Research Article

Dynamic Behaviour under Moving Distributed Masses of Nonuniform Rayleigh Beam with General Boundary Conditions

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Received 3 December 2013; Accepted 23 January 2014; Published 23 March 2014

Academic Editors: B. Sun and J. Sun

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This paper investigates the flexural vibration of a finite nonuniform Rayleigh beam resting on an elastic foundation and under travelling distributed loads. For the solution of this problem, in the first instance, the generalized Galerkin method was used. The resulting Galerkin's equations were then simplified using the modified asymptotic method of Struble. The simplified second-order ordinary differential equation was then solved using the method of integral transformation. The closed form solution obtained was analyzed and results show that, an increase in the values of foundation moduli K and rotatory inertia correction factor R_0 reduces the response amplitudes of both the clamped-clamped nonuniform Rayleigh beam and the clamped-free nonuniform Rayleigh beam. Also for the same natural frequency, the critical speed for the moving distributed mass problem is smaller than that for the moving distributed force problem. Hence resonance is reached earlier in the former. Furthermore, resonance conditions for the dynamical system are attained significantly by both R_0 and K for the illustrative end conditions considered.

1. Introduction

This paper is sequel to an earlier one by Oni and Ayankop-Andi in [1] that considered the response of a simply supported nonuniform Rayleigh beam to travelling distributed loads. In particular, this paper is a generalization of the theory advanced in [1]. For more than a century, the analysis of continuous elastic system subjected to moving systems has been the subject of interest in many fields, from structural to mechanical to aerospace engineering. Various structures ranging from bridges and roads to space vehicles and submarines are constantly acted upon by moving masses and hence the problem of analyzing the dynamic response of these structures under the action of moving masses continues to motivate a variety of investigations. In most of the studies available in literature, such as the works of Sadiku and Leipholz in [2], Oni in [3], Gbadeyan and Oni in [4], Huang and Thamiraratnam in [5], Lee and Ng in [6], Adams in [7], Chen and Li in [8], Savin in [9], Rao in [10], Shadnam et al. in [11], and Oni and Awodola in [12], the scope has been restricted to structural members having uniform cross-section whether the inertia of the moving load is considered or not and the load modelled as moving concentrated load.

In practice, cross-sections of elastic structures such as plates and beams are not usually uniform and the moving loads are commonly in distributed forms. To this end, in [13] an attempt was made on the studies concerning nonuniform structural members subjected to moving loads. In particular, he investigated the response of nonuniform beams resting on elastic foundation to several moving masses. He found that the response amplitudes of both moving force and moving mass problems decrease with increasing foundation constant and that the maximum transverse deflection of the beam is always greater than the displacement of the moving concentrated mass. For both clamped-clamped and clamped-free nonuniform Rayleigh beam under moving distributed loads, the response amplitudes of the beam not resting on elastic subgrade are greater than that of the beam supported by a subgrade. For these boundary conditions, the response amplitudes are lower than those of the simply supported boundary conditions. Other researchers who have worked on transverse motions under moving load of beams with nonuniform cross-sections include [14–17]. However, the scope of their consideration does not cover the wide range of application areas often encountered in engineering practice.

They simplified their investigation by modelling their loads as concentrated line loads.

Emphatically speaking, in order to accurately model such physical situations in realistic manners, an accurate representation of the moving load is that it is distributed over a portion or over the entire length of the structure. In the light of this, recently, some researchers have addressed this shortcoming. Among these are Esmailzadeh and Ghorashi in [18], Gbadeyan and Dada in [19], and Bogacz and Czyczula in [20], to mention a few who have tackled dynamical problems involving the response of elastic structures subjected to partially distributed moving masses. However their method of solution is either restricted to numerical development or suitable only for simply supported end conditions or both. In the context of this, this present paper considers the dynamic behaviour under moving distributed masses of nonuniform Rayleigh beams with general boundary conditions. The interest is in analytical development and as such often sheds valuable light on vital information about the physical system.

2. The Governing Equation

The nonuniform Rayleigh beam is a spatially one-dimensional elastic system. Thus, the problem of vibration of the finite nonuniform Rayleigh beam of length L , on a Winkler foundation and traversed by a travelling distributed load whose load inertia is not negligible with damping neglected, is governed by the fourth-order partial differential equation given in [21] by Fryba as follows:

$$\begin{aligned} & \frac{\partial^2}{\partial x^2} [EI(x)] \frac{\partial^2 V(x, t)}{\partial x^2} + \mu(x) \frac{\partial^2 V(x, t)}{\partial t^2} \\ & - \mu(x) R_0 \frac{\partial^4 V(x, t)}{\partial x^2 \partial t^2} + KV(x, t) \\ & = MgH(x - ct) \left[1 - \frac{1}{g} \frac{d^2 V(x, t)}{dt^2} \right], \end{aligned} \quad (1)$$

where x is the spatial coordinate, t is the time, $V(x, t)$ is the transverse displacement, E is Young's modulus, R_0 is the measure of rotatory inertia correction factor, and K is the elastic foundation moduli and the beam properties such as the moment of inertia $I(x)$ and mass per unit length $\mu(x)$ of the beam are considered as varying along the length L of the beam. c is the velocity of the distributed mass M , and the time t is assumed to be limited to that interval of time within which the mass μ is on the beam; that is,

$$0 \leq ct \leq L. \quad (2)$$

g is the acceleration due to gravity, and $H(x - ct)$ is the Heaviside function defined as

$$H(x - ct) = \begin{cases} 0, & x < ct \\ 1, & x > ct \end{cases} \quad (3)$$

with the following properties:

$$\begin{aligned} (i) \quad & \frac{d}{dx} \{H(x - ct)\} = \delta(x - ct) \\ (ii) \quad & f(x) H(x - ct) = \begin{cases} 0, & x < ct \\ f(x), & x > ct, \end{cases} \end{aligned} \quad (4)$$

where $\delta(x - ct)$ represents the Dirac delta function defined as follows:

$$(i) \quad \delta(x - ct) = \begin{cases} 0, & x \neq ct \\ \infty, & x = ct, \end{cases} \quad (5)$$

while

$$(ii) \quad \delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0, \end{cases} \quad (6)$$

with the following properties:

$$\begin{aligned} (i) \quad & \delta(-x) = \delta(x) \\ (ii) \quad & \int_a^b \delta(x - k) f(x) dx = \begin{cases} 0, & k < a < b \\ f(k), & a < k < b \\ 0, & a < b < k. \end{cases} \end{aligned} \quad (7)$$

d^2/dt^2 is the convective acceleration operator defined as follows:

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2c \frac{\partial^2}{\partial x \partial t} + c^2 \frac{\partial^2}{\partial x^2}. \quad (8)$$

when the mass M moves with a constant velocity.

The boundary conditions of the above problem are assumed to be general, while the initial conditions are

$$V(x, 0) = V_t(x, 0) = 0. \quad (9)$$

Adopting examples in [22], $I(x)$ and $\mu(x)$ are given by Williams as

$$\begin{aligned} I(x) &= I_0 \left(1 + \sin \frac{\pi x}{L} \right)^3 \\ \mu(x) &= \mu_0 \left(1 + \sin \frac{\pi x}{L} \right). \end{aligned} \quad (10)$$

3. Method of Solution

Substituting (8), (10) into (1), after some simplifications and rearrangement, one obtains

$$\begin{aligned}
 & EI_0 \left[\frac{\partial^4 V(x, t)}{\partial x^4} \left\{ \frac{5}{2} + \frac{15}{4} \sin \frac{\pi x}{L} - \frac{1}{4} \sin \frac{3\pi x}{L} \right. \right. \\
 & \quad \left. \left. - \frac{3}{2} \cos \frac{2\pi x}{L} \right\} + \frac{\partial^2 V(x, t)}{\partial x^2} \right. \\
 & \quad \left. \times \left\{ \frac{9\pi^2}{4L^2} \sin \frac{3\pi x}{L} - \frac{15\pi^2}{4L^2} \sin \frac{\pi x}{L} + \frac{6\pi^2}{L^2} \cos \frac{2\pi x}{L} \right\} \right] \\
 & + \mu_0 \left(1 + \sin \frac{\pi x}{L} \right) \frac{\partial^2 V(x, t)}{\partial t^2} - \mu_0 R_0 \\
 & \times \left[\left(1 + \sin \frac{\pi x}{L} \right) \frac{\partial^4 V(x, t)}{\partial x^2 \partial t^2} + \frac{\pi}{L} \cos \frac{\pi x}{L} \cdot \frac{\partial^3 V(x, t)}{\partial x \partial t^2} \right] \\
 & + KV(x, t) + MH(x - ct) \left[\frac{\partial^2 V(x, t)}{\partial t^2} + 2c \frac{\partial^2 V(x, t)}{\partial x \partial t} \right. \\
 & \quad \left. + c^2 \frac{\partial^2 V(x, t)}{\partial x^2} \right] \\
 & = MgH(x - ct). \tag{11}
 \end{aligned}$$

Equation (11) is the fourth-order partial differential equation with variable coefficients of the nonuniform Rayleigh beam under the action of travelling distributed loads. It is evident that exact closed form solution to this equation is impossible. To this end, an approximate analytical solution is desirable. An elegant approximation technique due to Galerkin referred to as Generalized Galerkin Method (GGM) discussed in [3] is employed to simplify and reduce (11) to a sequence of second-order ordinary differential equations called Galerkin equations. This technique requires that solution to (11) be written in the following form:

$$V_n(x, t) = \sum_{m=1}^n W_m(t) U_m(x), \tag{12}$$

where $U_m(x)$ is chosen such that the pertinent boundary conditions are satisfied. Equation (12) when substituted into (11) yields

$$\begin{aligned}
 & \sum_{m=1}^n \left\{ \frac{EI_0}{\mu_0} \left[\left(\frac{5}{2} + \frac{15}{4} \sin \frac{\pi x}{L} - \frac{1}{4} \sin \frac{3\pi x}{L} - \frac{3}{2} \cos \frac{2\pi x}{L} \right) \right. \right. \\
 & \quad \times U_m^{IV}(x) + \left(\frac{9\pi^2}{4L^2} \sin \frac{3\pi x}{L} - \frac{15\pi^2}{4L^2} \sin \frac{\pi x}{L} \right. \\
 & \quad \left. \left. + \frac{6\pi^2}{L^2} \cos \frac{2\pi x}{L} \right) U_m''(x) \right] \\
 & \quad \times W_m(t) + \left(1 + \sin \frac{\pi x}{L} \right) U_m(x) \ddot{W}_m(t)
 \end{aligned}$$

$$\begin{aligned}
 & - R_0 \left[\left(1 + \sin \frac{\pi x}{L} \right) U_m''(x) + \left(\frac{\pi}{L} \cos \frac{\pi x}{L} \right) U_m'(x) \right] \\
 & \times \ddot{W}_m(t) + \frac{K}{\mu_0} W_m(t) U_m(x) + \frac{MH(x - ct)}{\mu_0} \\
 & \times \left[\ddot{W}_m(t) U_m(x) + 2c \dot{W}_m(t) U_m'(x) \right. \\
 & \quad \left. + c^2 W_m(t) U_m''(x) \right] \Bigg\} - \frac{Mg}{\mu_0} H(x - ct) = 0. \tag{13}
 \end{aligned}$$

In order to determine $W_m(t)$, it is required that the expression on the left-hand side of (13) be orthogonal to the function $U_k(x)$. Hence,

$$\begin{aligned}
 & \int_0^L \left(\sum_{m=1}^n \left\{ \frac{EI_0}{\mu_0} \left[\left(\frac{5}{2} + \frac{15}{4} \sin \frac{\pi x}{L} - \frac{1}{4} \sin \frac{3\pi x}{L} \right. \right. \right. \right. \\
 & \quad \left. \left. - \frac{3}{2} \cos \frac{2\pi x}{L} \right) U_m^{IV}(x) \right. \right. \\
 & \quad \left. \left. + \left(\frac{9\pi^2}{4L^2} \sin \frac{3\pi x}{L} - \frac{15\pi^2}{4L^2} \sin \frac{\pi x}{L} \right. \right. \right. \\
 & \quad \left. \left. + \frac{6\pi^2}{L^2} \cos \frac{2\pi x}{L} \right) U_m''(x) \right] \right. \\
 & \quad \times W_m(t) + \left(1 + \sin \frac{\pi x}{L} \right) U_m(x) \ddot{W}_m(t) - R_0 \\
 & \quad \times \left[\left(1 + \sin \frac{\pi x}{L} \right) U_m''(x) + \left(\frac{\pi}{L} \cos \frac{\pi x}{L} \right) U_m'(x) \right] \\
 & \quad \times \ddot{W}_m(t) + \frac{K}{\mu_0} W_m(t) U_m(x) + \frac{MH(x - ct)}{\mu_0} \\
 & \quad \times \left[\ddot{W}_m(t) U_m(x) + 2c \dot{W}_m(t) U_m'(x) \right. \\
 & \quad \left. + c^2 W_m(t) U_m''(x) \right] \Bigg\} - \frac{Mg}{\mu_0} H(x - ct) \Bigg) \\
 & \times U_k(x) dx = 0. \tag{14}
 \end{aligned}$$

A rearrangement of the above equation yields

$$\begin{aligned}
 & \sum_{m=1}^n \left\{ [(Q_1(m, k) + Q_3(m, k)) \right. \\
 & \quad \left. - R_0(Q_4(m, k) + Q_2(m, k) \right. \\
 & \quad \left. + Q_5(m, k))] \ddot{W}_m(t) \right. \\
 & \quad \left. + \left[P_1(T_0 + T_1) + \frac{K}{\mu_0} Q_1(m, k) \right] W_m(t) + \frac{M}{\mu_0} \right. \\
 & \quad \times [Q_1(t) \ddot{W}_m(t) + 2c Q_2(t) \dot{W}_m(t) \\
 & \quad \left. + c^2 Q_3(t) W_m(t) \right] \Bigg\} - \frac{Mg}{\mu_0} Q_4(t) = 0, \tag{15}
 \end{aligned}$$

where

$$T_0 = Q_9(m, k) + Q_{10}(m, k) - (Q_{11}(m, k) + Q_{12}(m, k))$$

$$T_1 = Q_6(m, k) + Q_7(m, k) - Q_8(m, k),$$

$$P_1 = \frac{EI_0}{\mu_0}$$

$$Q_1(t) = \int_0^L H(x - ct) U_m(x) U_k(x) dx$$

$$Q_2(t) = \int_0^L H(x - ct) U'_m(x) U_k(x) dx$$

$$Q_3(t) = \int_0^L H(x - ct) U''_m(x) U_k(x) dx$$

$$Q_4(t) = \int_0^L H(x - ct) U_k(x) dx$$

$$Q_1(m, k) = \int_0^L U_m(x) U_k(x) dx$$

$$Q_2(m, k) = \int_0^L U''_m(x) U_k(x) dx$$

$$Q_3(m, k) = \int_0^L \sin \frac{\pi x}{L} U_m(x) U_k(x) dx$$

$$Q_4(m, k) = \frac{\pi}{L} \int_0^L \cos \frac{\pi x}{L} U'_m(x) U_k(x) dx$$

$$Q_5(m, k) = \int_0^L \sin \frac{\pi x}{L} U''_m(x) U_k(x) dx$$

$$Q_6(m, k) = \frac{9\pi^2}{4L^2} \int_0^L \sin \frac{3\pi x}{L} U''_m(x) U_k(x) dx$$

$$Q_7(m, k) = \frac{6\pi^2}{L^2} \int_0^L \cos \frac{2\pi x}{L} U''_m(x) U_k(x) dx$$

$$Q_8(m, k) = \frac{15\pi^2}{4L^2} \int_0^L \sin \frac{\pi x}{L} U''_m(x) U_k(x) dx$$

$$Q_9(m, k) = \frac{5}{2} \int_0^L U'^{\nu}_m(x) U_k(x) dx$$

$$Q_{10}(m, k) = \frac{15}{4} \int_0^L \sin \frac{\pi x}{L} U'^{\nu}_m(x) U_k(x) dx$$

$$Q_{11}(m, k) = \frac{1}{4} \int_0^L \sin \frac{3\pi x}{L} U'^{\nu}_m(x) U_k(x) dx$$

$$Q_{12}(m, k) = \frac{3}{2} \int_0^L \cos \frac{2\pi x}{L} U'^{\nu}_m(x) U_k(x) dx.$$

(16)

The ordinary differential equation (15) is valid for all variants of classical boundary conditions. Using the Fourier series representation of the Heaviside unit step function, namely,

$$H(x - ct)$$

$$= \frac{1}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi(x-ct))}{2n+1}, \quad 0 < x < L$$

(17)

in (15) and substituting (17) into (15) and simplifying yield

$$\begin{aligned} & \sum_{m=1}^n \left\{ [(Q_1(m, k) + Q_3(m, k)) - R_0(Q_4(m, k) + Q_2(m, k) \right. \\ & \quad \left. + Q_5(m, k))] \ddot{W}_m(t) \right. \\ & \quad + \left[P_1(T_0 + T_1) + \frac{K}{\mu_0} Q_1(m, k) \right] W_m(t) + \frac{M}{\mu_0} \\ & \quad \times \left[\left(Q_{1A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{1B}(n, m, k) \right. \right. \\ & \quad \left. \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{1C}(n, m, k) \right) \ddot{W}_m(t) \right. \\ & \quad + 2c \left(Q_{2A}(m, k) \right. \\ & \quad \left. + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{2B}(n, m, k) \right. \\ & \quad \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{2C}(n, m, k) \right) \dot{W}_m(t) \\ & \quad \left. + c^2 \left(Q_{3A}(m, k) \right. \right. \\ & \quad \left. \left. + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{3B}(n, m, k) \right. \right. \\ & \quad \left. \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{3C}(n, m, k) \right) \right. \\ & \quad \left. \times W_m(t) \right\} = \frac{Mg}{\mu_0} Q_4(t), \end{aligned} \quad (18)$$

where

$$Q_{1A}(m, k) = \frac{1}{4} \int_0^L U_m(x) U_k(x) dx$$

$$Q_{1B}(n, m, k) = \int_0^L \sin(2n+1)\pi x U_m(x) U_k(x) dx$$

$$Q_{1C}(n, m, k) = \int_0^L \cos(2n+1)\pi x U_m(x) U_k(x) dx$$

$$Q_{2A}(m, k) = \frac{1}{4} \int_0^L U'_m(x) U_k(x) dx$$

$$Q_{2B}(n, m, k) = \int_0^L \sin(2n+1)\pi x U'_m(x) U_k(x) dx$$

$$Q_{2C}(n, m, k) = \int_0^L \cos(2n+1)\pi x U'_m(x) U_k(x) dx$$

$$Q_{3A}(m, k) = \frac{1}{4} \int_0^L U''_m(x) U_k(x) dx$$

$$\begin{aligned}
Q_{3B}(n, m, k) &= \int_0^L \sin(2n+1)\pi x U_m''(x) U_k(x) dx \\
Q_{3C}(n, m, k) &= \int_0^L \cos(2n+1)\pi x U_m''(x) U_k(x) dx.
\end{aligned} \tag{19}$$

In order to solve this equation, an appropriate selection of function is the following beam function $U_m(x)$:

$$\begin{aligned}
U_m(x) &= \sin \frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} + B_m \sinh \frac{\lambda_m x}{L} \\
&\quad + C_m \cosh \frac{\lambda_m x}{L},
\end{aligned} \tag{20}$$

so that

$$\begin{aligned}
U_k(x) &= \sin \frac{\lambda_k x}{L} + A_k \cos \frac{\lambda_k x}{L} + B_k \sinh \frac{\lambda_k x}{L} \\
&\quad + C_k \cosh \frac{\lambda_k x}{L}.
\end{aligned} \tag{21}$$

The constants $A_m, B_m, C_m, A_k, B_k, C_k$ and the mode frequencies λ_m, λ_k are usually determined by using the appropriate classical boundary conditions. Substituting (20) and (21) into (18), after some simplification and rearrangement, one obtains

$$\begin{aligned}
&\sum_{m=1}^n \left\{ \Delta_1(m, k) \ddot{W}_m(t) + \Delta_2(m, k) W_m(t) + \epsilon_a L \right. \\
&\quad \times \left[\left(Q_{1A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{1B}(n, m, k) \right. \right. \\
&\quad \left. \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{1C}(n, m, k) \right) \ddot{W}_m(t) + 2c \right. \\
&\quad \times \left(Q_{2A}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{2B}(n, m, k) \right. \\
&\quad \left. \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{2C}(n, m, k) \right) \dot{W}_m(t) \right. \\
&\quad \left. + c^2 \left(Q_{3A}(m, k) \right. \right. \\
&\quad \left. \left. + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{3B}(n, m, k) \right. \right. \\
&\quad \left. \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{3C}(n, m, k) \right) \right. \\
&\quad \left. \times W_m(t) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\epsilon_a g L^2}{\lambda_k} \left[B_f(\lambda, k) + \cos \frac{\lambda_k ct}{L} - A_k \sin \frac{\lambda_k ct}{L} \right. \\
&\quad \left. - B_k \cosh \frac{\lambda_k ct}{L} - C_k \sinh \frac{\lambda_k ct}{L} \right],
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\Delta_1(m, k) &= Q_1(m, k) + Q_3(m, k) \\
&\quad - R_0(Q_4(m, k) + Q_2(m, k) + Q_5(m, k)) \\
\Delta_2(m, k) &= P_1(T_0 + T_1) + \frac{K}{\mu_0} Q_1(m, k), \\
\epsilon_a &= \frac{M}{\mu_0 L}
\end{aligned}$$

$$B_f(\lambda, k) = -\cos \lambda_k + A_k \sin \lambda_k + B_k \cosh \lambda_k + C_k \sinh \lambda_k. \tag{23}$$

Equation (22) is now the fundamental equation governing the problem of flexural vibrations of a finite nonuniform Rayleigh beam under travelling distributed loads. In what follows, two special cases of (22) are discussed.

4. Closed Form Solution

4.1. Case I. The differential equation describing the flexural vibrations of a finite nonuniform Rayleigh beam subjected to a moving distributed force may be obtained from (22) by setting $\epsilon_a = 0$. In this case, we obtain

$$\begin{aligned}
&\sum_{m=1}^n \left\{ \Delta_1(m, k) \ddot{W}_m(t) + \Delta_2(m, k) W_m(t) \right\} \\
&= \frac{MgL}{\mu_0 \lambda_k} \left[B_f(\lambda, k) + \cos \frac{\lambda_k ct}{L} - A_k \sin \frac{\lambda_k ct}{L} \right. \\
&\quad \left. - B_k \cosh \frac{\lambda_k ct}{L} - C_k \sinh \frac{\lambda_k ct}{L} \right].
\end{aligned} \tag{24}$$

Rearranging (24) and ignoring the summation sign yield

$$\begin{aligned}
&\ddot{W}_m(t) + \beta_m^2 W_m(t) \\
&= P_0 \left[B_f(\lambda, k) + \cos \frac{\lambda_k ct}{L} - A_k \sin \frac{\lambda_k ct}{L} \right. \\
&\quad \left. - B_k \cosh \frac{\lambda_k ct}{L} - C_k \sinh \frac{\lambda_k ct}{L} \right],
\end{aligned} \tag{25}$$

where

$$\beta_m^2 = \frac{\Delta_2(m, k)}{\Delta_1(m, k)}, \quad P_0 = \frac{MgL}{\mu_0 \lambda_k \Delta_1(m, k)}. \tag{26}$$

Solving (25) in conjunction with the initial condition, the solution is given by

$$\begin{aligned}
 W_m(t) = P_0 \left\{ B_f(\lambda, k) \left(\frac{1 - \cos \beta_m t}{\beta_m} \right) \right. \\
 + \frac{\cos \alpha_c t - \cos \beta_m t}{\beta_m^2 - \alpha_c^2} \\
 - A_k \left(\frac{\beta_m \sin \alpha_c t - \alpha_c \sin \beta_m t}{\beta_m (\beta_m^2 - \alpha_c^2)} \right) \\
 - B_k \left(\frac{\cosh \alpha_c t - \cos \beta_m t}{\alpha_c^2 + \beta_m^2} \right) \\
 \left. - C_k \left(\frac{\beta_m \sinh \alpha_c t - \alpha_c \sin \beta_m t}{\beta_m (\alpha_c^2 + \beta_m^2)} \right) \right\}, \quad (27)
 \end{aligned}$$

where

$$\alpha_c = \frac{\lambda_k c}{L}. \quad (28)$$

Thus using (27) in (12), one obtains

$$\begin{aligned}
 V_n(x, t) = \sum_{m=1}^{\infty} P_0 \left\{ B_f(\lambda, k) \left(\frac{1 - \cos \beta_m t}{\beta_m} \right) \right. \\
 + \frac{\cos \alpha_c t - \cos \beta_m t}{\beta_m^2 - \alpha_c^2} \\
 - A_k \left(\frac{\beta_m \sin \alpha_c t - \alpha_c \sin \beta_m t}{\beta_m (\beta_m^2 - \alpha_c^2)} \right) \\
 - B_k \left(\frac{\cosh \alpha_c t - \cos \beta_m t}{\alpha_c^2 + \beta_m^2} \right) \\
 \left. - C_k \left(\frac{\beta_m \sinh \alpha_c t - \alpha_c \sin \beta_m t}{\beta_m (\alpha_c^2 + \beta_m^2)} \right) \right\} \\
 \cdot \left\{ \sin \frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} \right. \\
 \left. + B_m \sinh \frac{\lambda_m x}{L} + C_m \cosh \frac{\lambda_m x}{L} \right\}. \quad (29)
 \end{aligned}$$

Equation (29) represents the transverse displacement response to a distributed force moving at constant velocity of a nonuniform Rayleigh beam having arbitrary end support conditions.

4.2. Case II. If the inertia term is retained, then $\epsilon_a \neq 0$. This is termed the moving mass problem. In this case, the solution to the entire equation (22) is required. As an exact solution to this problem is impossible, a modification of Struble's

technique discussed in [1] is employed. To this end, (22) is simplified and rearranged to take the form

$$\begin{aligned}
 \ddot{W}_m(t) + \frac{\epsilon_a L R_2(m, k, t)}{1 + \epsilon_a L R_1(m, k, t)} \dot{W}_m(t) \\
 + \frac{(\beta_m^2 + \epsilon_a L R_3(m, k, t))}{1 + \epsilon_a L R_1(m, k, t)} W_m(t) \\
 = \left(\frac{\epsilon_a L^2 g}{\lambda_k \Delta_1(m, k)} \right. \\
 \times \left(B_f(\lambda, k) + \cos \frac{\lambda_k c t}{L} \right. \\
 - A_k \sin \frac{\lambda_k c t}{L} - B_k \cosh \frac{\lambda_k c t}{L} \\
 \left. \left. - C_k \sinh \frac{\lambda_k c t}{L} \right) \right) \times (1 + \epsilon_a L R_1(m, k, t))^{-1}, \quad (30)
 \end{aligned}$$

where

$$\begin{aligned}
 R_1(m, k, t) &= Q_{1a}(m, k) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi c t}{2n+1} Q_{1b}(n, m, k) \\
 &\quad - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi c t}{2n+1} Q_{1c}(n, m, k), \\
 R_2(m, k, t) &= 2c Q_{2a}(m, k) + \frac{2c}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi c t}{2n+1} Q_{2b}(n, m, k) \\
 &\quad - \frac{2c}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi c t}{2n+1} Q_{2c}(n, m, k), \\
 R_3(m, k, t) &= c^2 Q_{3a}(m, k) + \frac{c^2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi c t}{2n+1} Q_{3b}(n, m, k) \\
 &\quad - \frac{c^2}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi c t}{2n+1} Q_{3c}(n, m, k),
 \end{aligned}$$

$$Q_{1a}(m, k) = \frac{Q_{1A}(m, k)}{\Delta_1(m, k)},$$

$$Q_{2a}(m, k) = \frac{Q_{2A}(m, k)}{\Delta_1(m, k)},$$

$$Q_{1b}(n, m, k) = \frac{Q_{1B}(n, m, k)}{\Delta_1(m, k)}$$

$$Q_{2b}(n, m, k) = \frac{Q_{2B}(n, m, k)}{\Delta_1(m, k)},$$

$$\begin{aligned}
Q_{3a}(m, k) &= \frac{Q_{3A}(m, k)}{\Delta_1(m, k)}, \\
Q_{3b}(n, m, k) &= \frac{Q_{3B}(n, m, k)}{\Delta_1(m, k)}, \\
Q_{1\bar{c}}(n, m, k) &= \frac{Q_{1C}(n, m, k)}{\Delta_1(m, k)}, \\
Q_{2\bar{c}}(n, m, k) &= \frac{Q_{2C}(n, m, k)}{\Delta_1(m, k)}, \\
Q_{3\bar{c}}(n, m, k) &= \frac{Q_{3C}(n, m, k)}{\Delta_1(m, k)}.
\end{aligned} \tag{31}$$

By means of this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the distributed moving mass. An equivalent free system operator defined by the modified frequency then replaces (30). Thus, the right-hand side of (30) is set to zero, and a parameter $\epsilon^* < 1$ is considered for any arbitrary mass ratio ϵ_a , defined as

$$\epsilon^* = \frac{\epsilon_a}{1 + \epsilon_a}. \tag{32}$$

Evidently,

$$\epsilon_a = \epsilon^* + o(\epsilon^*)^2, \tag{33}$$

which implies

$$\frac{1}{1 + \epsilon^* LR_1(m, k, t)} = 1 - \epsilon^* LR_1(m, k, t), \tag{33^*}$$

where

$$\begin{aligned}
&\left| \epsilon^* L \left(Q_{1a} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{1b} \right. \right. \\
&\quad \left. \left. - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{1\bar{c}} \right) \right| < 1.
\end{aligned} \tag{34}$$

When we set $\epsilon^* = 0$ a case corresponding to a situation when the inertia effect of the mass of the system is neglected is obtained and the solution of (30) can be written as

$$W_m(t) = \psi_m \cos(\beta_m t - \theta_m), \tag{35}$$

where ψ_m and θ_m are constants.

Since $\epsilon^* < 1$, Struble's technique requires that the asymptotic solution of the homogeneous part of (30) be of the form given in [23] by Nayfeh as

$$\begin{aligned}
W_m(t) &= \psi(m, t) \cos(\beta_m t - \theta(m, t)) \\
&\quad + \epsilon^* W_1(m, t) + o(\eta^2),
\end{aligned} \tag{36}$$

where $\psi(m, t)$ and $\theta(m, t)$ are slowly varying functions of time.

Substituting (36) and its derivatives into homogeneous part of (30) and taking into account (33), one obtains

$$\begin{aligned}
&-2\dot{\psi}(m, t) \beta_m \sin(\beta_m t - \theta(m, t)) \\
&+ 2\psi(m, t) \dot{\theta}(m, t) \beta_m \cos(\beta_m t - \theta(m, t)) \\
&- 2c\epsilon^* L Q_{2a}(m, k) \psi(m, t) \\
&\cdot \beta_m \sin(\beta_m t - \theta(m, t)) - \frac{2c\epsilon^* L \psi(m, t)}{\pi} \\
&\times \beta_m \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} \sin(\beta_m t - \theta(m, t)) \\
&\times Q_{2b}(n, m, k) + \frac{2c\epsilon^* L \psi(m, t)}{\pi} \beta_m \\
&\times \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} Q_{2\bar{c}}(n, m, k) \sin(\beta_m t - \theta(m, t)) \\
&- \beta_m^2 \epsilon^* L Q_{1a}(m, k) \psi(m, t) \\
&\cdot \cos(\beta_m t - \theta(m, t)) - \frac{\beta_m^2 \epsilon^* L \psi(m, t)}{\pi} \\
&\times \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{1b}(n, m, k) \cos(\beta_m t - \theta(m, t)) \\
&+ \frac{\epsilon^* L \psi(m, t)}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} \beta_m^2 \cos(\beta_m t - \theta(m, t)) \\
&\times Q_{1\bar{c}}(n, m, k) + c^2 \epsilon^* L Q_{3a}(m, k) \psi(m, t) \\
&\times \cos(\beta_m t - \theta(m, t)) \\
&+ \frac{c^2 \epsilon^* L \psi(m, t)}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} Q_{3b}(n, m, k) \\
&\cdot \cos(\beta_m t - \theta(m, t)) - \frac{c^2 \epsilon^* L \psi(m, t)}{\pi} \\
&\times \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} \cos(\beta_m t - \theta(m, t)) Q_{3\bar{c}}(n, m, k) = 0
\end{aligned} \tag{37}$$

retaining terms to $o(\epsilon^*)$.

The variational equations are obtained by equating the coefficients of $\sin(\beta_m t - \theta(m, t))$ and $\cos(\beta_m t - \theta(m, t))$ terms on both sides of (37) to zero. Thus, noting the following trigonometric identities:

$$\begin{aligned}
&\frac{\cos(2n+1)\pi ct}{2n+1} \sin[\beta_m t - \theta(m, t)] \\
&= \frac{1}{2} \sin \left[\frac{(2n+1)\pi ct}{2n+1} + \beta_m t - \theta(m, t) \right] \\
&\quad - \frac{1}{2} \sin \left[\frac{(2n+1)\pi ct}{2n+1} - \beta_m t + \theta(m, t) \right]
\end{aligned} \tag{38}$$

$$\begin{aligned} & \frac{\sin(2n+1)\pi ct}{2n+1} \sin[\beta_m t - \theta(m, t)] \\ &= \frac{1}{2} \cos\left[\frac{(2n+1)\pi ct}{2n+1} - \beta_m t + \theta(m, t)\right] \\ & \quad - \frac{1}{2} \cos\left[\frac{(2n+1)\pi ct}{2n+1} + \beta_m t - \theta(m, t)\right] \end{aligned} \quad (39)$$

$$\begin{aligned} & \frac{\cos(2n+1)\pi ct}{2n+1} \cos[\beta_m t - \theta(m, t)] \\ &= \frac{1}{2} \cos\left[\frac{(2n+1)\pi ct}{2n+1} + \beta_m t - \theta(m, t)\right] \\ & \quad + \frac{1}{2} \cos\left[\frac{(2n+1)\pi ct}{2n+1} - \beta_m t + \theta(m, t)\right] \end{aligned} \quad (40)$$

$$\begin{aligned} & \frac{\sin(2n+1)\pi ct}{2n+1} \cos[\beta_m t - \theta(m, t)] \\ &= \frac{1}{2} \sin\left[\frac{(2n+1)\pi ct}{2n+1} + \beta_m t - \theta(m, t)\right] \\ & \quad + \frac{1}{2} \sin\left[\frac{(2n+1)\pi ct}{2n+1} - \beta_m t + \theta(m, t)\right], \end{aligned} \quad (41)$$

and neglecting terms that do not contribute to the variational equations, (37) reduces to

$$\begin{aligned} & -2\dot{\psi}(m, t) \beta_m \sin(\beta_m t - \theta(m, t)) \\ & + 2\psi(m, t) \dot{\theta}(m, t) \beta_m \cos(\beta_m t - \theta(m, t)) \\ & - 2c\epsilon^* LQ_{2a}(m, k) \psi(m, t) \beta_m \\ & \times \sin(\beta_m t - \theta(m, t)) - \beta_m^2 \epsilon^* LQ_{1a}(m, k) \psi(m, t) \\ & \times \cos(\beta_m t - \theta(m, t)) \\ & + c^2 \epsilon^* LQ_{3a}(m, k) \psi(m, t) \cos(\beta_m t - \theta(m, t)) \\ & = 0. \end{aligned} \quad (42)$$

Then, the variational equations are, respectively,

$$-2\dot{\psi}(m, t) \beta_m - 2c\epsilon^* LQ_{2a}(m, k) \psi(m, t) \beta_m = 0, \quad (43)$$

$$\begin{aligned} & 2\psi(m, t) \dot{\theta}(m, t) \beta_m - \beta_m^2 \epsilon^* LQ_{1a}(m, k) \psi(m, t) \\ & + c^2 \epsilon^* LQ_{3a}(m, k) \psi(m, t) = 0. \end{aligned} \quad (44)$$

Solving (39) and (40), respectively, one obtains

$$\begin{aligned} \psi(m, t) &= \psi_m e^{-c\epsilon^* LQ_{2a}(m, k)t}, \\ \theta(m, t) &= \frac{\beta_m \epsilon^* LQ_{1a}(m, k)}{2} t - \frac{c^2 \epsilon^* LQ_{3a}(m, k)}{2\beta_m} t + \theta_m, \end{aligned} \quad (45)$$

where ψ_m and θ_m are constants.

Therefore, when the effects of the mass of the particle are considered, the first approximation to the homogeneous system is given as

$$W_m(t) = \psi(m, t) \cos(\omega_m t - \theta_m), \quad (46)$$

where

$$\omega_m = \frac{2\beta_m^2 - \epsilon^* L \{ \beta_m^2 Q_{1a}(m, k) - c^2 Q_{3a}(m, k) \}}{2\beta_m} \quad (47)$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass.

Thus, to solve the nonhomogeneous equation (30), the differential operator which acts on $W_m(t)$ is replaced by the equivalent free system operator defined by the modified frequency ω_m , that is,

$$\begin{aligned} & \ddot{W}_m(t) + \omega_m^2 W_m(t) \\ &= P_1 [B_f(\lambda, k) + \cos \alpha_c t - A_k \sin \alpha_c t \\ & \quad - B_k \cosh \alpha_c t - C_k \sinh \alpha_c t], \end{aligned} \quad (48)$$

where

$$P_1 = \frac{\epsilon^* L^2 g}{\lambda_k \Delta_1(m, k)}. \quad (49)$$

Solving (48) in conjunction with the initial condition, one obtains expression for $W_m(t)$. Thus, in view of (12)

$$\begin{aligned} V_n(x, t) &= \sum_{m=1}^{\infty} P_1 \left\{ B_f(\lambda, k) \left(\frac{1 - \cos \omega_m t}{\omega_m} \right) \right. \\ & \quad + \frac{\cos \alpha_c t - \cos \omega_m t}{\omega_m^2 - \alpha_c^2} \\ & \quad - A_k \left(\frac{\omega_m \sin \alpha_c t - \alpha_c \sin \omega_m t}{\omega_m (\omega_m^2 - \alpha_c^2)} \right) \\ & \quad - B_k \left(\frac{\cosh \alpha_c t - \cos \omega_m t}{\alpha_c^2 + \omega_m^2} \right) \\ & \quad \left. - C_k \left(\frac{\omega_m \sinh \alpha_c t - \alpha_c \sin \omega_m t}{\omega_m (\alpha_c^2 + \omega_m^2)} \right) \right\} \\ & \cdot \left\{ \sin \frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} \right. \\ & \quad \left. + B_m \sinh \frac{\lambda_m x}{L} + C_m \cosh \frac{\lambda_m x}{L} \right\}, \end{aligned} \quad (50)$$

which represents the transverse displacement response to a moving distributed mass of a nonuniform Rayleigh beam having arbitrary end support conditions.

5. Illustrative Examples

In this section, practical examples of classical boundary conditions are used to illustrate the analyses presented in this paper.

5.1. Clamped-Clamped Nonuniform Rayleigh Beam. The boundary conditions of a nonuniform Rayleigh beam clamped at both ends $x = 0$ and $x = L$ are given by

$$\begin{aligned} V_n(0, t) = 0 &= V_n(L, t), \\ \frac{\partial V_n(0, t)}{\partial x} = 0 &= \frac{\partial V_n(L, t)}{\partial x} \end{aligned} \quad (51)$$

and hence for normal modes

$$\begin{aligned} U_m(0) = 0 &= U_m(L), \\ \frac{\partial U_m(0)}{\partial x} = 0 &= \frac{\partial U_m(L)}{\partial x} \end{aligned} \quad (52)$$

which implies

$$\begin{aligned} U_k(0) = 0 &= U_k(L), \\ \frac{\partial U_k(0)}{\partial x} = 0 &= \frac{\partial U_k(L)}{\partial x}. \end{aligned} \quad (53)$$

Applications of (52) to (20) yield

$$\begin{aligned} A_m &= \frac{\sinh \lambda_m - \sin \lambda_m}{\cos \lambda_m - \cosh \lambda_m} = \frac{\cos \lambda_m - \cosh \lambda_m}{\sin \lambda_m - \sinh \lambda_m} = -C_m, \\ B_m &= -1. \end{aligned} \quad (54)$$

The frequency equation becomes

$$\cos \lambda_m \cosh \lambda_m = 1. \quad (55)$$

It follows that expression for A_k, B_k, C_k and the corresponding frequency equations are obtained by a simple interchange of m with k in (54) and (55).

Substituting (54) and (55) into (29) and (50), one obtains the displacement response, respectively, to a moving distributed force and a moving distributed mass of a clamped-clamped nonuniform Rayleigh beam.

5.2. One End Clamped and One End Free Condition-Cantilever Beam. As a second example, at end $x = 0$, the beam is taken to be clamped and, at the end $x = L$, the beam is free. Thus, the boundary conditions can be written as

$$\begin{aligned} V_n(0, t) = 0 &= \frac{\partial}{\partial x} V_n(0, t), \\ \frac{\partial^2}{\partial x^2} V_n(L, t) = 0 &= \frac{\partial^3}{\partial x^3} V_n(L, t). \end{aligned} \quad (56)$$

And for normal modes

$$\begin{aligned} U_m(0) = 0 &= \frac{d}{dx} U_m(0), \\ \frac{d^2}{dx^2} U_m(L) = 0 &= \frac{d^3}{dx^3} U_m(L) \end{aligned} \quad (57)$$

which implies that

$$\begin{aligned} U_k(0) = 0 &= \frac{d}{dx} U_k(0), \\ \frac{d^2}{dx^2} U_k(L) = 0 &= \frac{d^3}{dx^3} U_k(L). \end{aligned} \quad (58)$$

Application of (58) to (21) yields

$$\begin{aligned} A_k &= -\frac{\sin \lambda_k + \sinh \lambda_k}{\cos \lambda_k + \cosh \lambda_k} = \frac{\cos \lambda_k + \cosh \lambda_k}{\sin \lambda_k - \sinh \lambda_k} \\ &= -C_k, \\ B_k &= -1. \end{aligned} \quad (59)$$

And the frequency equation for both end conditions is

$$\cos \lambda_k \cosh \lambda_k = -1. \quad (60)$$

It follows that expression for A_m, B_m, C_m and the corresponding frequency equations are obtained by a simple interchange of k with m in (59) and (60).

Substituting (59) and (60) into (29) and (50), one obtains the displacement response, respectively, to a moving distributed force and a moving distributed mass of a clamped-free nonuniform Rayleigh beam.

6. Discussion of Closed Form Solutions

The response amplitude of a dynamical system such as this may grow without bound. Conditions under which this happens are termed resonance conditions. For both illustrative examples, we observe that the nonuniform Rayleigh beam traversed by a moving distributed force at constant velocity reaches a state of resonance whenever

$$\beta_m = \alpha_c, \quad (61)$$

while the same beam under the action of a moving distributed mass experiences resonance effect whenever

$$\omega_m = \alpha_c. \quad (62)$$

Evidently,

$$\omega_m = \beta_m \left[1 - \frac{\epsilon^* L}{2} \left\{ Q_{1a}(m, k) - \frac{c^2 Q_{3a}(m, k)}{\beta_m^2} \right\} \right] = \alpha_c. \quad (63)$$

Equations (61) and (63) show that for the same natural frequency, the critical speed for the same system consisting of a nonuniform Rayleigh beam resting on an elastic foundation and traversed by a moving distributed force is greater than that traversed by a moving distributed mass. Thus resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

FIGURE 1: Displacement response of a clamped-clamped nonuniform Rayleigh beam under distributed forces for various values of foundation moduli K .

FIGURE 2: Displacement response of a clamped-clamped nonuniform Rayleigh beam under distributed forces for various values of rotatory inertia correction factor R_0 .

7. Numerical Results and Discussion

In order to illustrate the foregoing analysis, the nonuniform Rayleigh beam of length $L = 12.192$ m is considered. The load velocity $c = 8.128 \text{ ms}^{-1}$, $E = 2109 \times 10^9 \text{ kg/m}$, and mass ratio $\epsilon_a = 0.2$. The values of the rotatory inertia correction factor R_0 are varied between 0.5 and 9.5, while the values of the foundation moduli K are varied between 0 and 4000000 Nm^2 . The displacement response of the nonuniform Rayleigh beam is calculated and graphs are plotted for beam response against time for values of rotatory inertia correction factor R_0 and foundation moduli constant K .

The transverse displacement response of the nonuniform clamped-clamped Rayleigh beam to distributed forces for various values of foundation moduli K and fixed value of rotatory inertia correction factor $R_0 = 5$ is displayed

FIGURE 3: Displacement response of a clamped-clamped nonuniform Rayleigh beam to distributed masses for various values of foundation moduli K .

FIGURE 4: Displacement response of a clamped-clamped nonuniform Rayleigh beam under distributed masses for various values of rotatory inertia correction factor R_0 .

in Figure 1. It is seen from the figure that as the values of the foundation moduli increase, the response amplitude of the nonuniform clamped-clamped Rayleigh beam under the action of distributed forces decreases. The same result is obtained when the same nonuniform clamped-clamped Rayleigh beam is traversed by moving distributed masses as shown in Figure 3. In Figure 2, the response of the clamped-clamped nonuniform Rayleigh beam to distributed forces for various values of rotatory inertia correction factor R_0 and fixed value of foundation moduli $K = 40000$ is depicted. It is seen that the deflection of the beam decreases with the increase in the rotatory inertia correction factor. The same behaviour characterizes the deflection profile of the nonuniform clamped-clamped Rayleigh beam when it is traversed by moving distributed masses as shown in Figure 4. Figure 5 depicts the comparison of the transverse displacement response for moving distributed force and moving

FIGURE 5: Comparison of the deflection of moving distributed force and moving distributed mass for nonuniform clamped-clamped Rayleigh beam, $K = 4000000$, $R_0 = 5$.

FIGURE 7: Displacement response of a nonuniform clamped-free Rayleigh beam under distributed forces for various values of rotatory inertia correction factor R_0 .

FIGURE 6: Displacement response of a nonuniform clamped-free Rayleigh beam under distributed forces for various values of foundation moduli K .

FIGURE 8: Displacement response of a nonuniform clamped-free Rayleigh beam under distributed masses for various values of foundation moduli K .

distributed mass cases of the nonuniform clamped-clamped Rayleigh beam for fixed values of foundation moduli $K = 4000000$ and rotatory inertia correction factor $R_0 = 5$. It is observed that the response amplitude of the moving distributed force is greater than that of the moving distributed mass problem.

For the second illustrative example, the deflection profile of the nonuniform clamped-free Rayleigh beam to moving distributed forces for various values of foundation moduli K and fixed value of rotatory inertia correction factor $R_0 = 5$ is displayed in Figure 6. It is seen that as the values of the foundation moduli increase, the deflection of the nonuniform clamped-free beam under the action of distributed forces decreases. The same results are obtained when the same nonuniform clamped-free Rayleigh beam is traversed by moving distributed masses as shown in Figure 8. In Figure 7, the response of the nonuniform clamped-free

Rayleigh beam to moving distributed forces for various values of rotatory inertia correction factor R_0 and fixed value of foundation moduli $K = 40000$ is depicted. It is observed that the deflection of the beam decreases with the increase in the rotatory inertia correction factor. The same behaviour characterizes the deflection profile of the same nonuniform clamped-free Rayleigh beam when it is traversed by moving distributed mass as displayed in Figure 9. Figure 10 depicts the comparison of the transverse displacement response for moving distributed force and moving distributed mass cases of the nonuniform clamped-free Rayleigh beam for fixed values of foundation moduli $K = 4000000$ and rotatory inertia correction factor $R_0 = 5$. Clearly, the response amplitude of the moving distributed mass is greater than that of the moving distributed force problem.

FIGURE 9: Deflection profile of a clamped-free nonuniform Rayleigh beam under distributed masses for various values of rotatory inertia correction factor R_0 .

FIGURE 10: Comparison of the deflection of moving distributed force and moving distributed mass cases of a nonuniform clamped-free Rayleigh beam, $K = 4000000$, $R_0 = 5$.

8. Conclusion

This paper investigated the flexural vibration of a finite nonuniform Rayleigh beam under travelling distributed loads. Both gravity and inertia effects of the distributed loads are taken into consideration. The versatile technique due to Galerkin suitable for all variants of classical boundary conditions was employed to reduce the governing fourth-order partial differential equation with variable coefficients to a sequence of second-order ordinary differential equations. These series of equations were treated using a modification of the asymptotic method of Struble and integral transformations. It is shown that an increase in the values of foundation moduli and rotatory inertia correction factor reduces the response amplitudes of both the clamped-clamped nonuniform Rayleigh beam and the clamped-free nonuniform Rayleigh beam.

Analysis of the closed form solutions for both clamped-clamped and clamped-free boundary conditions showed that the critical speed for the moving distributed mass problem is smaller than that for the moving distributed force problem. Hence resonance is reached earlier in the former. Furthermore, for fixed R_0 and K , the transverse deflections of the nonuniform Rayleigh beam with clamped-free end conditions which is under the action of moving distributed masses are higher than when the same elastic system is under the action of moving distributed forces, while for the same system with clamped-clamped end conditions, the deflection under the action of moving distributed forces is higher than when the system is under the action of moving distributed masses. Showing that the moving force solution is not always an upper bound to the moving mass problem, therefore, the inertia effect of the moving load must always be taken into consideration.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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