_s. \tag{41}$$

□

3.3. Henstock Approach to Define Stratonovich Integral with respect to Continuous Semimartingale. Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a continuous semimartingale such that

$$X_t = X_0 + M_t + C_t, \tag{42}$$

where M_t is a continuous local martingale and C_t is adapted, continuous process of bounded variation with $C_0 = 0$ a.s. We have

$$\begin{aligned}
&\frac{1}{2} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(X_{V_i} - X_{U_i}\right) \\
&\quad = \frac{1}{2} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(M_{V_i} - M_{U_i}\right) \\
&\qquad + \frac{1}{2} f\left(X_{\xi_{i+1}} + X_{\xi_i}\right)\left(C_{V_i} - C_{U_i}\right).
\end{aligned} \tag{43}$$

Now we consider a stopping time

$$T_m = \begin{cases} 0; & \text{if } |X_0| \geq m \\ \inf\{t \geq 0; |M_t| \geq m \text{ or} \\ \quad \langle M\rangle_t \geq m \text{ or } \check{C}_t \geq m\}; \\ \quad \text{if } |X_0| < m \\ \infty; & \text{if } |X_0| < m \ \& \\ \quad \inf\{t \geq 0; |M_t| \geq m \text{ or} \\ \qquad \langle M\rangle_t \geq m \text{ or } \check{C}_t \geq m\} = \infty, \end{cases} \tag{44}$$

where $\check{C}_t$ is the total variation of C_t on $[0, t]$. Then the sequence $\{T_m\}$ is nondecreasing with $P[\lim_{m\to\infty} T_m = \infty] = 1$. Thus, the stopping process $M_{t\wedge T_m}$ is a bounded martingale

for each $m = 1, 2, \ldots$. In addition, $C_t 1_{\{t \le T_m(w)\}}$ is of bounded variation. Under this situation, the values of f outside $[-3m, 3m]$ are irrelevant. We may assume without loss of generality that f has compact support, so f, f', and f'' are bounded. Given that the $\sum_{i=1}^{n}(1/2)f(X_{\xi_{i+1}} + X_{\xi_i})(C_{V_i} - C_{U_i})$ is classic Riemann-Stieltjes integral, $(1/2)f(X_{\xi_{i+1}} + X_{\xi_i})(A_{V_i} - A_{U_i})$ is WSHB integrable for every (δ, ε)-fine belated partial stochastic division $D = \{((U_i, V_i], \xi_i) : i = 1, 2, \ldots, n\}$ of $[0, t]$ (see [17]). Since the WSHB integral satisfies the linear property, we obtain the following result.

Theorem 16. *If $f(x) \in C^2(\mathbb{R})$ and X is a continuous semimartingale, then $f(X_t)$ is WSHB integrable with respect to X_t on $[0, t]$. Furthermore,*

$$(WSH) \int_0^t f(X_s) \circ dX_s = \int_0^t f(X_s) \circ dX_s. \tag{45}$$

4. Conclusion

From Theorem 16, if f is of class $C^2(\mathbb{R})$ and X is a continuous semimartingale, $f(X_t)$ is WSHB integrable to $\int_0^t f(X_s) \circ dX_s$ on $[0, t]$. Now we consider the Itô formula for Stratonovich integral; if $f \in C^3(\mathbb{R})$, as shown in Proposition 4,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s. \tag{46}$$

We substitute (WSH) $\int_0^t f'(X_s) \circ dX_s$ with $\int_0^t f'(X_s) \circ dX_s$. Then,

$$f(X_t) = f(X_0) + (\mathrm{WSH}) \int_0^t f'(X_s) \circ dX_s. \tag{47}$$

In conclusion, from the definition of Stratonovich integral with respect to continuous semimartingale using Henstock method, we manage to keep the important properties of the classical Stratonovich integral and also probably enlarge the scope of the integrands which satisfy classical Stratonovich integral Itô formula (46).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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