Research Article

Optimum Time-Censored Constant-Stress PALTSP for the Burr Type XII Distribution Using Tampered Failure Rate Model

P. W. Srivastava and D. Sharma

Department of Operational Research, University of Delhi, Delhi 110007, India

Correspondence should be addressed to P. W. Srivastava; preetisrivastava.saxena@gmail.com

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This paper presents optimum design of time-censored constant-stress partially accelerated life test sampling plan (PALTSP) in which each item runs either at use or at accelerated conditions and product life follows Burr type XII. The optimal plan consists in finding out sample proportions allocated to both use and accelerated conditions by minimizing the asymptotic variance of test statistic for deciding on acceptance/rejection of the lot such that producer's and consumer's interests are safeguarded. The method developed has been illustrated using an example. Sensitivity analysis has also been carried out.

1. Introduction

Quality control methods are commonly used to determine the acceptability of a product with regard to its usefulness at the time it is put into the service. Reliability is related to the concept of quality of performance. Achieving desirable standards of reliability requires careful analysis in the product design phase. Analysis of data, obtained on a timely basis during product performance, keeps the design and production parameters updated and ensures that the product continues to perform in an acceptable manner. Reliability is built in through quality of design.

Acceptance sampling plans (ASPs) are commonly used to determine the acceptability of a product. If the sample data are related to the reliability of items (e.g., lifetimes or number of failures), then the corresponding ASP is called life test sampling plan or reliability acceptance sampling plan.

Many modern high reliability products are designed to operate without failure for a very long time. Life testing for these products under normal operating conditions is time consuming as it takes a lot of time to obtain reasonable failure information. Introducing ALT in life test sampling plan can be a good way to overcome such a difficulty. This has necessitated the formulation of ALTSPs. ALTSPs differ in the assumed life distribution, censoring schemes, failure monitoring method, test condition, and so forth.

Life test under accelerated environmental conditions may be fully accelerated or partially accelerated. In fully accelerated life testing all the test units are run at accelerated condition, while in partially accelerated life testing they are run at both normal and accelerated conditions. Commonly used stress loadings are constant-stress and step-stress (see Nelson [1]). Under constant-stress PALT each item is tested either at normal operating condition or at accelerated condition only. However, in step-stress PALT, a test item is first run at normal operating condition and, if it does not fail then, it is run at accelerated condition until failure occurs or the observation is censored. Accelerated test condition includes stresses in the form of temperature, voltage, pressure, vibration, cycling rate, humidity, and so forth.

Several authors have studied the problem of designing PALT under constant-stress loading. Yang [2] has indicated that constant-stress accelerated life tests are widely used to save time and money. DeGroot and Goel [3] have considered a PALT and estimated the parameters of the exponential distribution and the acceleration factor using the Bayesian approach. Bai et al. [4] have used the maximum likelihood method to estimate the scale parameter and the acceleration factor for the log normally distributed lifetime, using type I censoring data. Ismail [5] has used maximum likelihood and Bayesian methods for estimating the acceleration factor

and the parameters of Pareto distribution of the second kind. Also, Bhattacharyya and Soejoeti [6] have estimated the parameters of the Weibull distribution and acceleration factor using maximum likelihood method. Bai and Chung [7] have considered optimal designs for both constant and step PALTs under type I censoring. Abdel-Hamid [8] has used the constant-stress PALT and estimated the parameters of the Burr type XII distribution and the acceleration factor under progressive type II censoring

However, no work seems to exist in the literature with PALT incorporated in acceptance sampling plan to facilitate formulation of decision rules for accepting or rejecting a lot of high reliability items satisfying producer's and consumer's requirements. In this paper optimum time-censored constant-stress PALT sampling plans have been designed using hazard acceleration and the Burr type XII life distribution. Single sample acceptance sampling plans by variables are used, and acceptance/rejection decision of a lot is based on Schneider's [9] approach to k -method wherein sample mean and standard deviation are replaced by MLEs of location parameter and scale parameter of log life distribution, respectively. The asymptotic variance of the test statistic so obtained is used for deciding on lot disposition. The optimal plan consists in finding out sample proportions allocated to both use and accelerated conditions for the constant PALT by minimizing the asymptotic variance of test statistic for deciding on acceptance/rejection of the lot such that producer's and consumer's interests are safeguarded.

2. Model Assumptions and Test Procedure

2.1. Assumptions

- The lifetime of an item tested either at normal operating condition or at accelerated conditions follows Burr type XII distribution.
- The lifetimes of test units are independent and identically distributed random variables.
- Tampered failure rate (TFR) model is assumed (Bhattacharyya and Soejoeti [6]).
- The log-lifetimes T_i , $i = 1, 2, \dots, n\phi_1$, of items allocated to normal operating condition and the log-lifetimes T_{1j} , $j = 1, 2, \dots, n\phi_2$, and T_{2j} , $j = 1, 2, \dots, n\phi_3$, of items allocated to accelerated conditions are mutually independent.

2.2. Test Procedure

- The test is conducted with three test chambers, namely, one in which items are tested at normal operating condition, the second in which acceleration factor, A_1 , is applied to the test items, and the third in which acceleration factor, A_2 , is applied to the test items. Out of total " n " items, " $n\phi_1$ " items randomly chosen are allocated to normal operating condition, " $n\phi_2$ " items are allocated to first accelerated condition, A_1 , and the remaining " $n\phi_3$ " items are allocated to second accelerated condition, A_2 , $\phi_1 + \phi_2 + \phi_3 = 1$.

- The test is continued until
 - failure of all test items, or
 - a prescribed censoring time η

whichever occurs earlier, and the test condition should remain the same.

3. Burr Type XII Life Distribution

The Burr type XII distribution has a nonmonotone hazard function, which can accommodate many shapes of hazard function.

The pdf and cdf, respectively, of Burr type XII distribution are

$$g(y; c, k, \alpha) = \left(\frac{kc}{\alpha}\right) \cdot \left(\frac{y}{\alpha}\right)^{c-1} \cdot \left(1 + \left(\frac{y}{\alpha}\right)^c\right)^{-(k+1)}, \quad (1)$$

$$y \geq 0, \quad c > 0, \quad k > 0, \quad \alpha > 0,$$

$$G(y; c, k, \alpha) = 1 - \left(1 + \left(\frac{y}{\alpha}\right)^c\right)^{-k}, \quad (2)$$

$$y \geq 0, \quad c > 0, \quad k > 0, \quad \alpha > 0,$$

where c and k are shape parameters and α is a scale parameter. The Burr (c, k, α) is unimodal, and its mode is $Y_{\text{mode}} = \alpha((c-1)/(ck+1))^{1/c}$ if $c > 1$ and the pdf is L -shaped if $c \leq 1$.

The reliability function and hazard rate function are given by

$$\bar{G}(y) = \left(1 + \left(\frac{y}{\alpha}\right)^c\right)^{-k}, \quad y \geq 0, \quad c > 0, \quad k > 0, \quad \alpha > 0,$$

$$h(y) = \frac{g(y)}{R(y)} = \left(\frac{kc}{\alpha}\right) \cdot \left(\frac{y}{\alpha}\right)^{c-1} \cdot \left(1 + \left(\frac{y}{\alpha}\right)^c\right)^{-1}, \quad (3)$$

$$y \geq 0, \quad c > 0, \quad k > 0, \quad \alpha > 0.$$

The exponential distribution follows from Burr type XII distribution as $k \rightarrow \infty$, such that $\alpha = k^{1/2}$ and $c = 1$. The Weibull life distribution also follows from Burr type XII distribution as $k \rightarrow \infty$, such that $\alpha = k^{1/2}$. For $k = 1$, the Burr type XII distribution reduces to log-logistic distribution.

4. Tampered Failure Rate Model

Bhattacharyya and Soejoeti [6] have proposed the TFR model for step-stress PALT. Their model assumes that a change in the stress has a multiplicative effect on the failure rate function over the remaining life. In this paper TFR model has been used in constant-stress PALT. Consider a constant-stress PALT wherein a test unit is tested either under use condition or under accelerated conditions till the censoring time. Let there be $(m+1)$ test chambers in all including one in which the items are tested under normal operating condition. The TFR model assumes that the effect of changing acceleration factor A_{i-1} in $(i-1)$ th test chamber to A_i in i th test chamber is to multiply the failure rate function $h_{i-1}(y)$ by A_i , $i = 1, 2, \dots, m$. Thus, the effect of changing stress

through acceleration factor in different test chambers has multiplicative effect on initial failure rate function. Hence, the tampered failure rate model is given by

$$h^*(y) = \begin{cases} h(y) & \text{under used condition} \\ h_1(y) = A_1 h(y) & \text{at 1st stress level} \\ h_2(y) = A_2 h_1(y) \\ \quad = A_1 A_2 h(y) & \text{at 2nd stress level} \\ h_3(y) = A_3 h_2(y) \\ \quad = A_1 A_2 A_3 h(y) & \text{at 3rd stress level} \\ \vdots & \vdots \\ h_j(y) = A_j h_{j-1}(y) \\ \quad = \prod_{i=1}^j A_i \cdot h(y) & \text{at } j\text{th stress level} \\ \vdots & \vdots \\ h_m(y) = A_m h_{m-1}(y) \\ \quad = \prod_{i=1}^m A_i \cdot h(y) & \text{at } m\text{th stress level,} \end{cases} \quad (4a)$$

where $A_i > 1 \forall i = 1, 2, \dots, m$.

Unlike fully accelerated life testing wherein a regression structure on the stress variable (e.g., temperature and voltage) is specified, the acceleration factor A_i , $i = 1, 2, \dots, m$, is assumed to be a parameter of the model in PALT.

Let $A_i^* = \prod_{k=1}^i A_k$; then (4a) can be rewritten as

$$h^*(y) = \begin{cases} h(y) & \text{under normal condition} \\ h_1(y) = A_1^* h(y) & \text{at 1st stress level} \\ h_2(y) = A_2^* h(y) & \text{at 2nd stress level} \\ h_3(y) = A_3^* h(y) & \text{at 3rd stress level} \\ \vdots & \vdots \\ h_j(y) = A_j^* h(y) & \text{at } j\text{th stress level} \\ \vdots & \vdots \\ h_m(y) = A_m^* h(y) & \text{at } m\text{th stress level,} \end{cases} \quad (5a)$$

where $A_m^* > A_{m-1}^* > \dots > A_1^* > 1$. Thus, $h(y) < h_1(y) < h_2(y) < \dots < h_m(y)$.

4.1. Life Distribution under TFR Model. Therefore, the pdf and cdf of TFR model when underlying life distribution is Burr type XII are given by

$$g_Y(y) = \left(\frac{kc}{\alpha}\right) \cdot \left(\frac{y}{\alpha}\right)^{c-1} \cdot \left(1 + \left(\frac{y}{\alpha}\right)^c\right)^{-(k+1)},$$

under normal condition,

$$g_{X_i}(y) = \left(\frac{k \prod_{j=1}^i A_j c}{\alpha}\right) \cdot \left(\frac{x_i}{\alpha}\right)^{c-1} \cdot \left(1 + \left(\frac{x_i}{\alpha}\right)^c\right)^{-(k \prod_{j=1}^i A_j + 1)},$$

under i th accelerated condition, $i = 1, 2, \dots, m$,

$$G_Y(y) = 1 - \left(1 + \left(\frac{y}{\alpha}\right)^c\right)^{-k},$$

under normal condition,

$$G_{X_i}(y) = 1 - \left(1 + \left(\frac{x_i}{\alpha}\right)^c\right)^{-k \prod_{j=1}^i A_j},$$

under i th accelerated condition, $i = 1, 2, \dots, m$. (6)

In this paper, we have taken three test chambers; that is, $m + 1 = 3$.

5. Lot Acceptance Sampling Procedure

Assume that one-sided lower specification limit L exists for the lifetime of a product. Let for convenience $T = \ln Y$, instead of using the actual life time Y . The lower specification limit on T is $L^T = \ln L$. Then, pdf and cdf of an item tested at normal operating condition and at accelerated conditions using $g_Y(y)$, $g_{X_i}(y)$, $G_Y(y)$, and $G_{X_i}(y)$ in Section 4.1 are given by

$$f_T(t; A_1, A_2, \alpha, c, k) = \frac{ke^{((t-\mu)/\sigma)}}{\left(\sigma(1 + e^{((t-\mu)/\sigma)})^{(k+1)}\right)}$$

under normal condition

$$f_{T_1}(t_1; A_1, A_2, \alpha, c, k) = \frac{kA_1 e^{((t_1-\mu)/\sigma)}}{\left(\sigma(1 + e^{((t_1-\mu)/\sigma)})^{(kA_1+1)}\right)}$$

at 1st stress level

$$f_{T_2}(t_2; A_1, A_2, \alpha, c, k) = \frac{kA_1 A_2 e^{((t_2-\mu)/\sigma)}}{\left(\sigma(1 + e^{((t_2-\mu)/\sigma)})^{(kA_1 A_2 + 1)}\right)}$$

at 2nd stress level,

$$F_T(t; A_1, A_2, \alpha, c, k) = 1 - \left(1 + e^{((t-\mu)/\sigma)}\right)^{-k}$$

under normal condition

$$F_{T_1}(t_1; A_1, A_2, \alpha, c, k) = 1 - \left(1 + e^{((t_1-\mu)/\sigma)}\right)^{-kA_1}$$

at 1st stress level

$$F_{T_2}(t_2; A_1, A_2, \alpha, c, k) = 1 - \left(1 + e^{((t_2-\mu)/\sigma)}\right)^{-kA_1 A_2}$$

at 2nd stress level. (7)

The reliability function of an item tested at normal and accelerated conditions A_1^* and A_2^* are, respectively, given by

$$\begin{aligned}\bar{F}_T(t; A_1, A_2, \alpha, c, k) &= \left(1 + e^{((t-\mu)/\sigma)}\right)^{-k} \\ &\text{under used condition} \\ \bar{F}_{T_1}(t_1; A_1, A_2, \alpha, c, k) &= \left(1 + e^{((t_1-\mu)/\sigma)}\right)^{-kA_1} \\ &\text{at 1st stress level} \\ \bar{F}_{T_2}(t_2; A_1, A_2, \alpha, c, k) &= \left(1 + e^{((t_2-\mu)/\sigma)}\right)^{-kA_1A_2} \\ &\text{at 2nd stress level,}\end{aligned}\quad (8)$$

where $\mu = \ln(\alpha)$ is location parameter and $\sigma = (c)^{-1}$.

The following lot acceptance sampling procedure is considered.

- (i) Samples of n items are randomly selected from the lot and are tested according to the above test procedure.
- (ii) Maximum likelihood estimators (MLEs) $\hat{\mu}$ and $\hat{\sigma}$, respectively, of location parameter μ and scale parameter σ at normal operating condition, respectively, of Burr type XII distribution are obtained from the test data.
- (iii) Schneider's [9] approach to k -method is used wherein sample mean and standard deviation are replaced by MLEs of location parameter and scale parameter of log life distribution, respectively. In this paper k_1 instead of k has been used as acceptability constant. The value of test statistic $T' = \hat{\mu} - k_1\hat{\sigma}$ is compared with L' ; the lot is accepted if $T' \geq L'$ and rejected otherwise. The sample size n and acceptability constant k_1 determined using variance optimality criterion subject to the constraints that the lots with fraction nonconforming $p \leq p_{\bar{\alpha}}$ are accepted with probability of at least $1 - \bar{\alpha}$ and lots with $p > p_{\bar{\beta}}$ are rejected with probability of at least $1 - \bar{\beta}$.

6. Model Formulation

Maximum likelihood method has been used to estimate the model parameters α , c , and acceleration factors, A_1 and A_2 , from the test data.

6.1. Likelihood Function. The likelihood function based on " n " observation is

$$\begin{aligned}L &\equiv L(\alpha, c, A_1, A_2) \\ &= \prod_{i=1}^{n\phi_1} L_{u_i}(\alpha, c; (t_i, \delta_{u_i})) \cdot \prod_{j=1}^{n\phi_2} L_{a_{1j}}(\alpha, c, A_1; (t_{1j}, \delta_{a_{1j}})) \\ &\quad \cdot \prod_{k=1}^{n\phi_3} L_{a_{2k}}(\alpha, c, A_1, A_2; (t_{2j}, \delta_{a_{2j}})),\end{aligned}\quad (9)$$

where

$$\begin{aligned}L_{u_i} &\equiv L_{u_i}(\alpha, c; (t_i, \delta_{u_i})) \\ &= \frac{ke^{(t_i-\mu)/\sigma}}{(\sigma(1 + e^{(t_i-\mu)/\sigma})^{k+1})} \cdot \left(1 + e^{(\eta'-\mu)/\sigma}\right)^{-k} \bar{\delta}_{u_i}, \\ L_{a_{1j}} &\equiv L_{a_{1j}}(\alpha, c, A_1; (t_{1j}, \delta_{a_{1j}})) \\ &= \left(\frac{kA_1e^{(t_{1j}-\mu)/\sigma}}{(\sigma(1 + e^{(t_{1j}-\mu)/\sigma})^{kA_1+1})}\right)^{\delta_{a_{1j}}} \\ &\quad \cdot \left(1 + e^{(\eta'-\mu)/\sigma}\right)^{-kA_1} \bar{\delta}_{a_{1j}}, \\ L_{a_{2j}} &\equiv L_{a_{2j}}(\alpha, c, A_1, A_2; t_{2j}, \delta_{a_{2j}}) \\ &= \left(\frac{kA_1A_2e^{(t_{2j}-\mu)/\sigma}}{(\sigma(1 + e^{(t_{2j}-\mu)/\sigma})^{kA_1A_2+1})}\right)^{\delta_{a_{2j}}} \\ &\quad \cdot \left(1 + e^{(\eta'-\mu)/\sigma}\right)^{-kA_1A_2} \bar{\delta}_{a_{2j}}, \\ \bar{\delta}_{u_i} &= 1 - \delta_{u_i}, \quad \bar{\delta}_{a_{1j}} = 1 - \delta_{a_{1j}}, \quad \bar{\delta}_{a_{2k}} = 1 - \delta_{a_{2k}}.\end{aligned}\quad (10)$$

The first-order partial derivatives of log-likelihood function of the i th unit with respect to A_1 , A_2 , α , and c are given by

$$\begin{aligned}\frac{\partial \ln L_{u_i}}{\partial A_1} &= 0, \\ \frac{\partial \ln L_{u_i}}{\partial A_2} &= 0, \\ \frac{\partial \ln L_{u_i}}{\partial \alpha} &= -\left(\frac{c}{\alpha}\right) \cdot \left\{ \delta_{u_i} \left(1 - \left(\frac{(k+1)e^{(t_i-\mu)/\sigma}}{(1 + e^{(t_i-\mu)/\sigma})}\right)\right) \right. \\ &\quad \left. - \bar{\delta}_{u_i} \left(\frac{ke^{(\eta'-\mu)/\sigma}}{(1 + e^{(\eta'-\mu)/\sigma})}\right) \right\},\end{aligned}$$

$$\frac{\partial \ln L_{u_i}}{\partial c} = \left(\frac{1}{c}\right) \cdot \left\{ \delta_{u_i} \left(1 + \left(\frac{(t_i - \mu)}{\sigma} \right) \right. \right. \\ \left. \left. - \left(\frac{(k+1)((t_i - \mu)/\sigma) e^{(t_i - \mu)/\sigma}}{(1 + e^{(t_i - \mu)/\sigma})} \right) \right) \right. \\ \left. - \bar{\delta}_{u_i} \left(\frac{k((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})} \right) \right\},$$

$$\frac{\partial \ln L_{a_{1j}}}{\partial A_1} = \delta_{a_{1j}} \left(\left(\frac{1}{A_1} \right) - k \ln(1 + e^{(k_{1j} - \mu)/\sigma}) \right) \\ - \bar{\delta}_{a_{1j}} \left(k \ln(1 + e^{(\eta' - \mu)/\sigma}) \right),$$

$$\frac{\partial \ln L_{a_{1j}}}{\partial A_2} = 0,$$

$$\frac{\partial \ln L_{a_{1j}}}{\partial \alpha} = -\left(\frac{c}{\alpha}\right) \cdot \left\{ \delta_{a_{1j}} \left(1 - \left(\frac{(kA_1 + 1) e^{(t_{1j} - \mu)/\sigma}}{(1 + e^{(t_{1j} - \mu)/\sigma})} \right) \right) \right. \\ \left. - \bar{\delta}_{a_{1j}} \left(\frac{kA_1 e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})} \right) \right\},$$

$$\frac{\partial \ln L_{a_{1j}}}{\partial c} = \left(\frac{1}{c}\right) \cdot \left\{ \delta_{a_{1j}} \left(1 + \left(\frac{(t_{1j} - \mu)}{\sigma} \right) \right. \right. \\ \left. \left. - \left(\frac{(kA_1 + 1)((t_{1j} - \mu)/\sigma) e^{(t_{1j} - \mu)/\sigma}}{(1 + e^{(t_{1j} - \mu)/\sigma})} \right) \right) \right. \\ \left. - \bar{\delta}_{a_{1j}} \left(\frac{kA_1((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})} \right) \right\},$$

$$\frac{\partial \ln L_{a_{2j}}}{\partial A_1} = \delta_{a_{2j}} \left(\left(\frac{1}{A_1} \right) - kA_2 \ln(1 + e^{(t_{2j} - \mu)/\sigma}) \right) \\ - \bar{\delta}_{a_{2j}} \left(kA_2 \ln(1 + e^{(\eta' - \mu)/\sigma}) \right),$$

$$\frac{\partial \ln L_{a_{2j}}}{\partial A_2} = \delta_{a_{2j}} \left(\left(\frac{1}{A_2} \right) - kA_1 \ln(1 + e^{(t_{2j} - \mu)/\sigma}) \right) \\ - \bar{\delta}_{a_{2j}} \left(kA_1 \ln(1 + e^{(\eta' - \mu)/\sigma}) \right),$$

$$\frac{\partial \ln L_{a_{2j}}}{\partial \alpha} = -\left(\frac{c}{\alpha}\right) \cdot \left\{ \delta_{a_{2j}} \left(1 - \left(\frac{(kA_1 A_2 + 1) e^{(t_{2j} - \mu)/\sigma}}{(1 + e^{(t_{2j} - \mu)/\sigma})} \right) \right) \right. \\ \left. - \bar{\delta}_{a_{2j}} \left(\frac{kA_1 A_2 e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})} \right) \right\},$$

$$\frac{\partial \ln L_{a_{2j}}}{\partial c} = \left(\frac{1}{c}\right) \\ \cdot \left\{ \delta_{a_{2j}} \left(1 + \left(\frac{(t_{2j} - \mu)}{\sigma} \right) \right. \right. \\ \left. \left. - \left(\frac{(kA_1 A_2 + 1)((t_{2j} - \mu)/\sigma) e^{(t_{2j} - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})} \right) \right) \right. \\ \left. - \bar{\delta}_{a_{2j}} \left(\frac{kA_1 A_2((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})} \right) \right\}. \quad (11)$$

On summing these partial derivatives and equating them to zero, likelihood equations are obtained. Since the closed form solutions of above likelihood equations are very hard to obtain, so further numerical treatment is required to obtain the MLEs of A_1 , A_2 , α , and c .

6.2. Fisher Information Matrix. The Fisher information matrix $F(A_1, A_2, \alpha, c) \equiv F$ is the 4×4 symmetric matrix of expectation of negative second-order partial derivatives of the log likelihood function with respect to A_1 , A_2 , α , and c . Consider

$$F = \begin{pmatrix} \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_1^2} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_1 \partial A_2} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_1 \partial \alpha} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_1 \partial c} \right] \\ \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_2 \partial A_1} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_2^2} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_2 \partial \alpha} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_2 \partial c} \right] \\ \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial \alpha \partial A_1} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial \alpha \partial A_2} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial \alpha^2} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial \alpha \partial c} \right] \\ \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c \partial A_1} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c \partial A_2} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c \partial \alpha} \right] & \sum_{i=1}^{n\phi_1} E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c^2} \right] \end{pmatrix}$$

$$\begin{aligned}
& + \left(\begin{array}{cccc} \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_1^2} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_1 \partial A_2} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_1 \partial \alpha} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_1 \partial c} \right] \\ \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_2 \partial A_1} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_2^2} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_2 \partial \alpha} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_2 \partial c} \right] \\ \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial \alpha \partial A_1} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial \alpha \partial A_2} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial \alpha^2} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial \alpha \partial c} \right] \\ \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial c \partial A_1} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial c \partial A_2} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial c \partial \alpha} \right] & \sum_{j=1}^{n\phi_2} E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial c^2} \right] \end{array} \right) \\
& + \left(\begin{array}{cccc} \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_1^2} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_1 \partial A_2} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_1 \partial \alpha} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_1 \partial c} \right] \\ \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_2 \partial A_1} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_2^2} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_2 \partial \alpha} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_2 \partial c} \right] \\ \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial \alpha \partial A_1} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial \alpha \partial A_2} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial \alpha^2} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial \alpha \partial c} \right] \\ \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c \partial A_1} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c \partial A_2} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c \partial \alpha} \right] & \sum_{k=1}^{n\phi_3} E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c^2} \right] \end{array} \right), \quad (12)
\end{aligned}$$

where the values of these elements are given in Appendix A.

6.3. *Asymptotic Variance of Test Statistic T'* . For any plan, the asymptotic variance-covariance matrix of the MLEs $\hat{A}_1$, $\hat{A}_2$, $\hat{\alpha}$, $\hat{c}$ is the inverse of the corresponding Fisher information matrix; that is,

$$\sum = F^{-1} = \begin{bmatrix} \text{Asvar}(\hat{A}_1) & \text{Ascov}(\hat{A}_1, \hat{A}_2) & \text{Ascov}(\hat{A}_1, \hat{\alpha}) & \text{Ascov}(\hat{A}_1, \hat{c}) \\ \text{Ascov}(\hat{A}_2, \hat{A}_1) & \text{Asvar}(\hat{A}_2) & \text{Ascov}(\hat{A}_2, \hat{\alpha}) & \text{Ascov}(\hat{A}_2, \hat{c}) \\ \text{Ascov}(\hat{\alpha}, \hat{A}_1) & \text{Ascov}(\hat{\alpha}, \hat{A}_2) & \text{Asvar}(\hat{\alpha}) & \text{Ascov}(\hat{\alpha}, \hat{c}) \\ \text{Ascov}(\hat{c}, \hat{A}_1) & \text{Ascov}(\hat{c}, \hat{A}_2) & \text{Ascov}(\hat{c}, \hat{\alpha}) & \text{Asvar}(\hat{c}) \end{bmatrix}. \quad (13)$$

The asymptotic variance of the test statistic T' is given by

$$\begin{aligned}
& \text{Asvar}(T') \\
& = \text{Asvar}(\hat{\mu}_0) - 2k_1 \text{Ascov}(\hat{\mu}_0, \hat{\sigma}_0) + k_1^2 \text{Asvar}(\hat{\sigma}_0) = \frac{V}{n}, \quad (14)
\end{aligned}$$

where V is obtained in Appendix B.

7. OC Curve

The operating characteristic (OC) curve measures the performance of a sampling plan and plots the probability of

accepting the lot versus the proportion nonconforming of the lot. Based on the asymptotic distribution theory,

$$\frac{(T' - (\mu - k_1\sigma))}{\sqrt{\text{AsVar}[T']}} \sim N(0, 1) \quad \text{as } n \rightarrow \infty. \quad (15)$$

The standardized variate using (14) is

$$U = \frac{(T' - (\mu - k_1\sigma)) \sqrt{n}}{\sqrt{V}} \sim N(0, 1) \quad \text{as } n \rightarrow \infty. \quad (16)$$

The OC curve is obtained by plotting the probability of accepting the lot, L_p , given as

$$L_p = \Pr[T' \geq L'] = 1 - \Phi \left[\frac{\sigma(u_p + k_1) \sqrt{n}}{\sqrt{V}} \right] \quad (17)$$

against the fraction nonconforming, p , where $u_p = (L' - \mu)/\sigma$ is the quantile of the standard logistic distribution corresponding to the fraction nonconforming p , and $\Phi(\cdot)$ is the standard normal distribution function. Therefore, the following equations have to be solved for k_1^* and n^* in order to obtain an optimal sampling plan for two given points $(p_{\tilde{\alpha}}, 1 - \tilde{\alpha})$ and $(p_{\tilde{\beta}}, \tilde{\beta})$ on the OC curve,

$$z_{\tilde{\alpha}} - \frac{(\sigma(u_{p_{\tilde{\alpha}}} + k_1)) \sqrt{n}}{\sqrt{V}} = 0, \quad (18)$$

$$z_{1-\tilde{\beta}} - \frac{(\sigma(u_{p_{\tilde{\beta}}} + k_1)) \sqrt{n}}{\sqrt{V}} = 0,$$

where $u_{p_{\tilde{\alpha}}}$ and $u_{p_{\tilde{\beta}}}$ denote the quantiles of the standard normal distribution and $z_{\tilde{\alpha}}$ and $z_{1-\tilde{\beta}}$ are the quantiles of the standard log-life distribution. Thus, on solving (18),

$$k_1^* = \frac{(z_{1-\tilde{\beta}}u_{p_{\tilde{\alpha}}} - z_{\tilde{\alpha}}u_{p_{\tilde{\beta}}})}{(z_{\tilde{\alpha}} - z_{1-\tilde{\beta}})}, \quad (19)$$

$$n^* = \frac{(k_2 V)}{\sigma^2}, \quad (20)$$

where

$$k_2 = \frac{(z_{\tilde{\alpha}} - z_{1-\tilde{\beta}})}{(u_{p_{\tilde{\alpha}}} - u_{p_{\tilde{\beta}}})} \quad (21)$$

and V is given in (14).

See Schneider [9] for reference.

8. Formulation of an Optimization Problem

V in (14) is a function of A_1 , A_2 , η , α , c , and k . Thus, the optimal design problem can be formulated as a nonlinear optimization problem:

$$\begin{aligned} &\text{Minimize } V \\ &\text{subject to } 0 < \phi_i < 1, \quad i = 1, 2, 3, \\ &\quad \sum_{i=1}^3 \phi_i = 1 \\ &\quad \phi_i p_i \geq 0.1, \quad i = 1, 2, 3. \end{aligned} \quad (22)$$

Since the sample size is unknown while minimizing V , therefore instead of giving minimum mean number of failures (MMNF) a prespecified value in advance, 0.1, is assigned to the ratio of MMNF and the sample size at each stress level. Another suitable number for MMNF can be specified according to cost, time, and precision implications (see Yang [2]). The optimum values of ϕ_i , $i = 1, 2, 3$, are found by minimizing V under type I censoring, using NMinimize option of *Mathematica* 8.

TABLE 1: Change in (ϕ_i^*, n^*) , $i = 1, 2, 3$ for different values of A_1 ($A_2 = 3$, $\eta = 5$, $\alpha = 0.9$, $c = 3.5$, and $k = 4$).

A_1	ϕ_1^*	ϕ_2^*	ϕ_3^*	n^*
1.5	0.29387	0.33916	0.36697	37.2889 $\approx$ 37
1.6	0.23162	0.23576	0.53262	36.6683 $\approx$ 37
1.7	0.19098	0.21097	0.59805	35.8638 $\approx$ 36
1.8	0.1584	0.20729	0.63431	35.1254 $\approx$ 35
1.9	0.13196	0.21318	0.65486	34.5078 $\approx$ 35
2	0.11085	0.22402	0.66513	34.0173 $\approx$ 34

9. A Numerical Example

In this section, a hypothetical constant-stress PALTSP experiment is considered to illustrate the methods described in this paper.

9.1. OC Curve. Choose two points $(p_{\tilde{\alpha}}, 1 - \tilde{\alpha})$ and $(p_{\tilde{\beta}}, \tilde{\beta})$ on the OC curve as (0.01, 0.99) and (0.10, 0.10), respectively.

9.2. Optimal Lot Acceptability Constant. Compute the acceptability constant k_1 , using (19).

Since $z_{\tilde{\alpha}} = -2.32635$, $z_{1-\tilde{\beta}} = 1.28155$, $u_{p_{\tilde{\alpha}}} = -5.98519$, and $u_{p_{\tilde{\beta}}} = -3.62346$, we have $k_1 = ((1.28155(-5.98519)) - ((-3.62346)(-2.32635)))/((-2.32635) - (1.28155)) = 4.46236$.

9.3. Optimal Plan. Optimal ϕ_i^* , $i = 1, 2, 3$, and n^* are using (22). These are given in Table 1 using the following hypothetical data set:

$$\begin{aligned} A_1 &= 1.5, & A_2 &= 3, & \eta &= 5, \\ \alpha &= 0.9, & c &= 3.5, & k &= 4. \end{aligned} \quad (23)$$

Table 1 shows that optimal sample size decreases as A_1 increases and optimal sample proportions allocated to accelerated condition A_2 increase as A_1 increases.

9.4. Optimal Acceptance Sampling Plan. Let us consider the lower specification limit of L as $L' (\ln L)$ indicating that items with lifetime shorter than $\ln L$ are nonconforming. For given lower specification limit L' , lot is accepted if $T' = \hat{\mu}_0 - k_1 \hat{\sigma}_0 > L'$, lot is accepted and otherwise rejected.

Figure 1 depicts the OC curve obtained from u_p , V , n^* and the two points $(p_{\tilde{\alpha}}, 1 - \tilde{\alpha})$ and $(p_{\tilde{\beta}}, \tilde{\beta})$.

Tables 2 and 3 present optimum sampling plan for various values of $p_{\tilde{\alpha}}$ and $p_{\tilde{\beta}}$ when $(1 - \tilde{\alpha}, \tilde{\beta}) = (0.95, 0.10)$ and $(0.99, 0.01)$. It is observed that for given $p_{\tilde{\alpha}}$ as $p_{\tilde{\beta}}$ increases optimal " n^* " decreases, optimal ϕ_1^* increases, and optimal ϕ_2^* and optimal ϕ_3^* decrease. Using the data in Table 2, Figures 2 and 3 depict the OC curve obtained from the two points $(p_{\tilde{\alpha}}, 1 - \tilde{\alpha})$ as (0.001, 0.95), (0.01, 0.95) and $(p_{\tilde{\beta}}, \tilde{\beta})$ as (0.01, 0.10), (0.05, 0.10), (0.10, 0.10); (0.05, 0.10), (0.01, 0.10), (0.15, 0.10), respectively. In a similar manner, data in Table 3

FIGURE 1: OC curve with $(p_{\bar{\alpha}}, 1 - \bar{\alpha})$ as (0.01, 0.99) and $(p_{\bar{\beta}}, \bar{\beta})$ as (0.10, 0.10).

FIGURE 2: OC curves for $(p_{\bar{\alpha}}, 1 - \bar{\alpha})$ as (0.001, 0.95) and $(p_{\bar{\beta}}, \bar{\beta})$ as (0.01, 0.10), (0.05, 0.10), and (0.10, 0.10).

FIGURE 3: OC curves for $(p_{\bar{\alpha}}, 1 - \bar{\alpha})$ as (0.01, 0.95) and $(p_{\bar{\beta}}, \bar{\beta})$ as (0.05, 0.10), (0.10, 0.10), and (0.15, 0.10).

FIGURE 4: OC curves for $(p_{\bar{\alpha}}, 1 - \bar{\alpha})$ as (0.001, 0.99) and $(p_{\bar{\beta}}, \bar{\beta})$ as (0.01, 0.01), (0.05, 0.01), and (0.10, 0.01).

FIGURE 5: OC curves for $(p_{\bar{\alpha}}, 1 - \bar{\alpha})$ as (0.01, 0.99) and $(p_{\bar{\beta}}, \bar{\beta})$ as (0.05, 0.01), (0.10, 0.01), and (0.15, 0.01).

has been used in plotting OC curve in Figures 4 and 5 by taking $1 - \bar{\alpha}$ as 0.99 and $\bar{\beta}$ as 0.01.

10. Sensitivity Analysis

To formulate an optimum sampling plan, the information about acceleration factors A_1 and parameters of the model α and c is needed as incorrect choice of these results in poor estimates of the parameters. The effects of incorrect preestimates of A_1 , A_2 , α , and c in terms of the relative increase in asymptotic variance of test statistic have been studied. The percentage deviations (PD) of the resulting optimal settings are measured by $PD = (|V^{**} - V^*|/V^*) \times 100$, where V^* is the setting obtained with the given design parameters, and V^{**} is the one obtained when the parameter is misspecified. Tables 4 and 5 show that irrespective of whether the incorrect variance is smaller or larger than the true variance, the proposed optimum plan is robust since the percentage deviation in variance is small.

TABLE 2: Effects of various values of $(p_{\bar{\alpha}}, 1 - \bar{\alpha})$ and $(p_{\bar{\beta}}, \bar{\beta})$ on optimal ALT plans ($A_1 = 1.5$, $A_2 = 3$, $\eta = 5$, $\alpha = 0.9$, $c = 3.5$, and $k = 4$).

$1 - \bar{\alpha} = 0.95, \bar{\beta} = 0.10$								
$p_{\bar{\alpha}}$	$p_{\bar{\beta}}$	k_1	k_2	V^*	ϕ_1^*	ϕ_2^*	ϕ_3^*	n^*
0.001	0.01	6.99603	1.60734	3.27656	0.16048	0.40322	0.43630	65
	0.05	6.07697	0.55073	2.45800	0.19573	0.38629	0.41798	17
	0.1	5.66856	0.39268	2.13194	0.21515	0.37697	0.40789	10
0.01	0.05	5.06613	3.20311	1.69332	0.24966	0.36039	0.38995	66
	0.1	4.65773	1.53536	1.42466	0.27827	0.34665	0.37508	27
	0.15	4.41008	1.09052	1.27305	0.29829	0.33703	0.36468	17

TABLE 3: Effects of various values of $(p_{\bar{\alpha}}, 1 - \bar{\alpha})$ and $(p_{\bar{\beta}}, \bar{\beta})$ on optimal ALT plans ($A_1 = 1.5$, $A_2 = 3$, $\eta = 5$, $\alpha = 0.9$, $c = 3.5$, and $k = 4$).

$1 - \bar{\alpha} = 0.99, \bar{\beta} = 0.01$								
$p_{\bar{\alpha}}$	$p_{\bar{\beta}}$	k_1	k_2	V^*	ϕ_1^*	ϕ_2^*	ϕ_3^*	n^*
0.001	0.01	7.13931	4.06301	3.41476	0.15582	0.40546	0.43872	170
	0.05	6.32175	1.39212	2.66454	0.18532	0.39129	0.42339	45
	0.1	5.95844	0.99262	1.47615	0.27219	0.34957	0.37824	18
0.01	0.05	5.16763	8.09678	1.76368	0.24326	0.36346	0.39328	175
	0.1	4.80432	3.88106	1.51843	0.26743	0.35186	0.38072	72
	0.15	4.58403	2.75661	1.37865	0.28400	0.34390	0.37211	47

TABLE 4: Effects of incorrect preestimate “ α ” on optimal ALT plans ($A_1 = 1.5$, $A_2 = 3$, $\eta = 5$, $\alpha = 0.9$, $c = 3.5$, and $k = 4$, $V^* = 1.30435$).

% change	Incorrect sample proportions				Incorrect variance	% deviation PD
	α	ϕ_1^*	ϕ_2^*	ϕ_3^*	V^{**}	
1%	0.909	0.36852	0.50889	0.12260	1.29050	1.06182
−1%	0.918	0.25565	0.28443	0.45992	1.30919	0.37156
2%	0.927	0.37624	0.52376	0.10000	1.26266	3.19600
−2%	0.936	0.22676	0.25874	0.51449	1.31231	0.61030
3%	0.945	0.37750	0.52250	0.10000	1.23615	5.22824
−3%	0.891	0.20252	0.24549	0.55199	1.31570	0.87059

TABLE 5: Effects of incorrect preestimate “ c ” on optimal ALT plans ($A_1 = 1.5$, $A_2 = 3$, $\eta = 5$, $\alpha = 0.9$, $c = 3.5$, and $k = 4$, $V^* = 1.30435$).

% change	Incorrect sample proportions				Incorrect variance	% deviation PD
	c	ϕ_1^*	ϕ_2^*	ϕ_3^*	V^{**}	
1%	3.535	0.23483	0.26473	0.50044	1.28556	1.44015
−1%	3.570	0.37585	0.52415	0.10000	1.29585	0.65107
2%	3.605	0.19437	0.24250	0.56313	1.26608	2.93382
−2%	3.640	0.37800	0.52200	0.10000	1.27416	2.31434
3%	3.675	0.16283	0.23735	0.59982	1.25043	4.13323
−3%	3.465	0.37927	0.52073	0.10000	1.25527	3.76273

11. Conclusion

In this paper we have formulated an optimum time-censored constant-stress PALTSP for the Burr type XII life distribution using hazard acceleration. Single sampling plan by variables has been used. The acceptance/rejection decision of the lot involves k -method in which Schneider's [9] approach has been used with sample mean replaced by MLE of location parameter and standard deviation replaced by MLE

of scale parameter of the log-lifetime distribution with lower specification limit specified. The optimum plan consists in finding optimum allocation at normal operating condition and accelerated conditions by minimizing the variance of the test statistic meant for deciding on lot acceptability. The sensitivity analysis has been carried out and it has been found that if the misspecified values of the parameters are not too far from the true values, the optimum plan is robust.

Appendices

A.

The elements of Fisher Information matrix are

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_1^2} \right] = 0,$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_1 \partial A_2} \right] = 0,$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial \alpha \partial A_1} \right] = 0,$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c \partial A_1} \right] = 0,$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial A_2^2} \right] = 0,$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial \alpha \partial A_2} \right] = 0,$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c \partial A_2} \right] = 0,$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial \alpha^2} \right]$$

$$= \left(\frac{c}{\alpha} \right)^2 \cdot \left\{ k(k+1) \right.$$

$$\times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{e^{2((t_i - \mu)/\sigma)}}{\sigma(1 + e^{(t_i - \mu)/\sigma})^{k+3}} \right) dt_i \\ \left. + k \left(\frac{e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{k+2}} \right) \right\},$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c \partial \alpha} \right]$$

$$= \left(-\frac{1}{\alpha} \right) \cdot \left\{ k(k+1) \right.$$

$$\times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_i - \mu)/\sigma) e^{2((t_i - \mu)/\sigma)}}{\sigma(1 + e^{(t_i - \mu)/\sigma})^{k+3}} \right) dt_i \\ \left. + \frac{k((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{k+2}} \right\},$$

$$E \left[-\frac{\partial^2 \ln L_{u_i}}{\partial c^2} \right]$$

$$= \left(\frac{1}{c^2} \right) \cdot \left\{ \left(1 - \left(1 + e^{(\eta' - \mu)/\sigma} \right)^{-k} \right) + k(k+1) \right. \\ \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_i - \mu)/\sigma)^2 e^{2((t_i - \mu)/\sigma)}}{\sigma(1 + e^{(t_i - \mu)/\sigma})^{k+3}} \right) dt_i \\ \left. + \frac{k((\eta' - \mu)/\sigma)^2 e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{k+2}} \right\},$$

$$E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial A_1^2} \right] = \left(\frac{1}{A_1} \right)^2 \cdot \left(1 - \left(1 + e^{(\eta' - \mu)/\sigma} \right)^{-kA_1} \right),$$

$$E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial \alpha \partial A_1} \right]$$

$$= \left(-\frac{c}{\alpha} \right) \cdot \left\{ k(kA_1) \right. \\ \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{e^{2((t_{1j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{1j} - \mu)/\sigma})^{kA_1+2}} \right) dt_{1j} \\ \left. + k \left(\frac{e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1+1}} \right) \right\},$$

$$E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial c \partial A_1} \right]$$

$$= \left(\frac{1}{c} \right) \cdot \left\{ k(kA_1) \right.$$

$$\times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_{1j} - \mu)/\sigma) e^{2((t_{1j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{1j} - \mu)/\sigma})^{kA_1+2}} \right) dt_{1j} \\ \left. + k \left(\frac{((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1+1}} \right) \right\},$$

$$E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial \alpha^2} \right]$$

$$\begin{aligned}
&= \left(\frac{c}{\alpha}\right)^2 \cdot \left\{ kA_1 (kA_1 + 1) \right. \\
&\quad \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{e^{2((t_{1j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{1j} - \mu)/\sigma})^{kA_1 + 3}} \right) dt_{1j} \\
&\quad \left. + kA_1 \left(\frac{e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 + 2}} \right) \right\}, \\
&E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial c \partial \alpha} \right] \\
&= \left(-\frac{1}{\alpha}\right) \cdot \left\{ kA_1 (kA_1 + 1) \right. \\
&\quad \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_{1j} - \mu)/\sigma) e^{2((t_{1j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{1j} - \mu)/\sigma})^{kA_1 + 3}} \right) dt_{1j} \\
&\quad \left. + kA_1 \frac{((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 + 2}} \right\}, \\
&E \left[-\frac{\partial^2 \ln L_{a_{1j}}}{\partial c^2} \right] \\
&= \left(\frac{1}{c^2}\right) \cdot \left\{ kA_1 (kA_1 + 1) \right. \\
&\quad \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_{1j} - \mu)/\sigma)^2 e^{2((t_{1j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{1j} - \mu)/\sigma})^{kA_1 + 3}} \right) dt_{1j} \\
&\quad + \left(1 - (1 + e^{(\eta' - \mu)/\sigma})^{-kA_1}\right) \\
&\quad \left. + kA_1 \frac{((\eta' - \mu)/\sigma)^2 e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 + 2}} \right\}. \\
&E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_1^2} \right] = \left(\frac{1}{A_1}\right)^2 \cdot \left(1 - (1 + e^{(\eta' - \mu)/\sigma})^{-kA_1 A_2}\right), \\
&E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_2 \partial A_1} \right] \\
&= \left(\frac{1}{c}\right) \\
&\quad \cdot \left\{ k(kA_1 A_2) \right.
\end{aligned}$$

$$\begin{aligned}
&\times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{\ln(1 + e^{(t_{2j} - \mu)/\sigma}) e^{(t_{2j} - \mu)/\sigma}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 1}} \right) dt_{2j} \\
&\quad + k \left(\frac{\ln(1 + e^{(\eta' - \mu)/\sigma}) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2}} \right) \Bigg\}, \\
&E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial \alpha \partial A_1} \right] \\
&= \left(-\frac{c}{\alpha}\right) \cdot \left\{ kA_2 (kA_1 A_2) \right. \\
&\quad \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{e^{2((t_{2j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 2}} \right) dt_{2j} \\
&\quad \left. + kA_2 \left(\frac{e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2 + 1}} \right) \right\}, \\
&E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c \partial A_1} \right] \\
&= \left(\frac{1}{c}\right) \cdot \left\{ kA_2 (kA_1 A_2) \right. \\
&\quad \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_{2j} - \mu)/\sigma) e^{2((t_{2j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 2}} \right) dt_{2j} \\
&\quad \left. + kA_2 \left(\frac{((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2 + 1}} \right) \right\}, \\
&E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial A_2^2} \right] = \left(\frac{1}{A_2}\right)^2 \cdot \left(1 - (1 + e^{(\eta' - \mu)/\sigma})^{-kA_1 A_2}\right), \\
&E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial \alpha \partial A_2} \right] \\
&= \left(-\frac{c}{\alpha}\right) \cdot \left\{ kA_1 (kA_1 A_2) \right. \\
&\quad \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{e^{2((t_{2j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 2}} \right) dt_{2j} \\
&\quad \left. + kA_1 \left(\frac{e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2 + 1}} \right) \right\}, \\
&E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c \partial A_2} \right]
\end{aligned}$$

$$= \left(\frac{1}{c}\right) \cdot \left\{ kA_1 (kA_1 A_2) \right. \\ \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_{2j} - \mu)/\sigma) e^{2((t_{2j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 2}} \right) dt_{2j} \\ \left. + kA_1 \left(\frac{((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2 + 1}} \right) \right\}, \\ E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial \alpha^2} \right] \\ = \left(\frac{c}{\alpha}\right)^2 \cdot \left\{ kA_1 A_2 (kA_1 A_2 + 1) \right. \\ \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{e^{2((t_{2j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 3}} \right) dt_{2j} \\ \left. + kA_1 A_2 \left(\frac{e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2 + 2}} \right) \right\}, \\ E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c \partial \alpha} \right] \\ = \left(-\frac{1}{\alpha}\right) \cdot \left\{ kA_1 A_2 (kA_1 A_2 + 1) \right. \\ \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_{2j} - \mu)/\sigma) e^{2((t_{2j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 3}} \right) dt_{2j} \\ \left. + \frac{kA_1 A_2 ((\eta' - \mu)/\sigma) e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2 + 2}} \right\}, \\ E \left[-\frac{\partial^2 \ln L_{a_{2j}}}{\partial c^2} \right] \\ = \left(\frac{1}{c^2}\right) \cdot \left\{ kA_1 A_2 (kA_1 A_2 + 1) \right. \\ \times \int_{-\infty}^{(\eta' - \mu)/\sigma} \left(\frac{((t_{2j} - \mu)/\sigma)^2 e^{2((t_{2j} - \mu)/\sigma)}}{\sigma(1 + e^{(t_{2j} - \mu)/\sigma})^{kA_1 A_2 + 3}} \right) dt_{2j} \\ \left. + \left(1 - \left(1 + e^{(\eta' - \mu)/\sigma} \right)^{-kA_1 A_2} \right) \right. \\ \left. + \frac{kA_1 A_2 ((\eta' - \mu)/\sigma)^2 e^{(\eta' - \mu)/\sigma}}{(1 + e^{(\eta' - \mu)/\sigma})^{kA_1 A_2 + 2}} \right\}. \quad (A.1)$$

B.

Using delta method, V in (20) has been obtained.

Let

$$\frac{V_1}{n} = \text{Asvar}(\hat{\mu}_0) = \text{Asvar}(\ln(\alpha)) \quad (B.1)$$

$$= \text{Asvar}[\hat{h}_1(\hat{A}_1, \hat{A}_2, \hat{\alpha}, \hat{c})] = \hat{H}_1' F^{-1} \hat{H}_1,$$

where $\hat{H}_1'$ is a transpose of $\hat{H}_1$ and

$$\hat{H}_1 = \left[\frac{\partial \hat{h}_1}{\partial \hat{A}_1} \quad \frac{\partial \hat{h}_1}{\partial \hat{A}_2} \quad \frac{\partial \hat{h}_1}{\partial \hat{\alpha}} \quad \frac{\partial \hat{h}_1}{\partial \hat{c}} \right]'. \quad (B.2)$$

Further, let

$$\frac{V_2}{n} = \text{Ascov}(\hat{\mu}_0, \hat{\sigma}_0) = \text{Ascov}(\ln(\alpha), (c)^{-1}) \\ = \text{Ascov}[\hat{h}_1(\hat{A}_1, \hat{A}_2, \hat{\alpha}, \hat{c}), \hat{h}_2(\hat{A}_1, \hat{A}_2, \hat{\alpha}, \hat{c})], \quad (B.3)$$

$$\frac{V_3}{n} = \text{Asvar}(\hat{\sigma}_0) = \text{Asvar}((c)^{-1}) \\ = \text{Asvar}[\hat{h}_2(\hat{A}_1, \hat{A}_2, \hat{\alpha}, \hat{c})] = \hat{H}_2' F^{-1} \hat{H}_2,$$

where $\hat{H}_2'$ is a transpose of $\hat{H}_2$ and

$$\hat{H}_2 = \left[\frac{\partial \hat{h}_2}{\partial \hat{A}_1} \quad \frac{\partial \hat{h}_2}{\partial \hat{A}_2} \quad \frac{\partial \hat{h}_2}{\partial \hat{\alpha}} \quad \frac{\partial \hat{h}_2}{\partial \hat{c}} \right]'. \quad (B.4)$$

Then,

$$\text{Asvar}(T') = \frac{(V_1 - 2kV_2 + k^2V_3)}{n} = \frac{V}{n}. \quad (B.5)$$

Acronyms

ALT:	Accelerated life test
ALTSP:	Accelerated life test sampling plan
PALT:	Partially accelerated life test
PALTSP:	Partially accelerated life test sampling plan
Asvar:	Asymptotic variance
Ascov:	Asymptotic covariance
cdf:	Cumulative distribution function
pdf:	Probability density function
MLE:	Maximum likelihood estimate.

Notations

A_1, A_2 :	Acceleration factors ($A_2 > A_1 > 1$)
A_i^* :	$\prod_{k=1}^i A_k, i = 1, 2$
ϕ_1, ϕ_2, ϕ_3 :	Proportion of sample allocated to normal operating condition, accelerated condition A_1^* , and accelerated condition A_2^* , respectively, where $\sum_{i=1}^3 \phi_i = 1$
ϕ_i^* :	Optimum proportion of sample allocated to normal operating condition, accelerated condition A_1 , and accelerated condition A_2 , respectively, where $\sum_{i=1}^3 \phi_i^* = 1$
n :	Total number of test items in a PALTSP
$n\phi_i$:	Items allocated to normal operating condition, accelerated condition A_1 , and accelerated condition A_2 , respectively, $i = 1, 2, 3$
n_u, n_{a_1}, n_{a_2} :	Number of items failed at normal operating condition, accelerated condition A_1 , and accelerated condition A_2^* , respectively
$n_{u_c}, n_{a_{1c}}, n_{a_{2c}}$:	Number of censored items at normal operating condition, accelerated condition A_1 , and accelerated condition A_2 , respectively
Y, X_1 & X_2 :	Lifetime of an item at normal operating condition, accelerated condition A_1 , and accelerated condition A_2 , respectively
T :	$\ln Y$
t_i :	Observed log-lifetime of i th item tested at normal operating condition
t_{1j} :	Observed log-lifetime of j th item tested at accelerated condition A_1
t_{2j} :	Observed log-lifetime of j th item tested at accelerated condition A_2
η :	Censoring time
η' :	Log-censoring time
α :	Scale parameter ($\alpha > 0$)
c, k :	Shape parameter ($c > 0, k > 0$)
$F(\cdot)$:	Cumulative distribution function
$f_1(\cdot)$:	Probability density function at normal operating condition
$R_1(\cdot)$:	Reliability function at normal operating condition
$h_1(\cdot)$:	Hazard (failure) rate at normal operating condition
$f_i(\cdot)$:	Probability density function at accelerated conditions $A_i, i = 1, 2$
$R_i(\cdot)$:	Reliability function at accelerated conditions $A_i, i = 1, 2$
$h_i(\cdot)$:	Hazard (failure) rate at accelerated conditions $A_i, i = 1, 2$
$\wedge$:	A maximum likelihood estimate

$\delta_{u_i}, \delta_{a_1}, \delta_{a_{2j}}$:	Indicator functions: $\delta_{u_i} = \{1, T_i \leq \eta', i = 1, 2, \dots, n_u, 0, \text{ otherwise}\};$ $\delta_{a_{1j}} = \{1, T_{1j} \leq \eta', j = 1, 2, \dots, n_{a_1}, 0, \text{ otherwise}\}$ and $\delta_{a_{2j}} = \{1, T_{2j} \leq \eta', j = 1, 2, \dots, n_{a_2}, 0, \text{ otherwise}\}.$
$t_{(1)} < t_{(2)} < \dots < t_{(n_u)} < \eta'$:	Ordered log-lifetimes times at normal operating condition
$t_{(11)} < t_{(12)} < \dots < t_{(1n_{a_1})} < \eta'$:	Ordered log-lifetimes at accelerated condition A_1
$t_{(21)} < t_{(22)} < \dots < t_{(2n_{a_2})} < \eta'$:	Ordered log-lifetimes at accelerated condition A_2 .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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