Hindawi Publishing Corporation
International Scholarly Research Notices
Volume 2014, Article ID 541524, 8 pages
http://dx.doi.org/10.1155/2014/541524

Research Article

Second Order Duality in Multiobjective Fractional Programming with Square Root Term under Generalized Univex Function

Arun Kumar Tripathy

Department of Mathematics, Trident Academy of Technology, F2/A, Chandaka Industrial Estate, Bhubaneswar, Odisha 751024, India

Correspondence should be addressed to Arun Kumar Tripathy; arun_tripathy06@rediffmail.com

Received 12 March 2014; Accepted 10 April 2014; Published 7 July 2014

Academic Editor: Majid Soleimani-damaneh

Copyright © 2014 Arun Kumar Tripathy. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Three approaches of second order mixed type duality are introduced for a nondifferentiable multiobjective fractional programming problem in which the numerator and denominator of objective function contain square root of positive semidefinite quadratic form. Also, the necessary and sufficient conditions of efficient solution for fractional programming are established and a parameterization technique is used to establish duality results under generalized second order ρ-univexity assumption.

1. Introduction

A fractional programming problem arises in many types of optimization problems such as portfolio selection, production, information theory, and numerous decision making problems in management science. More specifically, it can be used in engineering and economics to minimize a ratio of physical or economical function or both, such as cost/time, cost/volume, and cost/benefit, in order to measure the efficiency or productivity of the system. Many economic, noneconomic, and indirect applications of fractional programming problem have also been given by Bector [1], Bector and Chandra [2], Craven [3], Mond and Weir [4], Stancu-Minasian [5], Schaible and Ibaraki [6], Ahmad et al. [7], Ahmad and Sharma [8], and Gulati et al. [9].

The central concept in optimization is known as the duality theory which asserts that, given a (primal) minimization problem, the infimum value of the primal problem cannot be smaller than the supermom value of the associated (dual) maximization problem and the optimal values of primal and dual problems are equal. Duality in fractional programming is an important class of duality theory and several contributions have been made in the past [1, 5, 8, 10–14]. Second order duality provides a tighter bound for the value of the objective function when approximations are used. For more details, one can consult [15, page 93]. Another advantage of second order duality when applicable is that if a feasible point in the primal is given and first order duality does not apply, then we can use second order duality to provide a lower bound of the value of the primal problem (see [4]).

Multiobjective fractional programming duality has been of much interest in the recent past. Schaible [16] and Bector et al. [11] derived Fritz John and Karush-Kuhn Tucker necessary and sufficient optimality condition for a class of nondifferentiable convex multiobjective fractional programming problems and established some duality theorems. Liang et al. [17, 18] discussed the optimality condition and duality for nonlinear fractional programming. Santos et al. [19] discussed the optimality and duality for nonsmooth multiobjective fractional programming with generalized convexity. Bector et al. [20] and Xu [21] gave a mixed type duality for fractional programming, established some sufficient conditions, and obtained various duality results between the mixed dual and primal problem. Zhou and Wang [22] introduced a class of mixed type dual for nonsmooth multiobjective fractional programming and established the duality results under (V, ρ) invexity assumption.

Duality for various forms of mathematical problems involving square roots of positive semidefinite quadratic forms has been discussed by many authors [10, 23–25]. Mond [25] considered a nonlinear fractional programming problem

involving square roots of positive semidefinite quadratic form in the numerator and denominator and proved the necessary and sufficient condition for optimality. Kim et al. [26, 27] formulated a nondifferentiable multiobjective fractional problem in which numerators contain support function. One of the most known approaches used for solving nonlinear fractional programming problem is called parametric approach. Dinklebaeh [28] and Jagannathan [12] introduced this approach that was used later by Osuna-Gómez et al. [13] to characterize solution of a multiobjective fractional problem under generalized convexity. Tripathy [14] introduced three approaches given by Dinklebaeh [28] and Jagannathan [12] for both primal and mixed type dual of a nondifferentiable multiobjective fractional programming and established the duality results under generalized ρ-univexity.

To relax convexity assumption imposed on the function in theorems on optimality and duality, various generalized convexity concepts have been proposed. Hanson [29] introduced the class of invex functions. Bector et al. [30] introduced univex function. Mishra [31] derived the optimality condition for multiobjective programming with generalized univexity. Jayswal [32] presented minimax fractional programming under generalized ρ-univexity assumption.

Motivated by the earlier authors in this paper, we have introduced three approaches given by Dinklebaeh [28] and Jagannathan [12] for both primal and second order mixed type dual of a nondifferentiable multiobjective fractional programming problem in which the numerator and denominator of objective function contain square root of positive semidefinite quadratic form. Also we have established the necessary and sufficient optimality condition and used a parameterization technique to establish duality results under generalized ρ-univexity assumption.

2. Notations and Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ its nonnegative orthant. The following conventions for inequality will be used throughout this paper. For any $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, we denote the following.

(i) $x > y \Leftrightarrow x_i > y_i$, for all $i = 1, 2, \ldots, n$.

(ii) $x \geq y \Leftrightarrow x_i \geq y_i$, for all $i = 1, 2, \ldots, n$.

Throughout the paper, let X be a nonempty open subset of $\mathbb{R}^n$.

Consider the following nondifferentiable multiobjective fractional programming problem.

2.1. Multiobjective Fractional Primal Problem

(i) *MFP0*. Minimize

$$\frac{f(x) + \left(x^T B x\right)^{1/2}}{g(x) - \left(x^T C x\right)^{1/2}} = \left(K_1(x), K_2(x), \ldots, K_k(x)\right), \quad (1)$$

where

$$K_i(x) = \frac{f_i(x) + \left(x^T B_i x\right)^{1/2}}{g_i(x) - \left(x^T C_i x\right)^{1/2}}, \quad i = 1, 2, \ldots, k. \quad (2)$$

(ii) *MFP1*. Minimize

$$F(x) = \left(F_1(x), F_2(x), \ldots, F_k(x)\right), \quad (3)$$

where

$$F_i(x) = f_i(x) + \left(x^T B_i x\right)^{1/2} - \nu_i \left\{g_i(x) - \left(x^T C_i x\right)^{1/2}\right\}, \quad i = 1, 2, \ldots, k; \quad (4)$$

ν_i are fixed parameters.

(iii) *MFPλ*. Minimize $\lambda F(x)$; λ is k-dimensional strictly positive vector,

all subject to same constraint

$$h(x) \leq 0, \quad x \in X \subseteq \mathbb{R}^n, \quad (5)$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, k$ and $h = (h_1, \ldots, h_m)$; $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, m$, are differentiable functions, B_i and C_i, $i = 1, 2, \ldots, k$ are positive semidefinite matrices of order n. In the sequel, we assume that $f_i(x) \geq 0$ and $g_i(x) > 0$ on $\mathbb{R}^n$ for $i = 1, 2, \ldots, k$.

Let $X_0 = \{x \in X \subseteq \mathbb{R}^n : h_j(x) \leq 0, j = 1, 2, \ldots, m\}$ for all feasible solutions of MFP0, MFP1, and MFPλ and denote $I = \{1, 2, 3, \ldots, k\}$, $M = \{1, 2, 3, \ldots, m\}$, $J_1 = \{j \in M : h_j(x) = 0\}$, and $J_2 = \{j \in M : h_j(x) < 0\}$. It is obvious that $J_1 \cup J_2 = M$.

Throughout the paper, consider $f_i : X \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $p \in \mathbb{R}^n$, $\rho \in \mathbb{R}$.

Assume that $\psi : \mathbb{R} \to \mathbb{R}$ satisfying $a \leq 0 \Rightarrow \psi(a) \leq 0$ or $\psi(a) \leq 0 \Rightarrow a \leq 0$ and $\psi(-a) = -\psi(a)$, $K : X \times X \to \mathbb{R}_+$. For $x, \overline{x} \in X$ we can write $K(x, \overline{x}) = \lim_{\lambda \to 0} b(x, \overline{x}, \lambda) \geq 0$.

Definition 1. The real differentiable function f_i is said to be second order ρ-univex at $\overline{x} \in X$ with respect to η, ψ, and K, if

$$K(x, \overline{x})\, \psi\left[f_i(x) - f_i(\overline{x}) + \frac{1}{2} p^T \left(\nabla^2 f_i(\overline{x})\, p\right)\right] \geq \eta(x, \overline{x})^T \left[\nabla f_i(\overline{x}) + \nabla^2 f_i(\overline{x})\, p\right] + \rho \|x - \overline{x}\|^2, \quad \forall x \in X. \quad (6)$$

Definition 2. The real differentiable function f_i is said to be second order ρ-pseudounivex at $\overline{x} \in X$ with respect to η, ψ, and K, if

$$\eta(x, \overline{x})^T \left[\nabla f_i(\overline{x}) + \nabla^2 f_i(\overline{x})\, p\right] + \rho \|x - \overline{x}\|^2 \geq 0 \Longrightarrow K(x, \overline{x})\, \psi\left[f_i(x) - f_i(\overline{x}) + \frac{1}{2} p^T \left(\nabla^2 f_i(\overline{x})\, p\right)\right] \geq 0, \quad \forall x \in X. \quad (7)$$

Definition 3. The real differentiable function f_i is said to be second order ρ-quasiunivex at $\overline{x} \in X$ with respect to η, ψ, and K, if

$$K(x,\overline{x})\,\psi\left[f_i(x) - f_i(\overline{x}) + \frac{1}{2}p^T\left(\nabla^2 f_i(\overline{x})\,p\right)\right] \le 0$$
$$\Longrightarrow \eta(x,\overline{x})^T\left[\nabla f_i(\overline{x}) + \nabla^2 f_i(\overline{x})\,p\right] + \rho\|x-\overline{x}\|^2 \le 0, \quad \forall x \in X. \tag{8}$$

Remark 4. If $p = 0$, the above definitions reduce to the definitions of ρ-univex, ρ-pseudounivex, and ρ-quasiunivex as introduced in [14].

Definition 5. A feasible point $\overline{x}$ is said to be efficient for MFP1, if there exists no other feasible point x in MFP1 such that $F_i(x) \le F_i(\overline{x})$, $i = 1,2,\ldots,k$, and $F_r(x) < F_r(\overline{x})$ for some $r \in \{1,2,\ldots,k\}$.

Definition 6 (see [33]). A feasible point $\overline{x}$ is said to be properly efficient for MFP1, if it is efficient and there exist $M > 0$ such that, for each $i \in \{1,2,\ldots,k\}$ and for all feasible point x in MFP1 satisfying $F_i(x) < F_i(\overline{x})$, we have $F_i(\overline{x}) - F_i(x) \le M(F_r(x) - F_r(\overline{x}))$ for some $r \in \{1,2,\ldots,k\}$ such that $F_r(x) > F_r(\overline{x})$.

We assume that $f_i(x) + (x^T B_i x)^{1/2} \ge 0$, $g_i(x) - (x^T C_i x)^{1/2} > 0$, $i = 1,2,\ldots,k$ for all $x \in X$.

Definition 7 (generalized Schwarz Inequality). Let B be a positive semidefinite matrix of order n. Then, for all $x, w \in \mathbb{R}^n$, $x^T B w \le (x^T B x)^{1/2}(w^T B w)^{1/2}$.

The equality holds if $Bx = \lambda Bw$ for some $\lambda \ge 0$.

Let $J_1(x) = \{j \in M = \{1,2,\ldots,m\} : h_j(x) = 0\}$ and $\overline{v}_i = (f_i(\overline{x}) + (x^T B_i x)^{1/2})/(g_i(\overline{x}) - (x^T C_i x)^{1/2})$.

Then define the set $W(\overline{x}) = \{w \in \mathbb{R}^n : w^T \nabla h_j(\overline{x}) \le 0,\ j \in J(\overline{x})\}$ satisfying any one of the following conditions:

(a) $\overline{x}^T B_i \overline{x} > 0$, $\overline{x}^T C_i \overline{x} = 0 \Rightarrow w^T(\nabla f_i(\overline{x}) + B_i\overline{x}/(\overline{x}^T B_i \overline{x})^{1/2} - \overline{v}_i \nabla g_i(\overline{x})) + (w^T(\overline{v}_i^2 C_i)w)^{1/2} \ge 0$, $w \in W(\overline{x})$, $i = 1,2,\ldots,k$;

(b) $\overline{x}^T B_i \overline{x} = 0$, $\overline{x}^T C_i \overline{x} > 0 \Rightarrow w^T(\nabla f_i(\overline{x}) - \overline{v}_i\{\nabla g_i(\overline{x}) - C_i\overline{x}/(\overline{x}^T C_i \overline{x})^{1/2}\}) + (w^T B_i w)^{1/2} \ge 0$, $w \in W(\overline{x})$, $i = 1,2,\ldots,k$;

(c) $\overline{x}^T B_i \overline{x} = 0$, $\overline{x}^T C_i \overline{x} = 0 \Rightarrow w^T(\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})) + (w^T B_i w)^{1/2} + (w^T(\overline{v}_i^2 C_i)w)^{1/2} \ge 0$, $w \in W(\overline{x})$, $i = 1,2,\ldots,k$;

(d) $\overline{x}^T B_i \overline{x} > 0$, $\overline{x}^T C_i \overline{x} > 0 \Rightarrow w^T(\nabla f_i(\overline{x}) + B_i\overline{x}/(\overline{x}^T B_i \overline{x})^{1/2} - \overline{v}_i\{\nabla g_i(\overline{x}) - C_i\overline{x}/(\overline{x}^T C_i \overline{x})^{1/2}\}) \ge 0$, $w \in W(\overline{x})$, $i = 1,2,\ldots,k$.

Lemma 8 (see [33]). *If $x^0 \in X_0$ is an optimal solution of MFPλ, then x^0 is properly efficient for MFP1.*

Lemma 9 (see [12]). *$x^0 \in X_0$ is an efficient solution for MFP0 if and only if it is an efficient solution of MFP1 with $F(x^0) = 0$.*

Lemma 10 (see [10] necessary optimality condition). *If $\overline{x} \in X_0$ is an optimal solution of (MFPλ) such that $W(\overline{x}) = \phi$, then there exist $v_i \in \mathbb{R}_+$, $w, z \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$ such that*

$$\nabla\lambda F(\overline{x}) + y^T \nabla h(\overline{x}) = \sum_{i=1}^{k}\lambda_i\left[\nabla f_i(\overline{x}) + B_i w - v_i\{\nabla g_i(\overline{x}) - C_i z\}\right] + \sum_{j=1}^{m} y_j \nabla h_j(\overline{x}) = 0, \tag{9}$$

$$F_i(\overline{x}) = f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x}\right)^{1/2} - v_i\left(g_i(\overline{x}) - \left(\overline{x}^T C_i \overline{x}\right)^{1/2}\right) = 0, \quad i = 1,2,\ldots,k. \tag{10}$$

$$y^T h(\overline{x}) = 0, \tag{11}$$

$$w^T B_i w \le 1, \quad z^T C_i z \le 1, \quad i = 1,2,\ldots,k, \tag{12}$$

$$\left(\overline{x}^T B_i \overline{x}\right)^{1/2} = \overline{x}^T B_i w, \quad \left(\overline{x}^T C_i \overline{x}\right)^{1/2} = \overline{x}^T C_i z, \quad i = 1,2,\ldots,k, \tag{13}$$

$$y \ge 0, \tag{14}$$

$$v_i \ge 0, \quad i = 1,2,\ldots,k. \tag{15}$$

Theorem 11 (sufficient optimality condition). *Let $\overline{x} \in X_0$ be a feasible solution of MFP1 and there exist $\lambda_i \in \mathbb{R}_+$; $w, z \in \mathbb{R}^n$, $v_i \in \mathbb{R}_+$, and $y \in \mathbb{R}^m$ satisfying the condition in Lemma 10 at $\overline{x}$. Furthermore suppose that the following conditions hold.*

(i) *$P(x) = \sum_{i=1}^{k}\lambda_i[f_i(x) + (x^T B_i x)^{1/2} - v_i\{g_i(x) - (x^T C_i x)^{1/2}\}] + \sum_{j=1}^{m} y_j h_j(x)$ is second order ρ-pseudounivex with respect to η, ψ, and K at $\overline{x} \in X_0$, with $(\nabla^2 P(x))p = 0$, where $f_i : X \to \mathbb{R}$, $g_i : X \to \mathbb{R}$, $h_j : X \to \mathbb{R}$, $i = 1,2,\ldots,k$; $j = 1,2,\ldots,m$, $p \in \mathbb{R}^n$, $\eta : X \times X \to \mathbb{R}^n$, $K : X \times X \to \mathbb{R}_+$, and $\psi : \mathbb{R} \to \mathbb{R}$ satisfying $\psi(a) \le 0 \Rightarrow a \le 0$.*

(ii) *$\rho \ge 0$.*

Then $\overline{x}$ is an efficient solution of MFP1.

Proof. Suppose that the hypothesis holds.

Since the conditions of Lemma 10 are satisfied, from (9) and (13), we have

$$\nabla P(\overline{x}) = \nabla\left(\sum_{i=1}^{k}\lambda_i\left[f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x}\right)^{1/2} - v_i\left\{g_i(\overline{x}) - \left(\overline{x}^T C_i \overline{x}\right)^{1/2}\right\}\right] + \sum_{j=1}^{m} y_j h_j(\overline{x})\right)$$

$$= \nabla \left(\sum_{i=1}^{k} \lambda_i \left[f_i(\overline{x}) + \overline{x}^T B_i w - \nu_i \left\{ g_i(\overline{x}) - \overline{x}^T C_i z \right\} \right] + \sum_{j=1}^{m} y_j h_j(\overline{x}) \right) = 0. \tag{16}$$

Also from hypothesis (i), we have $\nabla^2 P(\overline{x}) p = 0$

So we can write $\nabla P(\overline{x}) + \nabla^2 P(\overline{x}) p = 0$.

Now for $\eta(x, \overline{x}) \in \mathbb{R}^n$, we can write $\eta(x, \overline{x})^T (\nabla P(\overline{x}) + \nabla^2 P(\overline{x}) p) = 0$.

For $\rho \geq 0$, we have $\eta(x, \overline{x})^T (\nabla P(\overline{x}) + \nabla^2 P(\overline{x}) p) + \rho \|x - \overline{x}\|^2 \geq 0$.

Since $P(x)$ is second order ρ-pseudounivex with respect to η, ψ, and K at $\overline{x} \in X_0$, we have $K(x, \overline{x}) \psi \{P(x) - P(\overline{x}) + (1/2) p^T (\nabla^2 P(\overline{x})) p\} \geq 0$, and using the properties of K, ψ, it gives

$$P(x) - P(\overline{x}) + \frac{1}{2} p^T \nabla^2 P(\overline{x}) p \geq 0 \Longrightarrow P(x) \geq P(\overline{x}) - \frac{1}{2} p^T \nabla^2 P(\overline{x}) p. \tag{17}$$

Since $\nabla^2 P(\overline{x}) p = 0$, the above inequality implies $P(x) \geq P(\overline{x})$

$$\Longrightarrow \sum_{i=1}^{k} \lambda_i \left[f_i(x) + \left(x^T B_i x \right)^{1/2} - \nu_i \left\{ g_i(x) - \left(x^T C_i x \right)^{1/2} \right\} \right] + \sum_{j=1}^{m} y_j h_j(x) \geq \sum_{i=1}^{k} \lambda_i \left[f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x} \right)^{1/2} - \nu_i \left\{ g_i(\overline{x}) - \left(\overline{x}^T C_i \overline{x} \right)^{1/2} \right\} \right] + \sum_{j=1}^{m} y_j h_j(\overline{x}). \tag{18}$$

Suppose that $\overline{x}$ is not efficient solution of MFP1; then there exist $\overline{x} \in X_0$ such that

$$f_i(x) + \left(x^T B_i x \right)^{1/2} - \nu_i \left\{ g_i(x) - \left(x^T C_i x \right)^{1/2} \right\} \leq f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x} \right)^{1/2} - \nu_i \left\{ g_i(\overline{x}) - \left(x^T C_i x \right)^{1/2} \right\}, \quad i = 1, 2, \ldots, k, \tag{19}$$

and $f_t(x) + (x^T B_t x)^{1/2} - \nu_t \{g_t(x) - (x^T C_t x)^{1/2}\} \leq f_t(\overline{x}) + (\overline{x}^T B_t \overline{x})^{1/2} - \nu_t \{g_t(\overline{x}) - (x^T C_t x)^{1/2}\}$, for some $t \in \{1, 2, \ldots, k\}$.

The above relation, together with the relation $\lambda_i > 0$, implies that

$$\sum_{i=1}^{k} \lambda_i \left[f_i(x) + \left(x^T B_i x \right)^{1/2} - \nu_i \left\{ g_i(x) - \left(x^T C_i x \right)^{1/2} \right\} \right] < \sum_{i=1}^{k} \lambda_i \left[f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x} \right)^{1/2} - \nu_i \left\{ g_i(\overline{x}) - \left(x^T C_i x \right)^{1/2} \right\} \right]. \tag{20}$$

From the relations (5), (11), and (14), we get

$$\sum_{j=1}^{m} y_j h_j(x) \leq \sum_{j=1}^{m} y_j h_j(\overline{x}). \tag{21}$$

Consequently (20) and (21) yield

$$\sum_{i=1}^{k} \lambda_i \left[f_i(x) + \left(x^T B_i x \right)^{1/2} - \nu_i \left\{ g_i(x) - \left(x^T C_i x \right)^{1/2} \right\} \right] + \sum_{j=1}^{m} y_j h_j(x) < \sum_{i=1}^{k} \lambda_i \left[f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x} \right)^{1/2} - \nu_i \left\{ g_i(\overline{x}) - \left(x^T C_i x \right)^{1/2} \right\} \right] + \sum_{j=1}^{m} y_j h_j(\overline{x}). \tag{22}$$

This contradicts (18). Hence $\overline{x}$ is an efficient solution for MFP1. □

Theorem 12 (sufficient optimality condition). *Let $\overline{x} \in X_0$ be a feasible solution of MFP1 and there exist $\lambda_i \in \mathbb{R}_+$; $w, z \in \mathbb{R}^n$, $\nu_i \in \mathbb{R}_+$, and $y \in \mathbb{R}^m$ satisfying the condition in Lemma 10 at $\overline{x}$. Furthermore suppose that the following conditions hold.*

(i) *$Q(x) = \sum_{i=1}^{k} \lambda_i [f_i(x) + (x^T B_i x)^{1/2} - \nu_i \{g_i(x) - (x^T C_i x)^{1/2}\}]$ is second order ρ-pseudounivex with respect to η, ψ_0, and K at $\overline{x} \in X_0$ and $H(x) = \sum_{j=1}^{m} y_j h_j(x)$ is second order σ-quasiunivex with respect to η, ψ_1, and K at $\overline{x} \in X_0$ with $(\nabla^2 Q(\overline{x})) p = 0$ and $(\nabla^2 H(\overline{x})) p = 0$ where $f_i : X \to \mathbb{R}$, $g_i : X \to \mathbb{R}$, $h_j : X \to \mathbb{R}, i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, m$, $p \in \mathbb{R}^n$, $\eta : X \times X \to \mathbb{R}^n$, $K : X \times X \to \mathbb{R}_+$, and $\psi_0, \psi_1 : \mathbb{R} \to \mathbb{R}$ satisfying $\psi_0(a) \geq 0 \Rightarrow a \geq 0$ and $\psi_1(a) \leq 0 \Rightarrow a \leq 0$.*

(ii) *$\rho + \sigma \geq 0$.*

Then $\overline{x}$ is an efficient solution of MFP1.

Proof. Suppose hypothesis holds.

From the relations (5), (11), and (14), we get

$$\sum_{j=1}^{m} y_j h_j(x) \leq \sum_{j=1}^{m} y_j h_j(\overline{x}) \Longrightarrow H(x) \leq H(\overline{x}) \Longrightarrow H(x) - H(\overline{x}) \leq 0. \tag{23}$$

Also from hypothesis (i), we get $(\nabla^2 H(\overline{x}))p = 0$.

So we have the following:

$$H(x) - H(\overline{x}) + \frac{1}{2}p^T\left(\nabla^2 H(\overline{x})\right)p \leq 0 \Longrightarrow K(x,\overline{x})\,\psi_1\left\{H(x) - H(\overline{x}) + \frac{1}{2}p^T\left(\nabla^2 H(\overline{x})\right)p\right\} \leq 0. \tag{24}$$

Hence, the σ-quasiunivexity of $H(x)$ with respect to η, ψ_1, and K implies the following:

$$\eta(x,\overline{x})^T\left\{\nabla H(\overline{x}) + \left(\nabla^2 H(\overline{x})\right)p\right\} + \sigma\|x-\overline{x}\|^2 \leq 0 \Longrightarrow \eta(x,\overline{x})^T\{\nabla H(\overline{x})\} + \sigma\|x-\overline{x}\|^2 \leq 0. \tag{25}$$

From (9), we get

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\left[\nabla f_i(\overline{x}) + B_i w - \nu_i\{\nabla g_i(\overline{x}) - C_i z\}\right] + \sum_{i=1}^{m} y_i \nabla h_i(\overline{x}) = 0\\ &\Longrightarrow \nabla Q(\overline{x}) + \nabla H(\overline{x}) = 0\\ &\Longrightarrow \eta(x,\overline{x})^T\left[\nabla Q(\overline{x}) + \nabla H(\overline{x})\right] = 0\\ &\Longrightarrow \eta(x,\overline{x})^T\nabla Q(\overline{x}) + \eta(x,\overline{x})^T\nabla H(\overline{x}) + \sigma\|x-\overline{x}\|^2 - \sigma\|x-\overline{x}\|^2 = 0.\end{aligned} \tag{26}$$

Using (25) in (26), we get

$$\eta(x,\overline{x})^T\nabla Q(\overline{x}) - \sigma\|x-\overline{x}\|^2 \geq 0. \tag{27}$$

Since $\rho + \sigma \geq 0$, we get

$$\rho\|x-\overline{x}\|^2 \geq -\sigma\|x-\overline{x}\|^2. \tag{28}$$

So, we have

$$\eta(x,\overline{x})^T\nabla Q(\overline{x}) + \rho\|x-\overline{x}\|^2 \geq 0 \Longrightarrow \eta(x,\overline{x})^T\left\{\nabla Q(\overline{x}) + \nabla^2 Q(\overline{x})\,p\right\} + \rho\|x-\overline{x}\|^2 \geq 0. \tag{29}$$

Since $Q(x)$ is ρ-pseudounivex with respect to η, ψ_0, and K, we obtained

$$K(x,\overline{x})\,\psi_0\left[Q(x) - Q(\overline{x}) + \frac{1}{2}p^T\nabla^2 Q(\overline{x})\,p\right] \geq 0. \tag{30}$$

Using the property of K and ψ_0, we get

$$\begin{aligned}&Q(x) - Q(\overline{x}) + \frac{1}{2}p^T\nabla^2 Q(\overline{x})\,p \geq 0 \Longrightarrow Q(x) \geq Q(\overline{x})\\ &\Longrightarrow \sum_{i=1}^{k}\lambda_i\left[f_i(x) + \left(x^T B_i x\right)^{1/2} - \nu_i\left\{g_i(x) - \left(x^T C_i x\right)^{1/2}\right\}\right]\\ &\geq \sum_{i=1}^{k}\lambda_i\left[f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x}\right)^{1/2} - \nu_i\left\{g_i(\overline{x}) - \left(\overline{x}^T C_i \overline{x}\right)^{1/2}\right\}\right].\end{aligned} \tag{31}$$

If $\overline{x}$ were not an efficient solution to MFP1, then, from (20), we have

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\left[f_i(x) + \left(x^T B_i x\right)^{1/2} - \nu_i\left\{g_i(x) - \left(x^T C_i x\right)^{1/2}\right\}\right]\\ &< \sum_{i=1}^{k}\lambda_i\left[f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x}\right)^{1/2} - \nu_i\left\{g_i(\overline{x}) - \left(\overline{x}^T C_i \overline{x}\right)^{1/2}\right\}\right].\end{aligned} \tag{32}$$

This contradicts (31).

Therefore, $\overline{x}$ is an efficient solution for MFP1. □

3. Second Order Mixed Type Multiobjective Fractional Duality

(i) *MMFD0*. Maximize

$$\begin{aligned}&L(u) = \left(L_1(u) - \frac{1}{2}p^T\nabla^2 L_1(u)\,p,\ L_2(u) - \frac{1}{2}p^T\nabla^2 L_2(u)\,p, \ldots, L_k(u) - \frac{1}{2}p^T\nabla^2 L_k(u)\,p\right),\\ &\text{where}\\ &L_i(u) = \frac{f_i(u) + y_{J_1}^T h_{J_1}(u) + u^T B_i w}{g_i(u) - u^T C_i z}, \quad i = 1,2,\ldots,k.\end{aligned} \tag{33, 34}$$

(ii) *MMFD1*. Maximize

$$\begin{aligned}&G(u) = \left(G_1(u) - \frac{1}{2}p^T\nabla^2 G_1(u)\,p,\ G_2(u) - \frac{1}{2}p^T\nabla^2 G_2(u)\,p, \ldots, G_k(u) - \frac{1}{2}p^T\nabla^2 G_k(u)\,p\right),\\ &\text{where } G_i(u) = f_i(u) + y_{J_1}^T h_{J_1}(u) + u^T B_i w - \nu_i\{g_i(u) - u^T C_i z\},\ i = 1,2,\ldots,k;\ \nu_i \text{ are fixed parameters.}\end{aligned} \tag{35}$$

(iii) *MMFDλ*. Maximize $\lambda G(u)$; λ is k-dimensional strictly positive vector,

all subject to same constraints

$$\sum_{i=1}^{k}\lambda_i\left[\nabla G_i(u)+\nabla^2 G_i(u)\,p\right]+y_{J_2}^{T}\left[\nabla h_{J_2}(u)+\nabla^2\nabla h_{J_2}(u)\,p\right]=0, \tag{36}$$

$$f_i(u)+y_{J_1}^{T}h_{J_1}(u)+u^{T}B_i w-v_i\left\{g_i(u)-u^{T}C_i z\right\}\geq 0, \quad \text{for } i=1,2,\ldots,k. \tag{37}$$

$$y_{J_2}^{T}h_{J_2}(u)-\frac{1}{2}p^{T}\nabla^2\left(y_{J_2}^{T}h_{J_2}(u)\right)p\geq 0, \quad y_{J_2}\in\mathbb{R}^{m-|J_1|}, \tag{38}$$

$$w^{T}B_i w\leq 1,\quad z^{T}C_i z\leq 1,\quad i=1,2,\ldots,k, \tag{39}$$

$$y\geq 0,\quad v_i\geq 0,\quad i=1,2,\ldots,k, \tag{40}$$

where $f_i: X\to\mathbb{R}, g_i: X\to\mathbb{R}, h_j: X\to\mathbb{R}, i=1,2,\ldots,k$; $j=1,2,\ldots,m$ are differentiable functions, $w,z\in\mathbb{R}^n$, $p\in\mathbb{R}^n$. B_i, and C_i, $i=1,2,\ldots,k$ are positive semidefinite matrices of order n.

For the following theorems, we assume that $\eta: X\times X\to\mathbb{R}^n$, $K: X\times X\to\mathbb{R}_+$, and $\psi_0,\psi_1:\mathbb{R}\to\mathbb{R}$ satisfying $\psi_0(a)\geq 0\Rightarrow a\geq 0$ and $b\leq 0\Rightarrow\psi_1(b)\leq 0$ and $\rho,\sigma\in\mathbb{R}$.

Theorem 13 (weak duality). *Let x be a feasible solution for the primal MFPλ and let (u,y,v,w) be feasible for dual SMMFDλ. If*

(i) $\sum_{i=1}^{k}\lambda_i G_i(\cdot)$ *is second order ρ-pseudounivex with respect to η, ψ_0, K, and for $y_{J_2}\in\mathbb{R}^{m-|J_1|}$, $y_{J_2}^{T}h_{J_2}(\cdot)$ is second order σ-quasiunivex with respect to η, ψ_1, and K along with*

(ii) $\rho+\sigma\geq 0$, *then* $\mathrm{Inf}(\lambda F(x))\geq\mathrm{Sup}(\lambda G(u))$.

Proof. Now from the primal and dual constraints, we have

$$h(x)\leq 0,\qquad y_{J_2}^{T}h_{J_2}(u)-\frac{1}{2}p^{T}\nabla^2\left(y_{J_2}^{T}h_{J_2}(u)\right)p\geq 0. \tag{41}$$

So

$$y_{J_2}^{T}h_{J_2}(x)-y_{J_2}^{T}h_{J_2}(u)+\frac{1}{2}p^{T}\nabla^2\left(y_{J_2}^{T}h_{J_2}(u)\right)p\leq 0$$
$$\Longrightarrow K(x,u)\,\psi_1\left[y_{J_2}^{T}h_{J_2}(x)-y_{J_2}^{T}h_{J_2}(u)+\frac{1}{2}p^{T}\nabla^2\left(y_{J_2}^{T}h_{J_2}(u)\right)p\right]\leq 0. \tag{42}$$

Since $y_{J_2}^{T}h_{J_2}$ is second order σ-quasiunivex with respect to η,ψ_1, and K and in view of (42), for $x,u\in\mathbb{R}^n$, we have

$$\eta(x,u)^{T}\left\{\nabla\left[y_{J_2}^{T}h_{J_2}(u)\right]+\nabla^2\left[y_{J_2}^{T}h_{J_2}(u)\right]p\right\}+\sigma\|x-u\|^2\leq 0. \tag{43}$$

Again from the dual constraint (36), we have

$$\sum_{i=1}^{k}\lambda_i\left[\nabla G_i(u)+\nabla^2 G_i(u)\,p\right]+y_{J_2}^{T\,T}\left[\nabla h_{J_2}(u)+\nabla^2 h_{J_2}(u)\,p\right]=0. \tag{44}$$

Since $\eta(x,u)\in\mathbb{R}^n$, we have

$$\begin{aligned}&\eta(x,u)^{T}\left\{\sum_{i=1}^{k}\lambda_i\left[\nabla G_i(u)+\nabla^2 G_i(u)\,p\right]\right\}\\&\quad+\eta(x,u)^{T}\left\{y_{J_2}^{T}\left[\nabla h_{J_2}(u)+\nabla^2 h_{J_2}(u)\,p\right]\right\}=0,\\&\Longrightarrow\eta(x,u)^{T}\left\{\sum_{i=1}^{k}\lambda_i\left[\nabla G_i(u)+\nabla^2 G_i(u)\,p\right]\right\}\\&\quad+\eta(x,u)^{T}\left\{y_{J_2}^{T}\left[\nabla h_{J_2}(u)+\nabla^2 h_{J_2}(u)\,p\right]\right\}\\&\quad+\sigma\|x-u\|^2-\sigma\|x-u\|^2=0.\end{aligned} \tag{45}$$

Using (43) in above equation, we get $\eta(x,u)^{T}\{\sum_{i=1}^{k}\lambda_i[\nabla G_i(u)+\nabla^2 G_i(u)p]\}-\sigma\|x-u\|^2\geq 0$.

Since $\rho+\sigma\geq 0$, we get $\rho\|x-u\|^2\geq-\sigma\|x-u\|^2$.

So, we have

$$\eta(x,u)^{T}\left\{\sum_{i=1}^{k}\lambda_i\left[\nabla G_i(u)+\nabla^2 G_i(u)\,p\right]\right\}+\rho\|x-u\|^2\geq 0. \tag{46}$$

Since $\sum_{i=1}^{k}\lambda_i G_i(u)$ is second order ρ-pseudounivex with respect to η,ψ_0, and K, by Definition 2 and (46), we get $K(x,u)\psi_0\{\sum_{i=1}^{k}\lambda_i G_i(x)-\sum_{i=1}^{k}\lambda_i G_i(u)+(1/2)p^{T}\sum_{i=1}^{k}\lambda_i G_i(u)p\}\geq 0$.

Using the property of ψ_0 and K, we get

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i G_i(x)-\sum_{i=1}^{k}\lambda_i G_i(u)+\frac{1}{2}p^{T}\sum_{i=1}^{k}\lambda_i G_i(u)\,p\geq 0\\&\Longrightarrow\sum_{i=1}^{k}\lambda_i G_i(x)\geq\sum_{i=1}^{k}\lambda_i G_i(u)\\&\Longrightarrow\sum_{i=1}^{k}\lambda_i\left[f_i(x)+y_{J_1}^{T}h_{J_1}(x)+x^{T}B_i w-v_i\left\{g_i(x)-x^{T}C_i z\right\}\right]\\&\geq\sum_{i=1}^{k}\lambda_i\left[f_i(u)+y_{J_1}^{T}h_{J_1}(u)+u^{T}B_i w-v_i\left\{g_i(u)-x^{T}C_i z\right\}\right].\end{aligned} \tag{47}$$

Equation (5) gives $h(x)\leq 0\Rightarrow y_{J_1}^{T}h_{J_1}(x)\leq 0$, for $y_{J_1}\geq 0$.

So (47) implies that

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x)+x^TB_iw-v_i\left\{g_i(x)-x^TC_iz\right\}\right]$$
$$\geq\sum_{i=1}^{k}\lambda_i\left[f_i(u)+y_{J_1}^Th_{J_1}(u)+u^TB_iw\right.$$
$$\left.-v_i\left\{g_i(u)-u^TC_iz\right\}\right]. \tag{48}$$

Now by Schwarz Inequality and (39), we have

$$x^TB_iw\leq\left(x^TB_ix\right)^{1/2}\left(w^TB_iw\right)^{1/2}\leq\left(x^TB_ix\right)^{1/2},$$
$$x^TC_iz\leq\left(x^TC_ix\right)^{1/2}\left(z^TC_iz\right)^{1/2}\leq\left(x^TC_ix\right)^{1/2}, \tag{49}$$
$$i=1,2,\ldots,k.$$

So both (48) and (49) imply that

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x)+\left(x^TB_ix\right)^{1/2}-v_i\left\{g_i(x)-\left(x^TC_ix\right)^{1/2}\right\}\right]$$
$$\geq\sum_{i=1}^{k}\lambda_i\left[f_i(u)+y_{J_1}^Th_{J_1}(u)+u^TB_iw\right.$$
$$\left.-v_i\left\{g_i(u)-u^TC_iz\right\}\right] \tag{50}$$
$$\Longrightarrow \operatorname{Inf}(\lambda F(x))\geq\operatorname{Sup}(\lambda F(u)).$$

□

Theorem 14 (strong duality). *Let $\overline{x}$ be optimal solution for MFPλ and let $W(\overline{x})=\phi$. Then there exist $\lambda_i\in\mathbb{R}_+$; $w,z\in\mathbb{R}^n$, $v_i\in\mathbb{R}_+$, and $y\in\mathbb{R}^m$ such that $(\overline{u},\lambda,y,v,w,z,p=0)$ is a feasible solution for dual and the objective values of both primal and dual are equal to zero. Furthermore if (i) $(\overline{u},\lambda,y,v,w,z)$ is feasible for dual, (ii) $\sum_{i=1}^{k}\lambda_iG_i(\cdot)$ is second order ρ-pseudounivex with respect to η, ψ_0, and K and for $y_{J_2}\in\mathbb{R}^{m-|J_1|}$, $y_{J_1}^Th_{J_2}(\cdot)$ is second order σ-quasiunivex with respect to η, ψ_1, and K along with (iii) $\rho+\sigma\geq 0$, then $(\overline{u},\lambda,y,v,w,z,p=0)$ is properly efficient for SMMFD0.*

Proof. Since $\overline{x}$ is optimal solution for (MFPλ), by Lemma 10, there exist $\lambda_i\in\mathbb{R}_+$; $w,z\in\mathbb{R}^n$, $v_i\in\mathbb{R}_+$, and $y\in\mathbb{R}^m$ such that

$$\sum_{i=1}^{k}\lambda_i\left[\nabla f_i(\overline{x})+B_iw-v_i\left\{\nabla g_i(\overline{x})-C_iz\right\}\right]+y^T\nabla h(\overline{x})=0,$$
$$f_i(\overline{x})+\left(\overline{x}^TB_i\overline{x}\right)^{1/2}-v_i\left(g_i(\overline{x})-\left(\overline{x}^TC_i\overline{x}\right)^{1/2}\right)=0,$$
$$i=1,2,\ldots,k,$$
$$y^Th(\overline{x})=0,$$
$$w^TB_iw\leq 1,\quad z^TC_iz\leq 1,\quad i=1,2,\ldots,k,$$
$$\left(\overline{x}^TB_i\overline{x}\right)^{1/2}=\overline{x}^TB_iw,\qquad\left(\overline{x}^TC_i\overline{x}\right)^{1/2}=\overline{x}^TC_iz,$$
$$i=1,2,\ldots,k,$$
$$y\geq 0,$$
$$v_i\geq 0,\quad i=1,2,\ldots,k, \tag{51}$$

which can be written as

$$\sum_{i=1}^{k}\lambda_i\left[\nabla f_i(\overline{x})+y_{J_1}^T\nabla h_{J_1}(\overline{x})+B_iw-v_i\left\{\nabla g_i(\overline{x})-C_iz\right\}\right]$$
$$+y_{J_1}^T\nabla h_{J_2}(\overline{x})=0,$$
$$f_i(\overline{x})+\left(\overline{x}^TB_i\overline{x}\right)^{1/2}-v_i\left\{g_i(\overline{x})-\left(\overline{x}^TC_i\overline{x}\right)^{1/2}\right\}$$
$$+y_{J_1}^Th_{J_1}(\overline{x})=0,$$
$$y_{J_1}^Th_{J_2}(\overline{x})=0,$$
$$w^TB_iw\leq 1,\quad z^TC_iz\leq 1,\quad i=1,2,\ldots,k,$$
$$\left(\overline{x}^TB_i\overline{x}\right)^{1/2}=\overline{x}^TB_iw,\qquad\left(\overline{x}^TC_i\overline{x}\right)^{1/2}=\overline{x}^TC_iz,$$
$$i=1,2,\ldots,k,$$
$$y\geq 0,$$
$$v_i\geq 0,\quad i=1,2,\ldots,k. \tag{52}$$

These are nothing but the dual constraints.

So $(\overline{u},\lambda,y,v,w,z,p=0)$ is feasible solution for dual problem.

And the objective values of MFPλ and SMMFDλ are equal to zero. It follows from Theorem 13 and for any feasible solution $(\overline{u},\lambda,y,v,w,z,p=0)$ of dual that $\lambda G(\overline{u})\leq\lambda G(\overline{x})$.

So $(\overline{u},\lambda,y,v,w,z,p=0)$ is optimal solution of SMMFDλ. Then applying Lemmas 8 and 9, we conclude that $(\overline{u},\lambda,y,v,w,z,p=0)$ is properly efficient for (SMMFD0). □

4. Special Case

If $p=0$, $C_i=0$, $i=1,2,\ldots,k$, then our dual programming reduces to the dual programming proposed by Tripathy [14].

5. Conclusion

In this paper, three approaches given by Dinklebaeh [28] and Jagannathan [12] for both primal and second order mixed type dual of a nondifferentiable multiobjective fractional programming problem are introduced and the necessary and sufficient optimality conditions are established and a parameterization technique is used to establish duality results

under generalized second order ρ-univexity assumption. The results developed in this paper can be further extended to higher order mixed type fractional problem containing square root term. Also the present work can be further extended to a class of nondifferentiable minimax mixed fractional programming problems.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] C. R. Bector, "Duality in nonlinear fractional programming," *Zeitschrift für Operations Research*, vol. 17, no. 5, pp. 183–193, 1973.

[2] C. R. Bector and S. Chandra, "First and Second order duality for a class of non differentiable fractional programming," *Journal of Information & Optimization Sciences*, vol. 7, pp. 335–348, 1986.

[3] B. D. Craven, "Fractional programming," in *Sig Ma Series in Applied Mathematics*, vol. 4, pp. 145–152, Heldermann, Berlin, Germany, 1998.

[4] B. Mond and T. Weir, "Duality for fractional programming with generalized convexity condition," *Journal of Information & Optimization Sciences*, vol. 3, pp. 105–124, 1982.

[5] I. M. Stancu-Minasian, *Fractional Programming; Theory, Method and Application*, Kluwer Academic Publishers, Dordrecht, The Netherland, 1997.

[6] S. Schaible and T. Ibaraki, "Fractional programming," *European Journal of Operational Research*, vol. 12, no. 4, pp. 325–338, 1983.

[7] I. Ahmad, S. K. Gupta, N. Kailey, and R. P. Agarwal, "Duality in nondifferentiable minimax fractional programming with B-(p,r)-invexity," *Journal of Inequalities and Applications*, vol. 2011, article 75, 15 pages, 2011.

[8] I. Ahmad and S. Sharma, "Second-order duality for nondifferentiable multiobjective programming problems," *Numerical Functional Analysis and Optimization*, vol. 28, no. 9-10, pp. 975–988, 2007.

[9] T. R. Gulati, I. Ahmad, and D. Agarwal, "Sufficiency and duality in multiobjective programming under generalized type i functions," *Journal of Optimization Theory and Applications*, vol. 135, no. 3, pp. 411–427, 2007.

[10] I. Ahmad, Z. Husain, and S. Al-Homidan, "Second-order duality in nondifferentiable fractional programming," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 2, pp. 1103–1110, 2011.

[11] C. R. Bector, S. Chandra, and I. Husain, "Optimality conditions and duality in subdifferentiable multiobjective fractional programming," *Journal of Optimization Theory and Applications*, vol. 79, no. 1, pp. 105–125, 1993.

[12] R. Jagannathan, "Duality for nonlinear fractional programs," *Zeitschrift für Operations Research*, vol. 17, no. 1, pp. 1–3, 1973.

[13] R. Osuna-Gómez, A. Ruíz-Canales, and P. Rufián-Lizana, "Multiobjective fractional programming with generalized convexity," *Trabajos de Investigación Operativa*, vol. 8, no. 1, pp. 97–110, 2000.

[14] A. K. Tripathy, "Mixed type duality in multiobjective fractional programming under generalized ρ-univex function," *Journal of Mathematical Modelling and Algorithms in Operations Research*, 2013.

[15] S. K. Mishra, S. Wang, and K. K. Lai, "Role of α-pseudo-univex functions in vector variational-like inequality problems," *Journal of Systems Science and Complexity*, vol. 20, no. 3, pp. 344–349, 2007.

[16] S. Schaible, "Fractional programming," in *Handbook of Global Optimization*, R. Horst and P. M. Pardalos, Eds., pp. 495–608, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1995.

[17] Z. A. Liang, H. X. Huang, and P. M. Pardalos, "Optimality conditions and duality for a class of nonlinear fractional programming problems," *Journal of Optimization Theory and Applications*, vol. 110, no. 3, pp. 611–619, 2001.

[18] Z.-A. Liang, H.-X. Huang, and P. M. Pardalos, "Efficiency conditions and duality for a class of multiobjective fractional programming problems," *Journal of Global Optimization*, vol. 27, no. 4, pp. 447–471, 2003.

[19] L. B. Santos, R. Osuna-Gomez, and A. Rojas-Medar, "Nonsmooth multiobjective fractional programming with generalized convexity," *Revista Integración*, vol. 26, no. 1, pp. 1–12, 2008.

[20] C. R. Bector, S. Chandra, and A. Abha, "On mixed duality in mathematical programming," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 346–356, 2001.

[21] Z. Xu, "Mixed type duality in multiobjective programming problems," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 3, pp. 621–635, 1996.

[22] H. Zhou and Y. Wang, "Optimality and mixed duality for nonsmooth multi-objective fractional programming," *Pure Mathematics and Applications*, vol. 14, no. 3, pp. 263–274, 2003.

[23] I. Ahmad and Z. Husain, "Duality in nondifferentiable minimax fractional programming with generalized convexity," *Applied Mathematics and Computation*, vol. 176, no. 2, pp. 545–551, 2006.

[24] I. Ahmad and Z. Husain, "Optimality conditions and duality in nondifferentiable minimax fractional programming with generalized convexity," *Journal of Optimization Theory and Applications*, vol. 129, no. 2, pp. 255–275, 2006.

[25] B. Mond, "A class of nondifferentiable fractional programming problem," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 58, no. 8, pp. 337–341, 1978.

[26] D. S. Kim, S. J. Kim, and M. H. Kim, "Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems," *Journal of Optimization Theory and Applications*, vol. 129, no. 1, pp. 131–146, 2006.

[27] D. S. Kim, Y. J. Lee, and K. D. Bae, "Duality in nondifferentiable multiobjective fractional programs involving cones," *Taiwanese Journal of Mathematics*, vol. 13, no. 6 A, pp. 1811–1821, 2009.

[28] W. Dinklebaeh, "On nonlinear fractional programming," *Management Science*, vol. 13, no. 7, pp. 492–498, 1967.

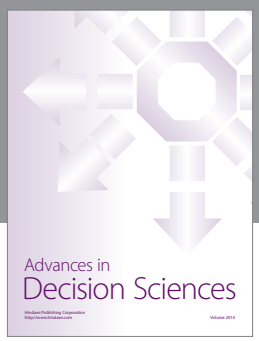
[29] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

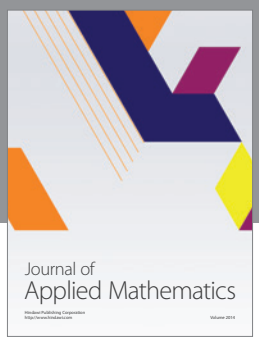
[30] C. R. Bector, S. K. Suneja, and S. Gupta, "Univex functions and univex nonlinear programming," in *Proceeding of the Administrative Science Association of Canada*, pp. 115–124, 1992.

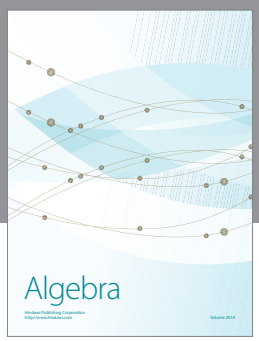
[31] S. K. Mishra, "On multiple-objective optimization with generalized univexity," *Journal of Mathematical Analysis and Applications*, vol. 224, no. 1, pp. 131–148, 1998.

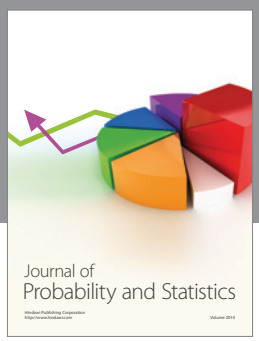
[32] A. Jayswal, "Non-differentiable minimax fractional programming with generalized α-univexity," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 121–135, 2008.

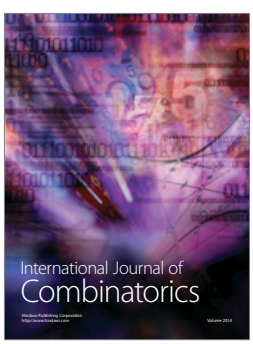
[33] A. M. Geoffrion, "Proper efficiency and the theory of vector maximization," *Journal of Mathematical Analysis and Applications*, vol. 22, no. 3, pp. 618–630, 1968.

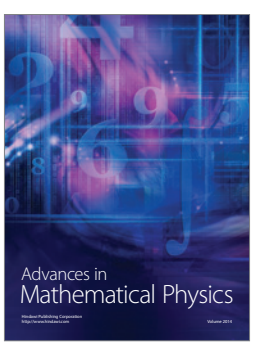


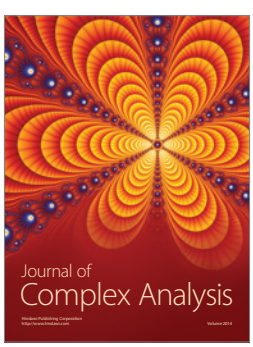


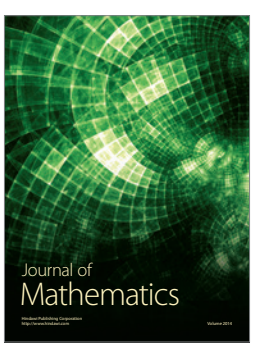


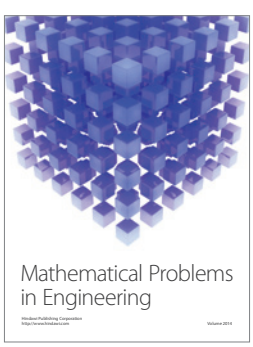


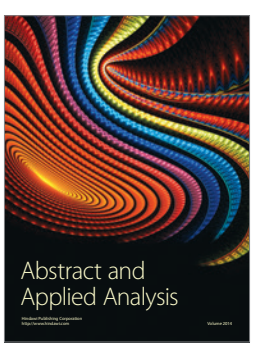


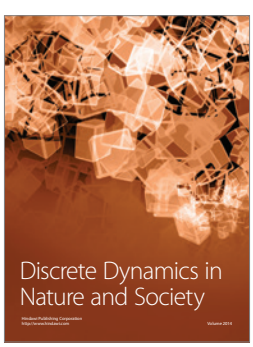


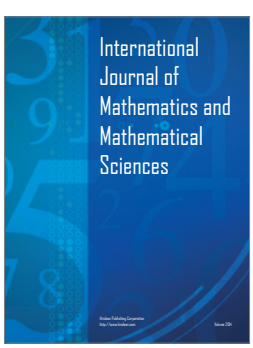


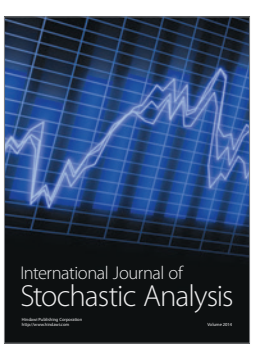












Hindawi

Submit your manuscripts at
http://www.hindawi.com