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Research Article

Two-Sided Annihilator Condition with Generalized Derivations on Multilinear Polynomials

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Let R be a prime ring of characteristic different from 2, with extended centroid C, U its two-sided Utumi quotient ring, F a nonzero generalized derivation of R, $f(x_1,\dots,x_n)$ a noncentral multilinear polynomial over C in n noncommuting variables, and $a,b \in R$ such that $a[F(f(r_1,\dots,r_n)), f(r_1,\dots,r_n)]b = 0$ for any $r_1,\dots,r_n \in R$. Then one of the following holds: (1) $a = 0$; (2) $b = 0$; (3) there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$; (4) there exist $q \in U$ and $\lambda \in C$ such that $F(x) = (q+\lambda)x + xq$, for all $x \in R$, and $f(x_1,\dots,x_n)^2$ is central valued on R; (5) there exist $q \in U$ and $\lambda, \mu \in C$ such that $F(x) = (q+\lambda)x + xq$, for all $x \in R$, and $aq = \mu a$, $qb = \mu b$.

1. Introduction

Let R be a prime ring with center $Z(R)$. We denote by $[a,b] = ab - ba$ the simple commutator of the elements $a,b \in R$ and by $[a,b]_k = [[a,b]_{k-1}, b]$, for $k > 1$, the kth commutator of a, b. Throught this paper we will use the following notation: U will be the (two-sided) Utumi quotient ring of a ring R (sometimes, as in [1], U is called the symmetric ring of quotients). The definition, the axiomatic formulation, and the properties of this quotient ring U can be found in [1–3].

In any case, when R is a prime ring, all that we need here about this object is that

(1) $R \subseteq U$;

(2) U is a prime ring;

(3) the center of U, denoted by C is a field which is called the extended centroid of R.

A well known result of Posner [4] says that if d is a derivation of R such that $[d(x),x] \in Z(R)$, for all $x \in R$, then R is commutative. In [5] Lanski generalizes the result of Posner, by replacing the element $x \in R$ with an element of a noncentral Lie ideal L of R. More precisely he proves that if $[d(x),x]_k = 0$ for all $x \in L$ and $k \geq 1$ a fixed integer; then $\mathrm{char}(R) = 2$ and R satisfies s_4, the standard identity of degree 4.

Let $f(x_1,\dots,x_n)$ be a multilinear polynomial over C in n noncommuting variables and denote by $f(X)$ the set of all evaluations of $f(x_1,\dots,x_n)$ in $X \subseteq R$. In case $f(x_1,\dots,x_n)$ is not central valued on R, it is well known that the additive subgroup generated by $f(R)$ contains a noncentral Lie ideal of R. Moreover any noncentral Lie ideal of R contains all the commutators $[x,y]$ for x, y in some nonzero ideal of R, unless $\mathrm{char}(R) = 2$ and $\dim_C RC = 4$.

In light of this and following the line of investigation of the previous cited papers, in [6] P. H. Lee and T. K. Lee consider the Engel-condition $[d(x),x]_k = 0$, in case $x \in f(I)$, where I is a two-sided ideal of R. They show that either $f(x_1,\dots,x_n)$ is central valued in R or $\mathrm{char}(R) = 2$ and R satisfies s_4.

These results indicate that the global structure of a prime ring R is often tightly connected to the behaviour of additive mappings defined on R, which act on suitable subsets of the whole ring. In [7] de Filippis and di Vincenzo study the left annihilator of the set $\{d(u)u - ud(u),\ u \in f(R)\}$, where d is a derivation. In case the annihilator is not zero, the conclusion is that $f(x_1,\dots,x_n)$ is central valued on R. These facts in a prime ring are natural tests which evidence that the set $\{d(u)u - ud(u),\ u \in f(R)\}$ is rather large in R.

More recently, Liu [8] and Wang [9] have examined the identity $a[d(u),u]_k = 0$, where d is a derivation of R and $u \in$

$f(I)$, where I is a one-sided ideal of R. In particular, for $I = R$, if $a \neq 0$ and $f(x_1, \ldots, x_n)$ is not central valued on R, then $\mathrm{char}(R) = 2$ and R satisfies s_4.

In [10] de Filippis considers a similar situation, in the case the derivation d is replaced by a generalized derivation F. An additive map $F : R \to R$ is said to be a generalized derivation if there is a derivation d of R such that, for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. A significative example is a map of the form $F(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see, e.g., [11]).

The main result in [10] is the following.

Theorem A. *Let R be a prime ring of characteristic different from 2, with extended centroid C, U its two-sided Utumi quotient ring, $F \neq 0$ a nonzero generalized derivation of R, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C in n noncommuting variables, and $a \in R$ such that*

$$a\left[F\left(f\left(r_1, \ldots, r_n\right)\right), f\left(r_1, \ldots, r_n\right)\right] = 0 \tag{1}$$

for any $r_1, \ldots, r_n \in R$. Then either $a = 0$ or one of the following holds:

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$;*

(2) *there exist $q \in U$ and $\lambda \in C$ such that $F(x) = (q+\lambda)x + xq$, for all $x \in R$, and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

We would like to remark that the same conclusions hold in case we consider the right annihilator, more precisely.

Theorem B. *Let R be a prime ring of characteristic different from 2, with extended centroid C, U its two-sided Utumi quotient ring, $F \neq 0$ a nonzero generalized derivation of R, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C in n noncommuting variables, and $a \in R$ such that*

$$\left[F\left(f\left(r_1, \ldots, r_n\right)\right), f\left(r_1, \ldots, r_n\right)\right] a = 0 \tag{2}$$

for any $r_1, \ldots, r_n \in R$. Then either $a = 0$ or one of the following holds:

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$;*

(2) *there exist $q \in U$ and $\lambda \in C$ such that $F(x) = (q+\lambda)x + xq$, for all $x \in R$, and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

Here we will consider a more general situation, involving a two-sided annihilating condition. More specifically, we study simultaneously left and right annihilators of the set $\{[F(x), x] : x \in f(R)\}$ and prove the following.

Theorem 1. *Let R be a prime ring of characteristic different from 2, with extended centroid C, U its two-sided Utumi quotient ring, F a nonzero generalized derivation of R, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C in n noncommuting variables, and $a, b \in R$ such that*

$$a\left[F\left(f\left(r_1, \ldots, r_n\right)\right), f\left(r_1, \ldots, r_n\right)\right] b = 0 \tag{3}$$

for any $r_1, \ldots, r_n \in R$. Then one of the following holds:

(1) *$a = 0$;*

(2) *$b = 0$;*

(3) *there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$;*

(4) *there exist $q \in U$ and $\lambda \in C$ such that $F(x) = (q+\lambda)x + xq$, for all $x \in R$, and $f(x_1, \ldots, x_n)^2$ is central valued on R;*

(5) *there exist $q \in U$ and $\lambda, \mu \in C$ such that $F(x) = (q + \lambda)x + xq$, for all $x \in R$, and $aq = \mu a$, $qb = \mu b$.*

Remark 2. By the primeness of R and in light of Theorems A and B, we may assume that R is not a domain. Moreover, since the center of a prime ring cannot contain nonzero zero-divisor, then neither $a \in Z(R)$ nor $b \in Z(R)$. Finally in all that follows we always suppose $\mathrm{char}(R) \neq 2$.

In the sequel we will make a frequent use of the following.

Remark 3. If B is a basis of U over C then any element of $T = U *_C C\{x_1, \ldots, x_n\}$, the free product over C of the C-algebra U and the free C-algebra $C\{x_1, \ldots, x_n\}$, is called generalized polynomial and can be written in the form $g = \sum_i \alpha_i m_i$. In this decomposition the coefficients α_i are in C and the elements m_i are B-monomials; that is, $m_i = q_0 y_1 q_1 \cdots y_h q_h$, with $q_i \in B$ and $y_i \in \{x_1, \ldots, x_n\}$. In [12] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if all α_i are zero. Let $a_1, \ldots, a_k \in U$ be linearly independent over C and $a_1 g_1(x_1, \ldots, x_n) + \cdots + a_k g_k(x_1, \ldots, x_n) = 0 \in T$, for some $g_1, \ldots, g_k \in T$. If, for any i, $g_i(x_1, \ldots, x_n) = \sum_{j=1}^n x_j h_j(x_1, \ldots, x_n)$ and $h_j(x_1, \ldots, x_n) \in T$, then $g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)$ are the zero element of T. The same conclusion holds if $g_1(x_1, \ldots, x_n)a_1 + \cdots + g_k(x_1, \ldots, x_n)a_k = 0 \in T$, and $g_i(x_1, \ldots, x_n) = \sum_{j=1}^n h_j(x_1, \ldots, x_n)x_j$ for some $h_j(x_1, \ldots, x_n) \in T$.

We refer the reader to [1, 12] for more details on generalized polynomial identities.

2. An Independent Result

We will dedicate this section to the proof of the following proposition on linear identities with commutators in matrix rings. This result will be useful in the sequel.

Proposition 4. *Let C be a field and $R = M_t(C)$ the algebra of $t \times t$ matrices over C and $S = [R, R]$. Let $a, b, c \in R$, such that $c \notin Z(R)$ and $a[c, x]b = 0$ for all $x \in S$. Then there exists $\lambda \in Z(R)$ such that $ac = \lambda a$ and $cb = \lambda b$.*

In order to prove Proposition 4, we need several lemmas.

Lemma 5. *Let K be an infinite field and $t \geq 2$. If $A_1, \ldots, A_k$ are not scalar matrices in $M_t(K)$, then there exists some invertible matrix $B \in M_t(K)$ such that each matrix $BA_1B^{-1}, \ldots, BA_kB^{-1}$ has all nonzero entries.*

Proof. See Lemma 1.5 in [13]. □

Lemma 6. *Let R be a prime ring with extended centroid C. Suppose $\sum_{i=1}^{n} a_i x b_i + \sum_{j=1}^{m} c_j x d_j = 0$, for all $x \in R$, where $a_i, b_i, c_j, d_j \in R$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. If $a_1, \ldots, a_n$ are C-independent then each b_i is C-dependent on $d_1, \ldots, d_m$. Analogously if $b_1, \ldots, b_n$ are C-independent then each a_i is C-dependent on $c_1, \ldots, c_m$.*

Proof. It is Martindale's result contained in [14]. □

Lemma 7. *Let R be a prime ring with extended centroid C. Suppose $a[x, y] + [x, y]b = 0$, for all $x, y \in R$, where $a, b \in R$. Then $a = -b \in C$.*

Proof. It is an easy consequence of Lemma 6. □

Lemma 8. *Let K be an infinite field, $R = M_m(K)$ the algebra of $m \times m$ matrices over K, $Z(R)$ the center of R, and $S = [R, R]$. Assume that there exist a, b, c, q nonzero elements of R such that $axq + cxb = 0$ for all $x \in S$. If $q \in Z(R)$ then one of the following holds:*

(1) *a, b, c are central matrices and $aq + bc = 0$;*

(2) *b is a central matrix and $aq + bc = 0$.*

Proof. Since $q \in Z(R)$, by the assumption, we have that $aqx + cxb = 0$ for all $x \in S$. Clearly if $c \in Z(R)$ then $aqx + xbc = 0$ for all $x \in S$, and by Lemma 7 we get $aq = -bc \in Z(R)$; that is, $a, b, c \in Z(R)$. On the other hand, if $b \in Z(R)$, then $(aq+bc)x = 0$ for all $x \in S$ and it follows easily that $aq+bc = 0$.

In light of this, we consider c and b both nonscalar matrices. We will prove that in this case we get a contradiction.

Here we denote by e_{ij} the usual matrix unit with 1 in the (i, j)-entry and zero elsewhere.

By Lemma 5, we can assume that c and b have all nonzero entries, say $c = \sum_{kl} c_{kl} e_{kl}$ and $b = \sum_{kl} b_{kl} e_{kl}$, for $0 \neq c_{kl}$, $0 \neq b_{kl} \in K$.

Since $e_{ji} \in S$ for all $i \neq j$, then, for any $i \neq j$,

$$X = aqe_{ji} + ce_{ji}b = 0 \tag{4}$$

in particular the (i, j)-entry of X is $c_{ij}b_{ij} = 0$, a contradiction. □

For sake of clearness, we may write the previous lemma as follows.

Lemma 9. *Let K be an infinite field, $R = M_m(K)$ the algebra of $m \times m$ matrices over K, $Z(R)$ the center of R, and $S = [R, R]$. Let a_1, a_2, a_3, a_4 be nonzero elements of R such that $a_1 x a_2 + a_3 x a_4 = 0$ for all $x \in S$. Assume there exists $i \in \{1, 2, 3, 4\}$ such that $a_i \in Z(R)$. Then $a_1 = \alpha a_3$ and $a_2 = -\alpha a_4$, for a suitable $\alpha \in Z(R)$.*

Lemma 10. *Let K be an infinite field, $R = M_m(K)$ the algebra of $m \times m$ matrices over K, and $Z(R)$ the center of R. Assume that there exist a, b, c, q nonzero elements of R such that $axq + cxb = 0$ for all $x \in S = [R, R]$. If $q \notin Z(R)$ and $b - \alpha q \in Z(R)$, for a suitable $\alpha \in K$, then $b - \alpha q = a + \alpha c = 0$.*

Proof. Assume that $a + \alpha c$ is not a scalar matrix. By Lemma 5, we can assume that $a + \alpha c$ and q have all nonzero entries, say $a + \alpha c = \sum_{kl} t_{kl} e_{kl}$ and $q = \sum_{kl} q_{kl} e_{kl}$, for $0 \neq t_{kl}$, $0 \neq q_{kl} \in K$.

Since $b = \beta I + \alpha q$, for a suitable $\beta \in K$, by our assumption we have that

$$axq + cx(\beta + \alpha q) = 0; \tag{5}$$

that is,

$$\beta cx + (a + \alpha c)xq = 0, \tag{6}$$

for all $x \in S$. In particular for $x = [e_{ii}, e_{ij}] = e_{ij}$, with $i \neq j$,

$$0 = X = \beta ce_{ij} + (a + \alpha c)e_{ij}q. \tag{7}$$

By calculations one has that the (j, i)-entry of X is $0 = t_{ji}q_{ji}$, a contradiction.

Therefore $a + \alpha c$ must be a central matrix. In light of this, there exist $\beta, \gamma \in K$ such that $b = \alpha q + \beta$ and $a = -\alpha c + \gamma$, so that $0 = (-\alpha c + \gamma)xq + cx(\alpha q + \beta) = (\beta c)x + x(\gamma q)$, for all $x \in S = [R, R]$. Once again by Lemma 7 and since $q \notin Z(R)$, it follows that $\beta = \gamma = 0$; that is, $b = \alpha q$ and $a = -\alpha c$. □

Lemma 11. *Let K be an infinite field, $R = M_m(K)$ the algebra of $m \times m$ matrices over K, and $S = [R, R]$. Suppose there exist a, b, c, $q \in R$ such that $axq + cxb = 0$ for all $x \in S$. Denote*

$$a = \sum_{kl} a_{kl} e_{kl}, \qquad b = \sum_{kl} b_{kl} e_{kl},$$
$$c = \sum_{kl} c_{kl} e_{kl}, \qquad q = \sum_{kl} q_{kl} e_{kl} \tag{8}$$

for suitable a_{kl}, b_{kl}, c_{kl}, and q_{kl} elements of K. If there are $i \neq j$ such that $q_{ji} \neq 0$, $c_{ji} \neq 0$, and $b_{ji} = 0$, then $a_{ri} = 0$ and $b_{rk} = 0$ for all $r \neq i$ and $k \neq r$ (i.e., the only nonzero off-diagonal elements of b fall in the ith row).

Proof. Consider the assumption

$$axq + cxb = 0 \quad \forall x \in [R, R]. \tag{9}$$

In particular, for $x = e_{ij}$, we have

$$X = ae_{ij}q + ce_{ij}b = 0 \tag{10}$$

so that, for all $r \neq i$, the (r, i)-entry of the matrix X is $0 = a_{ri}q_{ji} + c_{ri}b_{ji} = a_{ri}q_{ji}$. Since $q_{ji} \neq 0$, one has $a_{ri} = 0$ for all $r \neq i$, in particular $a_{ji} = 0$. Thus, in case $m = 2$ we are done (since $b_{ji} = a_{ji} = 0$).

Assume in what follows that $m \geq 3$, and choose $x = e_{it}$, with $t \neq i, j$. Hence we also have

$$Y = ae_{it}q + ce_{it}b = 0. \tag{11}$$

From the previous equalities it follows that

(1) for all $s \neq i$, the (j, s)-entry of the matrix X is $a_{ji}q_{js} + c_{ji}b_{js} = 0$;

(2) for all $s \neq i, j$, the (j, s)-entry of the matrix Y is $a_{ji}q_{ts} + c_{ji}b_{ts} = 0$;

(3) the (j,i)-entry of the matrix Y is $a_{ji}q_{ti} + c_{ji}b_{ti} = 0$;

(4) for all $k \neq i, t$, the (j,k)-entry of the matrix Y is $a_{ji}q_{tk} + c_{ji}b_{tk} = 0$ (note that this holds also in case $k = j$).

By (1) and (2) and since $a_{ji} = 0$ and $c_{ji} \neq 0$, we have both $b_{js} = 0$, for all $s \neq i$, and $b_{ts} = 0$ for all $t \neq i, j$ and $s \neq i, j$. So by (3) $b_{ti} = 0$ for all $t \neq i$. Finally by (4), $b_{tk} = 0$ for all $t \neq i, j$ and $k \neq t$. □

Lemma 12. *Let K be an infinite field, $R = M_m(K)$ the algebra of $m \times m$ matrices over K, and $S = [R,R]$. Suppose there exist $a, b, c, q \in R$ such that $axq + cxb = 0$ for all $x \in S$. Denote*

$$b = \sum_{kl} b_{kl}e_{kl}, \qquad c = \sum_{kl} c_{kl}e_{kl}, \qquad q = \sum_{kl} q_{kl}e_{kl} \tag{12}$$

for suitable b_{kl}, c_{kl}, and q_{kl} elements of K. Assume there are $i \neq j$ such that $b_{ji} = 0$. If $q_{rs} \neq 0$, $c_{rs} \neq 0$ for all $r \neq s$, then one of the following holds:

(1) *$a = b = 0$;*

(2) *$m = 2$, $cq = 0$, and there exists $0 \neq \lambda \in K$ such that*

$$a = \begin{bmatrix} 0 & \lambda c_{12} \\ 0 & \lambda c_{22} \end{bmatrix}, \qquad b = \begin{bmatrix} -\lambda q_{11} & -\lambda q_{12} \\ 0 & 0 \end{bmatrix}. \tag{13}$$

Proof. Firstly we consider the case $m \geq 3$. The first step is to apply twice Lemma 11: this forces b to be a diagonal matrix. In fact $b_{ji} = 0$, $q_{ji} \neq 0$, and $c_{ji} \neq 0$ imply that $b_{rk} = 0$ for all $r \neq i$ and $k \neq r$; in particular, since $m \geq 3$, there exists $t \neq i$ such that $b_{lt} = 0$, for all $l \neq t, i$. Since $q_{lt} \neq 0$, $c_{lt} \neq 0$, we have $b_{rk} = 0$ for all $r \neq t$ and $k \neq r$, so $b_{ik} = 0$ for all $k \neq i$, as required. Say $b = \sum_k b_{kk}e_{kk}$.

Consider now the inner automorphism of R induced by the invertible matrix $P = I + e_{rj}$, for $r \neq i, j$: $\varphi(x) = P^{-1}xP$. Of course $\varphi(a)x\varphi(q) + \varphi(c)x\varphi(b) = 0$, for all $x \in S$. Moreover the (j,i)-entries of $\varphi(q)$, $\varphi(c)$, and $\varphi(b)$ are, respectively, $q_{ji} \neq 0$, $c_{ji} \neq 0$, and $b_{ji} = 0$. Therefore, again by Lemma 11, any (r,j)-entry of $\varphi(b)$ is zero, for all $r \neq i$. By calculations $0 = (\varphi(b))_{rj} = b_{jj} - b_{rr}$; that is, $b_{jj} = b_{rr}$.

On the other hand, if χ is the inner automorphisms induced by the invertible matrix $Q = I + e_{ri}$, as above $\chi(a)x\chi(q) + \chi(c)x\chi(b) = 0$, for all $x \in R$. Since the (i,j)-entries of $\chi(q)$, $\chi(c)$, and $\chi(b)$ are, respectively, $q_{ij} \neq 0$, $c_{ij} \neq 0$, and $b_{ij} = 0$, and again any (r,i)-entry of $\chi(b)$ is zero, for all $r \neq j$; that is, $0 = (\varphi(b))_{ri} = b_{ii} - b_{rr}$ and $b_{ii} = b_{rr} = b_{jj} = \beta$, for all $r \neq i, j$. Thus $b = \beta I$ is a central matrix in R. By Lemma 9, either $b = \alpha q$ for some $\alpha \in K$ or $b = 0$. Since the first case cannot occur, we get $b = 0$ and also $a = 0$ which follows from $a[R,R]q = 0$ and $q \neq 0$.

Let now $m = 2$; that is, $R = M_2(K)$. In this case it is well known that for any element $x \in [R,R]$ there exist $\alpha, \beta, \gamma \in K$ such that $x = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right]$. Without loss of generality we may assume $b_{21} = 0$. In case $b_{12} = 0$, then by the same above argument we show that $b \in Z(R)$ and we are done again. Thus we consider the case $b_{12} \neq 0$. Moreover, by applying Lemma 11 it follows $a_{21} = 0$. Hence we may write

$$a = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \qquad b = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}. \tag{14}$$

For $x = e_{12} \in [R,R]$ we have

$$X = ae_{12}q + ce_{12}b = 0 \tag{15}$$

so that the $(2,2)$-entry of the matrix X is $0 = c_{21}b_{22}$; that is, $b_{22} = 0$ and the $(1,1)$-entry of the matrix X is $0 = a_{11}q_{21}$; that is, $a_{11} = 0$. On the other hand, for $x = e_{21} \in [R,R]$, we have

$$Y = ae_{21}q + ce_{21}b = 0. \tag{16}$$

The $(1,2)$-entry of the matrix Y is $0 = a_{12}q_{12} + c_{12}b_{12}$; that is, $a_{12} \neq 0$ and $b_{12}/q_{12} = -a_{12}/c_{12}$. Moreover the $(2,2)$-entry of the matrix Y is $0 = a_{22}q_{12} + c_{22}b_{12}$. Therefore, if denoted $\lambda = -b_{12}/q_{12}$, one has $a_{22} = \lambda c_{22}$ and $a_{12} = \lambda c_{12}$.

Analogously, the $(1,1)$-entry of the matrix Y is $0 = a_{12}q_{11} + c_{12}b_{11}$. Thus $b_{11} = -\lambda q_{11}$ and $b_{12} = -\lambda q_{12}$.

Finally, by our assumption and for $x = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right]$, with $\alpha \neq 0$, we also have

$$\begin{aligned} &\begin{bmatrix} 0 & \lambda c_{12} \\ 0 & \lambda c_{22} \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \cdot \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \\ &\quad + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \cdot \begin{bmatrix} -\lambda q_{11} & -\lambda q_{12} \\ 0 & 0 \end{bmatrix} = 0, \end{aligned} \tag{17}$$

and by easy calculations it follows $cq = 0$. □

Lemma 13. *Let K be an infinite field, $R = M_2(K)$ the algebra of $m \times m$ matrices over K, and $S = [R,R]$. Let $a, b, c \in R$ and denote*

$$\begin{aligned} a &= \sum_{kl} a_{kl}e_{kl}, \qquad & b &= \sum_{kl} b_{kl}e_{kl}, \\ c &= \sum_{kl} c_{kl}e_{kl}, \qquad & cb &= \sum_{kl} p_{kl}e_{kl} \end{aligned} \tag{18}$$

for suitable a_{kl}, b_{kl}, c_{kl}, and p_{kl} elements of K. Suppose $c \notin Z(R)$ and $a[c,x]b = 0$ for all $x \in S$. Assume there are $i \neq j$ such that $p_{ji} = 0$. If $a_{rs} \neq 0$, $b_{rs} \neq 0$, for all $r \neq s$, then $ac = cb = 0$.

Proof. By our hypothesis, we have $acxb - axcb = 0$ for all $x \in S$. By Lemma 12 it follows that either $ac = cb = 0$ or $ab = 0$ and there exists $0 \neq \lambda \in K$ such that

$$ac = \begin{bmatrix} 0 & \lambda a_{12} \\ 0 & \lambda a_{22} \end{bmatrix}, \qquad cb = \begin{bmatrix} \lambda b_{11} & \lambda b_{12} \\ 0 & 0 \end{bmatrix}. \tag{19}$$

Notice that $ab = 0$ implies that the following holds:

$$a_{11}b_{11} + a_{12}b_{21} = 0, \tag{20}$$

$$a_{21}b_{11} + a_{22}b_{21} = 0. \tag{21}$$

Moreover, by computing the product ac we get

$$a_{11}c_{11} + a_{12}c_{21} = 0, \tag{22}$$

$$a_{21}c_{11} + a_{22}c_{21} = 0, \tag{23}$$

$$a_{11}c_{12} + a_{12}c_{22} = \lambda a_{12}. \tag{24}$$

Finally, by computing the product cb we also have

$$c_{21}b_{11} + c_{22}b_{21} = 0, \tag{25}$$

$$c_{21}b_{12} + c_{22}b_{22} = 0. \tag{26}$$

Notice that, in case $a_{11} = 0$, by (20) it follows the contradiction $a_{12}b_{21} = 0$. Thus $a_{11} \neq 0$ and multiply (25) by a_{11}, so that $c_{21}b_{11}a_{11} + c_{22}b_{21}a_{11} = 0$. Again by (20) we have $b_{21}(c_{22}a_{11} - c_{21}a_{12}) = 0$ and using (22) it follows $b_{21}(c_{22}a_{11} + c_{11}a_{11}) = 0$. Since $b_{21} \neq 0$ and $a_{11} \neq 0$, then $c_{11} = -c_{22} = \mu$.

Assume $\mu \neq 0$, denoted by I the identity matrix in R, and let $c' = c - \mu I = \left[\begin{smallmatrix} 0 & c_{12} \\ c_{21} & -2\mu \end{smallmatrix}\right]$.

Since c and c' induce the same inner derivation, then by our assumptions we have that $a[c', x]b = 0$ for all $x \in S$. By applying again Lemma 12, it follows that either $ac' = c'b = 0$ or $ab = 0$ and there exists $0 \neq \nu \in K$ such that

$$ac' = \begin{bmatrix} 0 & \nu a_{12} \\ 0 & \nu a_{22} \end{bmatrix}, \qquad c'b = \begin{bmatrix} \nu b_{11} & \nu b_{12} \\ 0 & 0 \end{bmatrix}. \tag{27}$$

In the latter case, by using the same above argument, the matrix c' satisfies the equalities (22) and (25); that is, respectively,

$$a_{12}c_{21} = 0 \tag{28}$$

implying $c_{21} = 0$, and

$$-2\mu b_{21} = 0 \tag{29}$$

which is a contradiction.

Therefore

$$ac' = c'b = 0, \quad \text{i.e., } ac = \mu a, \ cb = \mu b. \tag{30}$$

In this case, by using both (22) and (30), the $(1,1)$-entry of the matrix ac should be

$$0 = a_{11}c_{11} + a_{12}c_{21} = \mu a_{11} \neq 0. \tag{31}$$

The previous contradiction implies $\mu = 0$; that is, $c_{22} = 0$ and by (26) also $c_{21} = 0$. Hence $c = \left[\begin{smallmatrix} 0 & c_{12} \\ 0 & 0 \end{smallmatrix}\right]$.

Now consider the following elements in S:

$$x_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad y_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{32}$$

Thus

$$Z = a\left[c, x_0\right]b = 0 \qquad T = a\left[c, y_0\right]b = 0 \tag{33}$$

and in particular the $(1,1)$-entry of Z is

$$c_{12}\left(-a_{11}b_{11} - 2a_{11}b_{21} + a_{21}b_{21}\right) = 0, \tag{34}$$

and the $(1,1)$-entry of T is

$$c_{12}\left(a_{11}b_{11} - 2a_{11}b_{21} - a_{21}b_{21}\right) = 0. \tag{35}$$

Since $c \neq 0$, then $c_{12} \neq 0$. Therefore the sum of (34) and (35) forces the contradiction $-4a_{11}b_{21} = 0$. □

Lemma 14. *Let K be an infinite field, $R = M_t(K)$ the algebra of $t \times t$ matrices over K, and $S = [R, R]$. Let $a, b, c \in R$ and denote*

$$\begin{aligned} a &= \sum_{kl} a_{kl}e_{kl}, & b &= \sum_{kl} b_{kl}e_{kl}, \\ cb &= \sum_{kl} p_{kl}e_{kl}, & ac &= \sum_{kl} q_{kl}e_{kl} \end{aligned} \tag{36}$$

for suitable a_{kl}, b_{kl}, p_{kl}, and q_{kl} elements of K. Suppose $c \notin Z(R)$ and $a[c, x]b = 0$ for all $x \in S$. Then there exists $\lambda \in Z(R)$ such that $ac = \lambda a$ and $cb = \lambda b$.

Proof. Clearly if one of a, b, ac, or cb is a scalar matrix we are done by Lemma 9. In order to prove this lemma, we may assume that a, b, ac, and cb are noncentral matrices.

By Lemma 5, there exists some invertible matrix $Q \in M_t(K)$ such that $QaQ^{-1} = a'$, $QbQ^{-1} = b'$, $Q(ac)Q^{-1} = (ac)'$, and $Q(cb)Q^{-1} = (cb)'$ have all nonzero entries.

Notice that $\{ac, a\}$ are linearly $Z(R)$-dependent if and only if $\{(ac)', a'\}$ are linearly $Z(R)$-dependent; analogously $\{cb, b\}$ are linearly $Z(R)$-dependent if and only if $\{(cb)', b'\}$ are linearly $Z(R)$-dependent. Moreover $ac = cb = 0$ if and only if $(ac)' = (cb)' = 0$. Therefore, in order to prove our result, we may replace a, b, ac, cb, respectively, by a', b', $(ac)'$, $(cb)'$, so that a, b, ac, and cb have all nonzero entries.

For $x = e_{ij} \in S$ we have

$$X = ace_{ij}b - ae_{ij}cb = 0; \tag{37}$$

in particular the (j,i)-entry of X is $q_{ji}b_{ji} - a_{ji}p_{ji} = 0$. Denote $0 \neq \eta = q_{ji}/a_{ji}$, so that $p_{ji} = \eta b_{ji}$. Let I be the identity matrix in R and $u = c - \eta I$. Since u and c induce the same inner derivation in R, then $a[u, x]b = 0$; that is, $a(c - \eta I)xb - ax(c - \eta I)b = 0$, for all $x \in S$. Moreover a and b have all nonzero entries, and the (j,i)-entry of $(c - \eta I)b$ is zero. Thus we may apply Lemmas 12 and 13 and obtain $ac = \eta a$ and $cb = \eta b$, as required. □

Proof of Proposition 4. If one assumes that C is infinite, the conclusion follows from Lemma 14.

Now let K be an infinite field which is an extension of the field C and let $\overline{R} = M_t(K) \cong R \otimes_C K$. Consider the generalized polynomial

$$P(x_1, x_2) = a\left(c\left[x_1, x_2\right] - \left[x_1, x_2\right]c\right)b \tag{38}$$

which is a generalized polynomial identity for R. Since $P(x_1, x_2)$ is a multilinear generalized polynomial in the indeterminates x_1, x_2, then it is a generalized polynomial identity for $\overline{R}$ and the conclusion follows again from Lemma 14. □

3. The Inner-Case in Prime Rings

In this section we consider $f(R)$, the set of all evaluations of the noncentral multilinear polynomial $f(x_1, \ldots, x_n)$ over C, and assume that F is an inner generalized derivation, so that

there exist $c, q \in U$ such that $F(x) = cx + xq$, for all $x \in R$, and $f(R)$ satisfies

$$a\left[cx + xq, x\right]b, \tag{39}$$

where a, b are nonzero elements of R.

In order to prove the first result we premit the following.

Fact 1. Let $R = M_t(C)$ be the algebra of $t \times t$ matrices over C of characteristic different from 2. Notice that the set $f(R) = \{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in R\}$ is invariant under the action of all inner automorphisms of R. Hence if denoted by $r = (r_1, \ldots, r_n) \in R \times R \times R \times \cdots \times R = R^n$, then for any inner automorphism φ of $M_t(C)$, we have that $\underline{r} = (\varphi(r_1), \ldots, \varphi(r_n)) \in R^n$ and $\varphi(f(r)) = f(\underline{r}) \in f(R)$.

Since $f(x_1, \ldots, x_n)$ is not central then, by [15] (see also [16]), there exist $u_1, \ldots, u_n \in M_t(C)$ and $\alpha \in C - \{0\}$, such that $f(u_1, \ldots, u_n) = \alpha e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(v_1, \ldots, v_n) : v_1, \ldots, v_n \in M_t(C)\}$ is invariant under the action of all C-automorphisms of $M_t(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_t(C)$ such that $f(r_1, \ldots, r_n) = \alpha e_{ij}$.

Now we may start with the following.

Proposition 15. *Let C be a field, $R = M_t(C)$ the algebra of $t\times t$ matrices over C, and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. Let $0 \neq a$, $0 \neq b$, $c, q \in R$ and denote*

$$\begin{aligned} a &= \sum_{kl} a_{kl} e_{kl}, \qquad & b &= \sum_{kl} b_{kl} e_{kl}, \\ c &= \sum_{kl} c_{kl} e_{kl}, \qquad & q &= \sum_{kl} q_{kl} e_{kl} \end{aligned} \tag{40}$$

for suitable a_{kl}, b_{kl}, c_{kl}, and q_{kl} elements of C. Suppose that

$$a\left[cf\left(r_1, \ldots, r_n\right) + f\left(r_1, \ldots, r_n\right)q, f\left(r_1, \ldots, r_n\right)\right]b = 0 \tag{41}$$

for all $r_1, \ldots, r_n \in R$. Then one of the following holds:

(1) *$c, q \in Z(R)$;*

(2) *there exists $\lambda \in Z(R)$ such that $c - q = \lambda$, and $f(x_1, \ldots, x_n)^2$ is central valued on R;*

(3) *there exist $\lambda, \mu \in Z(R)$ such that $c - q = \lambda$, $ac = \mu a$ and $cb = \mu b$.*

Proof. By our assumption, R satisfies the following generalized polynomial identity:

$$a\left[cf\left(x_1, \ldots, x_n\right) + f\left(x_1, \ldots, x_n\right)q, f\left(x_1, \ldots, x_n\right)\right]b. \tag{42}$$

As in the previous section e_{ij} denotes the matrix unit with 1 in (i, j)-entry and zero elsewhere.

Firstly we assume C is an infinite field.

Since $f(x_1, \ldots, x_n)$ is not central then, by Fact 1, for any $i \neq j$, there exist $r_1, \ldots, r_n \in M_t(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$.

Then we obtain

$$0 = a\left[ce_{ji} + e_{ji}q, e_{ji}\right]b = ae_{ji}qe_{ji}b - ae_{ji}ce_{ji}b. \tag{43}$$

In particular,

$$a_{ij}\left(q_{ij} - c_{ij}\right)b_{ij} = 0. \tag{44}$$

In light of Remark 2, we assume that a and b are not central matrices. Denote $w = q - c$ and suppose that w is not scalar. By Lemma 5 there exists an C-automorphism φ of $M_t(C)$ such that $w' = \varphi(w)$, $a' = \varphi(a)$, and $b' = \varphi(b)$ have all nonzero entries. Clearly w', a', and b' must satisfy the condition (44) and this is a contradiction.

This means that $q - c = \gamma I$, for some $\gamma \in C$, and the main condition is now

$$a\left[cf\left(r_1, \ldots, r_n\right) + f\left(r_1, \ldots, r_n\right)c, f\left(r_1, \ldots, r_n\right)\right]b = 0, \tag{45}$$

for all $r_1, \ldots, r_n \in R$; that is, $a[c, f(r_1, \ldots, r_n)^2]b = 0$, for all $r_1, \ldots, r_n \in R$.

Consider the additive subgroup of R, generated by the set $S = \{x^2 : x \in f(R)\}$. By [17], either $S \subseteq Z(R)$ or the noncentral Lie ideal $[R, R]$ of R is contained in S. In the first case we conclude that $f(x_1, \ldots, x_n)^2$ is central valued in R and we are done. In either case we have $a[c, [r_1, r_2]]b = 0$, for all $r_1, r_2 \in R$, and by Proposition 4 we get the required conclusions.

Now let K be an infinite field which is an extension of the field C and let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} P\left(x_1, \ldots, x_n\right) = a\Big(&cf\left(x_1, \ldots, x_n\right)^2 b - f\left(x_1, \ldots, x_n\right)^2 qb \\ &+ af\left(x_1, \ldots, x_n\right)\left(q - c\right)f\left(x_1, \ldots, x_n\right)\Big)b \end{aligned} \tag{46}$$

which is a generalized polynomial identity for R. Moreover it is multihomogeneous of multidegree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$.

Hence the complete linearization of $P(x_1, \ldots, x_n)$ is a multilinear generalized polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $2n$ indeterminates; moreover

$$\Theta\left(x_1, \ldots, x_n, x_1, \ldots, x_n\right) = 2^n P\left(x_1, \ldots, x_n\right). \tag{47}$$

Clearly the multilinear polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $\mathrm{char}(C) \neq 2$ we obtain $P(r_1, \ldots, r_n) = 0$, for all $r_1, \ldots, r_n \in \overline{R}$, and the conclusion follows from the argument contained in the first part of this proposition. □

Lemma 16. *If there exist $0 \neq a \in R$, $0 \neq b \in R$, $c, q \in U$ such that $a[cx + xq, x]b = 0$, for all $x \in f(R)$, then R satisfies a nontrivial generalized polynomial identity, unless when one of the following holds:*

(1) *$c, q \in C$;*

(2) *$c - q \in C$ and there exists $\lambda \in C$ such that $ac = \lambda a$, $qc = \lambda b$.*

Proof. Assume that R does not satisfy any nontrivial generalized polynomial identity with coefficients in U. Therefore,

$$\Phi(x_1,\dots,x_n) = a\left[cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q, f(x_1,\dots,x_n)\right]b \tag{48}$$

is a trivial generalized polynomial identity for R. By calculations

$$\begin{aligned}\Phi(x_1,\dots,x_n) = a\Big(&cf(x_1,\dots,x_n)^2 + f(x_1,\dots,x_n)\\ &\times(q-c)f(x_1,\dots,x_n)\\ &-f(x_1,\dots,x_n)^2 q\Big)b = 0\end{aligned} \tag{49}$$

for all $x_1,\dots,x_n \in R$. If $c \in C$ and $q \in C$, the proof is completed; hence we suppose that c and q are not simultaneously central. By Remark 3 and by (49), if $\{b, qb\}$ are linearly C-independent then R satisfies the trivial generalized polynomial identity $af(x_1,\dots,x_n)^2 qb = 0$. It means, since $a \neq 0$, $qb = 0$, a contradiction. Analogously, if we suppose $\{a, ac\}$ linearly C-independent, we get $ac = 0$, a contradiction.

Therefore there exist $\alpha, \beta \in C$ such that $qb = \beta b$ and $ac = \alpha a$; now (49) becomes

$$\begin{aligned}a\Big(&f(x_1,\dots,x_n)(q-c)f(x_1,\dots,x_n)\\ &+(\alpha-\beta)f(x_1,\dots,x_n)^2\Big)b = 0\end{aligned} \tag{50}$$

for all $x_1,\dots,x_n \in R$. Since it is a trivial generalized polynomial identity, then $c - q = \alpha - \beta$. Moreover, $\beta b = qb = cb + \beta b - \alpha b$; that is, $cb = \alpha b$. □

Proposition 17. *Let $0 \neq a, 0 \neq b, c, q \in R$ such that*

$$a\left[cf(r_1,\dots,r_n) + f(r_1,\dots,r_n)q, f(r_1,\dots,r_n)\right]b = 0 \tag{51}$$

for all $r_1,\dots,r_n \in R$. Then one of the following holds:

(1) *$c, q \in Z(R)$;*

(2) *there exists $\lambda \in C$ such that $c - q = \lambda$, and $f(x_1,\dots,x_n)^2$ is central valued on R;*

(3) *there exist $\lambda, \mu \in C$ such that $c - q = \lambda$, $ac = \mu a$, and $cb = \mu b$.*

Proof. By Remark 2 we assume that R is not a domain.

Moreover, by Lemma 16, R satisfies the nontrivial generalized polynomial identity:

$$\begin{aligned}P(x_1,\dots,x_n) = a\Big[&cf(x_1,\dots,x_n) + f(x_1,\dots,x_n)q,\\ &f(x_1,\dots,x_n)\Big]b.\end{aligned} \tag{52}$$

By a theorem due to Beidar (Theorem 2 in [18]) this generalized polynomial identity is also satisfied by U. In case C is infinite, we have $P(r_1,\dots,r_n) = 0$ for all $r_1,\dots,r_n \in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U\otimes_C\overline{C}$ are centrally closed [19, Theorems 2.5 and 3.5], we may replace R by U or $U\otimes_C\overline{C}$ according to C being finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [14], R is a primitive ring having a nonzero socle H with C as the associated division ring, and eHe is a simple central algebra finite dimensional over C, for any minimal idempotent element $e \in RC$.

In light of Jacobson's theorem [20, page 75] R is isomorphic to a dense ring of linear transformations on some vector space V over C.

Assume first that V is finite-dimensional over C. Then the density of R on V implies that $R \cong M_k(C)$, the ring of all $k\times k$ matrices over C. Since R is not commutative we assume $k \geq 2$. In this case the conclusion follows by Proposition 15.

Assume next that V is infinite-dimensional over C. As in Lemma 2 in [21], the set $f(R)$ is dense on R and so from $P(r_1,\dots,r_n) = 0$, for all $r_1,\dots,r_n \in R$, we have that R satisfies the generalized identity $P(x) = a[cx + xq, x]b$. We remark that H satisfies $P(x) = a(cx^2 - x^2q + x(q-c)x)b = 0$ (see, e.g., [5, proof of Theorem 1]); that is, for all $r \in H$,

$$a\left(cr^2 - r^2q + r(q-c)r\right)b = 0. \tag{53}$$

In this equality we substitute r with $ex(1-e)$, for any nontrivial idempotent element $e = e^2 \in H$, and obtain

$$aex(1-e)(q-c)ex(1-e)b = 0. \tag{54}$$

By the primeness of R, it follows that either $ae = 0$ or $(1-e)b = 0$ or $(1-e)(q-c)e = 0$. Here our aim is to prove that in any case $(1-e)(q-c)e = 0$. To do this, we firstly assume that $ae = 0$. In (53) replace r by ex, so that $ac(ex)^2b = 0$, which implies $ace = 0$.

Moreover we substitute in (53) r with $ex + y(1-e)$ and by easy computation it follows $ay(1-e)(q-c)exb = 0$; that is, $(1-e)(q-c)e = 0$.

On the other hand, if one supposes $(1-e)b = 0$ and replacing in (53) r by $x(1-e)$, one has $a(x(1-e))^2qb = 0$, which implies $(1-e)qb = 0$. Finally, if substituted in (53) r with $x(1-e) + ey$, as above we have $ax(1-e)(q-c)eyb = 0$. Thus in any case it follows $(1-e)(q-c)e = 0$.

Similarly one can prove also that $e(q-c)(1-e) = 0$.

Hence $[q-c, e] = 0$, for any idempotent element $e \in H$. Since H is not a domain, then H is generated by its minimal idempotent elements; therefore $[q-c, H] = (0)$; that is, $q - c \in C$. Let $\lambda \in C$ such that $q = c + \lambda$. By our assumption it follows that H satisfies $a[cx + xc, x]b$ that is H satisfies $a[c, x^2]b$. In this last replace x by $x+1$ and obtain that H satisfies $a[c, 2x]b$. Since $\mathrm{char}(H) \neq 2$, then $acrb - arcb = 0$, for all $r \in H$. By [14, Lemma 1] it follows that there exists $\mu \in C$ such that $ac = \mu a$ and $cb = \mu b$, unless $ac = cb = 0$. □

Corollary 18. *Let $a, b, c \in R$ such that $c \notin C$ and $f(x_1,\dots,x_n)$ be a noncentral multilinear polynomial over C. If $a[c, f(r_1,\dots,r_n)]_2 b = 0$, for all $r_1,\dots,r_n \in R$, then either $a = 0$ or $b = 0$.*

4. The Main Result

In [11] Lee proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U and obtained the following result.

Theorem 19 (Theorem 3 in [11]). *Every generalized derivation F on a dense right ideal of R can be uniquely extended to U and assumes the form $F(x) = cx + d(x)$, for some $c \in U$ and a derivation d on U.*

In this section we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$, for all $r_1, \ldots, r_n$ in R.

In light of this, we finally prove our main result.

Proof of Theorem 1. Suppose both $a \neq 0$ and $b \neq 0$. Since R satisfies the generalized differential identity

$$a\left[F\left(f\left(x_1, \ldots, x_n\right)\right), f\left(x_1, \ldots, x_n\right)\right] b, \tag{55}$$

the above cited Lee's result says that R satisfies

$$a\left[cf\left(x_1, \ldots, x_n\right) + d\left(f\left(x_1, \ldots, x_n\right)\right), f\left(x_1, \ldots, x_n\right)\right] b. \tag{56}$$

If d is an inner derivation induced by an element $q \in U$, then R satisfies the generalized polynomial identity:

$$\begin{aligned} &a\left[cf\left(x_1, \ldots, x_n\right) + qf\left(x_1, \ldots, x_n\right) - f\left(x_1, \ldots, x_n\right) q,\right. \\ &\quad \left. f\left(x_1, \ldots, x_n\right)\right] b \end{aligned} \tag{57}$$

which is

$$a\left[(c+q) f\left(x_1, \ldots, x_n\right) - f\left(x_1, \ldots, x_n\right) q, f\left(x_1, \ldots, x_n\right)\right] b. \tag{58}$$

In this case we are done by Proposition 17.

Hence let d be an outer derivation of R. In this case R satisfies the differential identity:

$$\begin{aligned} a\Bigg[&cf\left(x_1, \ldots, x_n\right) + f^d\left(x_1, \ldots, x_n\right) \\ &+ \sum_i f\left(x_1, \ldots, d\left(x_i\right), \ldots, x_n\right), \\ &f\left(x_1, \ldots, x_n\right)\Bigg] b. \end{aligned} \tag{59}$$

By Kharchenko's theorem (see [16, 22]), R satisfies the generalized polynomial identity:

$$\begin{aligned} a\Bigg[&cf\left(x_1, \ldots, x_n\right) + f^d\left(x_1, \ldots, x_n\right) \\ &+ \sum_i f\left(x_1, \ldots, y_i, \ldots, x_n\right), \\ &f\left(x_1, \ldots, x_n\right)\Bigg] b, \end{aligned} \tag{60}$$

and in particular, for all $i = 1, \ldots, n$, R satisfies the blended component

$$a\left[f\left(x_1, \ldots, y_i, \ldots, x_n\right), f\left(x_1, \ldots, x_n\right)\right] b. \tag{61}$$

Let $q \in R - Z(R)$ and replace any y_i by $[q, x_i]$. Thus R satisfies

$$a\left[\sum_i f\left(x_1, \ldots, [q, x_i], \ldots, x_n\right), f\left(x_1, \ldots, x_n\right)\right] b. \tag{62}$$

that is,

$$a[q, f\left(x_1, \ldots, x_n\right)]_2 b. \tag{63}$$

By Corollary 18, we get the contradiction $q \in Z(R)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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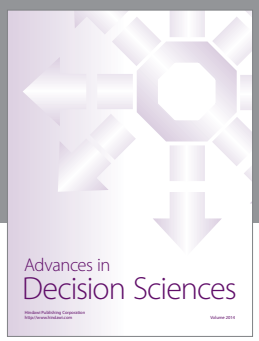

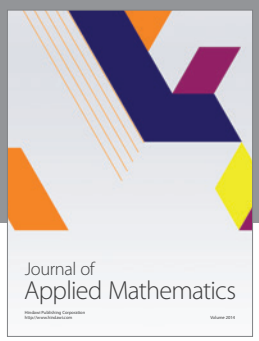

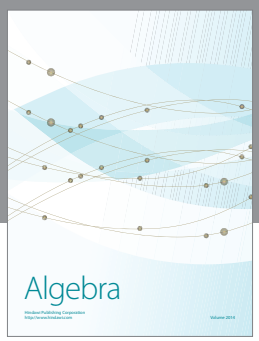

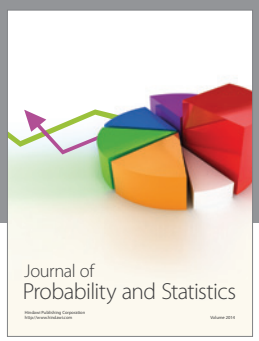

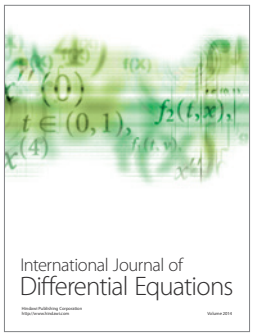

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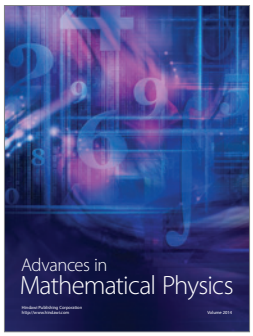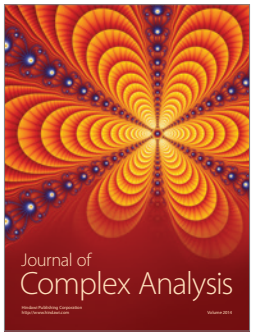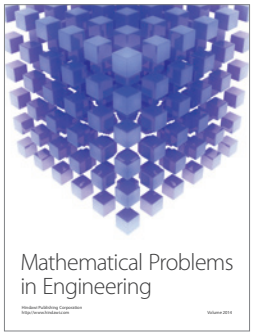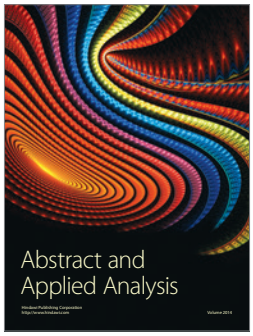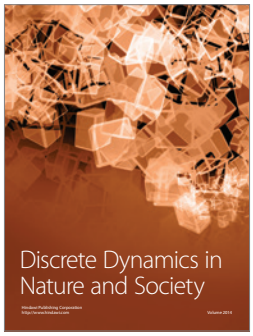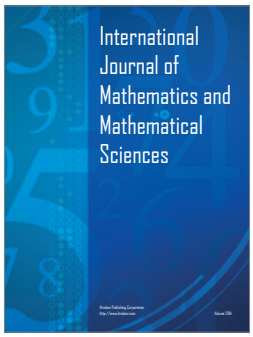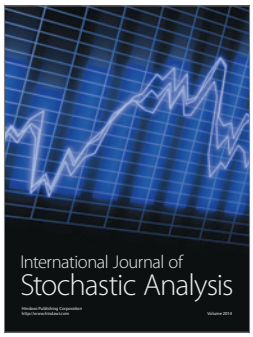
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