Research Article

Existence and Multiplicity Results for the Scalar Curvature Problem on the Half-Sphere $\mathbb{S}_+^3$

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In this paper we deal with the scalar curvature problem under minimal boundary mean curvature condition on the standard 3-dimensional half-sphere. Using tools related to the theory of critical points at infinity, we give existence results under perturbative and nonperturbative hypothesis, and with the help of some “Morse inequalities at infinity”, we provide multiplicity results for our problem.

1. New Results on Scalar Curvature Problem

In this paper, we revisit a problem having a geometric origin. Namely, let

$$\mathbb{S}_+^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \text{ s.t. } |x| = 1, x_4 > 0\} \quad (1)$$

be the standard 3-dimensional half-sphere endowed with its standard Riemannian metric g . Given a function $K : \mathbb{S}_+^3 \rightarrow \mathbb{R}$, we consider the problem of finding a metric $\tilde{g}$ in the conformal class of g such that $\mathcal{R}_{\tilde{g}} = K$ and $h_{\tilde{g}} = 0$, where $\mathcal{R}_{\tilde{g}}$ is the scalar curvature of $\mathbb{S}_+^3$ and $h_{\tilde{g}}$ is the mean curvature of $\partial\mathbb{S}_+^3$ with respect to $\tilde{g}$. Let $\tilde{g} = u^4 g$ be such a metric “conformal” to g ; then the above problem amounts to find a smooth positive solution to the following PDE:

$$\begin{aligned} -\Delta_g u + \frac{3}{4}u &= \frac{1}{8}Ku^5 \text{ in } \mathbb{S}_+^3, \\ u &> 0 \text{ in } \mathbb{S}_+^3, \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\mathbb{S}_+^3, \end{aligned} \quad (2)$$

where ν is the outward normal vector with respect to the metric g . Problems related to (2) were widely studied by various authors [1–12].

Note that, to solve the problem (2), the function K has to be positive somewhere. Moreover, there exist topological

obstructions, as Kazdan-Warner obstructions for the scalar curvature problem on $\mathbb{S}^n$ (see [13]). Therefore we are led to seek sufficient conditions to set on K , so that the problem (2) has solutions. In addition to the existence problem, we address, in this paper, the question of the number of such metrics $\tilde{g}$ in the conformal class of g , with prescribed scalar curvature K and zero boundary mean curvature.

In [3, 4, 6, 11], the authors have studied the problem (2). The methods of [6, 11] involve a fine blow-up analysis of some subcritical approximations and the use of topological degree tools. However, the methods of [3, 4] make use of algebraic topological and dynamical tools, coming from the theory of critical points at infinity (see [14]); we also have addressed this problem in [12], using similar tools.

The main contribution of the present work is to generalize certain previous existence results of [3, 12] and to give new existence results to which we add multiplicity results, using tools coming from the theory of critical points at infinity.

In the first part of this paper, we provide existence and multiplicity results under perturbative hypothesis.

In order to state our results, we introduce the following notations and assumptions.

Through the whole of this paper, we assume that $K_1 = K|_{\partial\mathbb{S}_+^3}$ has a finite set of nondegenerate critical points,

$$\mathcal{K} = \{y \in \partial\mathbb{S}_+^3 \text{ s.t. } \nabla K_1(y) = 0\} = \{y_0, y_1, \dots, y_\ell\}, \quad (3)$$

ordered such that

$$K_1(y_0) \geq K_1(y_1) \geq \dots \geq K_1(y_\ell). \quad (4)$$

We define the following sets:

$$\begin{aligned} \mathcal{F}_+ &= \left\{ y \in \mathcal{K} \text{ s.t. } \frac{\partial K}{\partial \nu}(y) > 0 \right\}, \\ \mathcal{F}_+^0 &= \left\{ y \in \mathcal{K} \text{ s.t. } \frac{\partial K}{\partial \nu}(y) = 0, -\Delta_g K(y) > 0 \right\}. \end{aligned} \quad (5)$$

Let Z be a pseudogradient of K_1 of Morse-Smale type; that is, the intersections of unstable and stable manifolds of the critical points of K_1 , with respect to Z , are transverse. For $y \in \mathcal{K}$, we denote by $W_u(y)$ and $W_s(y)$, respectively, the unstable manifold of y and the stable manifold of y with respect to Z , and we denote by $\text{ind}(K_1, y)$ the Morse index of K_1 at y , that is, the dimension of the submanifold $W_u(y)$.

We introduce the following assumptions:

- (A₀) $W_u(y_j) \cap W_s(y_i) = \emptyset$ for each $y_j \in \mathcal{K} \setminus (\mathcal{F}_+ \cup \mathcal{F}_+^0)$ and $y_i \in \mathcal{F}_+ \cup \mathcal{F}_+^0$.
- (A₁) For each critical point y of K_1 such that $(\partial K / \partial \nu)(y) = 0$ we have $\Delta_g K(y) \neq 0$; furthermore, there is a constant $\bar{r} > 0$ such that $-\Delta_g K(y)(\partial K / \partial \nu)(a) \geq 0$, for all $a \in B(y, \bar{r}) \cap \partial \mathbb{S}_+^3$.
- (A₂) For all $y \in \mathcal{F}_+ \cup \mathcal{F}_+^0$, $\text{ind}(K_1, y) \geq 1$.

We then have the following perturbative result.

Theorem 1. *Under assumptions, (A₀), (A₁), and (A₂), there exists a constant $c_0 > 0$ independent of the function K such that if*

$$\|K_1 - 1\|_{L^\infty(\partial \mathbb{S}_+^3)} < c_0, \quad (6)$$

and if

$$\sum_{y \in \mathcal{F}_+ \cup \mathcal{F}_+^0} (-1)^{2-\text{ind}(K_1, y)} \neq 1, \quad (7)$$

then the problem (2) has at least one solution.

Further, if one assumes that all the solutions of (2) are nondegenerate, then one has

$$\#\mathcal{C} \geq \left| 1 - \sum_{y \in \mathcal{F}_+ \cup \mathcal{F}_+^0} (-1)^{2-\text{ind}(K_1, y)} \right|, \quad (8)$$

where $\#\mathcal{C}$ is the cardinality of the set of solutions of (2).

Note 1. We recall here that for a generic function, K , it follows from the Sard-Smale theorem that all the solutions of (2) are nondegenerate. See, for example [15], for a related discussion on this. Thereby, in the assumptions of multiplicity results, one may replace expression ‘‘Further, if we assume that all the solutions of (2) are nondegenerate, then we have...’’ by expression ‘‘Further, for a generic function K , we have...’’.

We recall also that $w \in \mathcal{C}$ is said to be a nondegenerate solution, if zero is not eigenvalue of the associated linearized operator $-\Delta_g + 3/4 - (1/8)Kw^4$.

Remark 2. The existence result of Theorem 1 is slightly different from the one of Theorem 1.3 of [3], since the required assumptions by the two theorems are not all the same. Further, Theorem 1 also gives us a multiplicity result.

We will provide a more general result than Theorem 1. For this, we define, for each index i , $0 \leq i \leq \ell$, the set

$$X_i = \bigcup_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{F}_+ \cup \mathcal{F}_+^0}} \overline{W_s}(y_j). \quad (9)$$

Under the assumption (A₂), we see that $X_i \not\subseteq \partial \mathbb{S}_+^3 = \mathbb{S}^2$; hence X_i is contractible in $\mathbb{S}^2$. Let us denote by $\theta : [0, 1] \times X_i \rightarrow \mathbb{S}^2$ this contraction and let

$$c_i = \min K \circ \theta. \quad (10)$$

We now introduce the following assumptions:

- (A₃) $|K(y_0)/c_i - 1| < c_0$, where c_0 is given in Theorem 1.
- (A₄) For all $y \in \{y_{i+1}, \dots, y_\ell\} \cap \mathcal{F}_+ \cup \mathcal{F}_+^0$, $K(y) < c_i$.

We then have the following existence and multiplicity result.

Theorem 3. *Under the conditions, (A₀), (A₁), (A₂), (A₃), and (A₄), if*

$$\sum_{\substack{y_j \in \mathcal{F}_+ \cup \mathcal{F}_+^0 \\ 0 \leq j \leq i}} (-1)^{2-\text{ind}(K_1, y_j)} \neq 1, \quad (11)$$

then the problem (2) has at least one solution.

Further, if one assumes that all the solutions of (2) are nondegenerate, then one has

$$\#\mathcal{C} \geq \left| 1 - \sum_{\substack{y_j \in \mathcal{F}_+ \cup \mathcal{F}_+^0 \\ 0 \leq j \leq i}} (-1)^{2-\text{ind}(K_1, y_j)} \right|. \quad (12)$$

Remark 4. Theorem 3 is more general than Theorem 1 for two reasons. First, because, in Theorem 3, K is close to a constant only in a prescribed region of $\partial \mathbb{S}_+^3$ and not in all $\partial \mathbb{S}_+^3$. Secondly, the count-index formula of Theorem 3 is more general than the one of Theorem 1, since the formula of Theorem 1 is obtained for $i = \ell$ from the one of Theorem 3.

In the second part of this work, we will establish nonperturbative results. For this, denoting by d the geodesic distance on $\mathbb{S}_+^3$, let G_1 be the function defined, for $x \neq y$, by

$$G_1(x, y) = \frac{1}{\sqrt{1 - \cos d(x, y)}}. \quad (13)$$

And now, denoting by $\#\mathcal{F}_+$ the cardinality of $\mathcal{F}_+$, let us introduce, for any integer p , $1 \leq p \leq \#\mathcal{F}_+$, the set

$$\mathcal{F}_p = \left\{ (y_{i_1}, \dots, y_{i_p}) \in (\mathcal{F}_+)^p, y_{i_j} \neq y_{i_k} \text{ for } j \neq k \right\}. \quad (14)$$

For all p -tuple $(y_{i_1}, \dots, y_{i_p}) \in \mathcal{F}_p$ we define the matrix $M(y_{i_1}, \dots, y_{i_p}) = (m_{jk})_{\substack{1 \leq p \leq j \\ 1 \leq k \leq p}}$ by

$$\begin{aligned} m_{jj} &= \frac{1}{K(y_{i_j})^{3/2}} \frac{\partial K}{\partial v}(y_{i_j}), \\ m_{jk} &= -\frac{4\sqrt{2}G_1(y_{i_j}, y_{i_k})}{(K(y_{i_j})K(y_{i_k}))^{1/4}} \end{aligned} \quad (15)$$

for $j \neq k$.

Now we formulate the following assumptions:

- (A₅) $\forall (y_{i_j}, y_{i_k}) \in \mathcal{F}_2$; we have $m_{jj}m_{kk} < m_{jk}^2$.
 (A₆) $\forall$ critical point y of K_1 , we have $(\partial K/\partial v)(y) \leq 0$.

We then have the following.

Theorem 5. *Under the conditions, (A₀), (A₁), and (A₅) or (A₆), if there exists an index i , $1 \leq i \leq \ell$, such that*

$$\begin{aligned} (H_i) \quad & \forall y \in (\mathcal{F}_+ \cup \mathcal{F}_+^0) \cap \{y_{i+1}, \dots, y_\ell\}, \\ & \frac{1}{K(y)^{1/2}} > \frac{1}{K(y_0)^{1/2}} + \frac{1}{K(y_i)^{1/2}}, \end{aligned} \quad (16)$$

and if

$$\sum_{\substack{y_j \in \mathcal{F}_+ \cup \mathcal{F}_+^0 \\ 0 \leq j \leq i}} (-1)^{2-\text{ind}(K_1, y_j)} \neq 1, \quad (17)$$

then the problem (2) has at least one solution.

If in addition one assumes that all the solutions of (2) are nondegenerate, then

$$\#\mathcal{C} \geq \left| 1 - \sum_{\substack{y_j \in \mathcal{F}_+ \cup \mathcal{F}_+^0 \\ 0 \leq j \leq i}} (-1)^{2-\text{ind}(K_1, y_j)} \right|. \quad (18)$$

As a corollary of Theorem 5, one has the following.

Corollary 6. *Under the conditions, (A₀), (A₁), and (A₅) or (A₆), if*

$$\sum_{y \in \mathcal{F}_+ \cup \mathcal{F}_+^0} (-1)^{2-\text{ind}(K_1, y)} \neq 1, \quad (19)$$

then the problem (2) has at least one solution.

Furthermore, if we assume that all the solutions of (2) are nondegenerate, then

$$\#\mathcal{C} \geq \left| 1 - \sum_{y \in \mathcal{F}_+ \cup \mathcal{F}_+^0} (-1)^{2-\text{ind}(K_1, y)} \right|. \quad (20)$$

Remark 7. The result of Corollary 6 can be recovered by Theorem 1.1 and Corollary 1.2 in [3] and by Theorem 1.2 and Corollary 1 in [4], and it completes the existence result of Corollary 1.1 in [12] by a multiplicity result.

Commentary. We point out that the main new contribution of Theorem 5 (as well as that of Theorem 3) is that we address here the case where the total sum

$$\sum_{y \in \mathcal{F}_+ \cup \mathcal{F}_+^0} (-1)^{2-\text{ind}(K_1, y)} \quad (21)$$

is equal to 1, but a partial sum

$$\sum_{\substack{y_j \in \mathcal{F}_+ \cup \mathcal{F}_+^0 \\ 0 \leq j \leq i}} (-1)^{2-\text{ind}(K_1, y_j)} \quad (22)$$

is not equal to 1. The main issue is to take advantage of such information to prove the existence of solutions to the problem (2). Notice that an interpretation of the fact that the total sum is different from one is that the topological contribution of the *critical points at infinity* to the topology of the level sets of the associated Euler-Lagrange functional is not trivial.

In view of such an interpretation, we raise the following question: what happens if the total contribution is trivial, but a subset of *critical points at infinity* induce a nontrivial difference of topology; can we still use such a topological information to prove existence of solutions?

With respect to the above question, Theorem 5 (and also Theorem 3) gives a sufficient condition to be able to derive from such local information the existence of solutions for the problem (2).

As pointed out above, our result does not only give existence results but also, under generic conditions, gives a lower bound on the number of solutions of (2). Such a result is reminiscent to the celebrated Morse Theorem, which states that the number of critical points of a Morse function defined on a compact manifold is lower-bounded in terms of the topology of the underlying manifold. Our result can be seen as a kind of *Morse Inequality at Infinity*. Indeed, it gives a lower bound on the number of metrics with prescribed scalar curvature and zero boundary mean curvature, in terms of the *topology at infinity*.

In what follows we show a situation where Corollary 6 does not work, while Theorem 5 allows having solutions to problem (2).

Example. Let $K_1 : \mathbb{S}^2 \rightarrow \mathbb{R}$ be such that

$$\begin{aligned} \mathcal{F}_+^0 &= \{y_0, y_1, y_2, y_3\}, \\ \mathcal{F}_+ &= \{y_4\}, \\ \text{ind}(K_1, y_0) &= \text{ind}(K_1, y_3) = \text{ind}(K_1, y_4) = 2, \end{aligned}$$

and

$$\text{ind}(K_1, y_1) = \text{ind}(K_1, y_2) = 1.$$

It is easy to compute that

$$\sum_{y \in \mathcal{F}_+ \cup \mathcal{F}_+^0} (-1)^{2-\text{ind}(K_1, y)} = 1, \quad (23)$$

and then Corollary 6 does not work. But if we have

$$\frac{1}{K(y_3)^{1/2}} > \frac{1}{K(y_0)^{1/2}} + \frac{1}{K(y_2)^{1/2}}, \quad (24)$$

using Theorem 5 with $i = 2$, then we have

$$\left| 1 - \sum_{\substack{y_j \in \mathcal{I}_+ \cup \mathcal{I}_+^0 \\ 0 \leq j \leq 2}} (-1)^{2 - \text{ind}(K_1, y_j)} \right| = 2 \neq 1, \quad (25)$$

and thus (2) has at least a solution. If solutions of (2) are assumed to be nondegenerate, we derive the existence of at least two solutions.

The rest of our paper is organized as follows. In Section 2, we set the general framework and recall some basic known facts. Section 3 is devoted to the proofs of Theorem 1 and Theorem 3. Finally, we prove Theorem 5 and Corollary 6 in Section 4.

2. Known Facts about Scalar Curvature Problem

In this Section we recall the variational formulation of the problem (2), as well as some previous useful results. We introduce on $H^1(\mathbb{S}_+^3, \mathbb{R})$ the norm

$$\|u\|^2 = \int_{\mathbb{S}_+^3} \left(-\Delta_g u + \frac{3}{4}u \right) u \, dv_g \quad (26)$$

associated with the Yamabe operator $-\Delta_g + 3/4$, where dv_g is the volume element of g on $\mathbb{S}_+^3$. Now we define $\Sigma = \{u \in H^1, \|u\| = 1\}$ to be the unit sphere of $H^1(\mathbb{S}_+^3, \mathbb{R})$ and $\Sigma^+ = \{u \in \Sigma, u \geq 0\}$. The Euler functional on $H^1(\mathbb{S}_+^3, \mathbb{R})$ associated with the problem (2) is

$$J(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}_+^3} K u^6 \, dv_g \right)^{1/3}}. \quad (27)$$

The problem (2) then amounts to find a critical point of J under the constraint $u \in \Sigma^+$. The difficulty in this problem comes from the fact that the functional J fails to satisfy the Palais-Smale condition on Σ^+ . This failure was studied by various authors; see for example [16–18]. To characterize the sequences violating the Palais-Smale condition on Σ^+ , we need to fix some notations. For $(a, \lambda) \in \overline{\mathbb{S}_+^3} \times (0, +\infty)$ let

$$\delta_{(a, \lambda)}(x) = \sigma \frac{\sqrt{\lambda}}{\sqrt{\lambda^2 + 1 - (\lambda^2 - 1) \cos d(a, x)}}, \quad (28)$$

where d is the geodesic distance on $(\mathbb{S}_+^3, g)$ ($\delta_{(a, \lambda)}$ is known to be the solution of the Yamabe problem on the Sphere $\mathbb{S}^3$) and σ is chosen so that

$$-\Delta_g \delta_{(a, \lambda)} + \frac{3}{4} \delta_{(a, \lambda)} = \delta_{(a, \lambda)}^5 \quad (29)$$

is satisfied on $\mathbb{S}_+^3$. Observe that, if $a \in \partial \mathbb{S}_+^3$, we have $\partial \delta_{(a, \lambda)} / \partial \nu = 0$ on $\partial \mathbb{S}_+^3$. However, $\partial \delta_{(a, \lambda)} / \partial \nu \neq 0$ if $a \notin \partial \mathbb{S}_+^3$; then in this case we need to introduce another function $\varphi_{(a, \lambda)}$ which satisfies

$$\begin{aligned} -\Delta_g \varphi_{(a, \lambda)} + \frac{3}{4} \varphi_{(a, \lambda)} &= \delta_{(a, \lambda)}^5 \text{ in } \mathbb{S}_+^3, \\ \frac{\partial \varphi_{(a, \lambda)}}{\partial \nu} &= 0 \text{ on } \partial \mathbb{S}_+^3. \end{aligned} \quad (30)$$

We will write in the sequel δ_i for $\delta_{(a_i, \lambda_i)}$ and φ_i for $\varphi_{(a_i, \lambda_i)}$.

Let w be a nondegenerate solution of (2) or zero. Then, for $\epsilon > 0$ and integers p, q such that $1 \leq q + p$, we define

$$\begin{aligned} V(p, q, \epsilon, w) &= \left\{ u \in \Sigma / \exists a_1, \dots, a_{p+q} \in \overline{\mathbb{S}_+^3}, \right. \\ &\quad \exists \lambda_1, \dots, \lambda_{p+q} > \frac{1}{\epsilon}, \\ &\quad \exists \alpha_0, \dots, \alpha_{p+q} > 0, \\ &\quad \text{s.t. } |\alpha_0^4 J(u)^3 - 1| < \epsilon, \\ &\quad \left\| u - \alpha_0 w - \sum_{i=1}^p \alpha_i \delta_i - \sum_{i=p+1}^{p+q} \alpha_i \varphi_i \right\| < \epsilon, \\ &\quad \epsilon_{ij} < \epsilon, \left| \frac{\alpha_i^4 K(a_i)}{\alpha_j^4 K(a_j)} - 1 \right| < \epsilon \\ &\quad \text{for } 1 \leq i \neq j \leq p+q, \\ &\quad \lambda_i d_i < \epsilon \text{ for } 1 \leq i \leq p, \\ &\quad \left. \lambda_i d_i > \frac{1}{\epsilon} \text{ for } p+1 \leq i \leq p+q \right\}, \end{aligned} \quad (31)$$

where $d_i = d(a_i, \partial \mathbb{S}_+^3)$ and $\epsilon_{ij} = (\lambda_i / \lambda_j + \lambda_j / \lambda_i + \lambda_i \lambda_j (1 - \cos d(a_i, a_j)))^{-1/2}$. Note that, when $w = 0$, to write the definition of $V(p, q, \epsilon, 0)$, we replace w by 0 and we remove α_0 and the condition $|\alpha_0^4 J(u)^3 - 1| < \epsilon$ in the definition of $V(p, q, \epsilon, w)$ above.

The failure of the Palais-Smale condition can be described (see Proposition 1 in [4]), as follows.

Proposition 8. *Let (u_k) be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k) \rightarrow 0$. Then, there exist integers p, q such that $1 \leq q + p$, a sequence $\epsilon_k > 0$ ($\epsilon_k \rightarrow 0$), and an extracted subsequence of (u_k) , again denoted by (u_k) , such that $\forall k \in \mathbb{N}$, $u_k \in V(p, q, \epsilon_k, w)$, where w is either solution of (2) or zero.*

Here ∂J is the gradient of J with respect of the H^1 -inner product:

$$(u, v) = \int_{\mathbb{S}_+^3} \left(-\Delta_g u + \frac{3}{4}u \right) v \, dv_g. \quad (32)$$

We consider the following minimization problem for a function u belonging to $V(p, q, \epsilon, w)$, with ϵ small:

$$\min \left\| u - \alpha_0(w + h) - \sum_{i=1}^p \alpha_i \delta_i - \sum_{i=p+1}^{p+q} \alpha_i \varphi_i \right\|, \quad (33)$$

for $\begin{cases} \alpha_0, \alpha_i, \lambda_i > 0, & 1 \leq i \leq p+q \\ a_i \in \partial \mathbb{S}_+^3, & 1 \leq i \leq p \\ a_i \in \mathbb{S}_+^3, & p+1 \leq i \leq p+q \\ h \in T_w(W_u(w)), \end{cases}$

where $T_w(W_u(w))$ is the tangent space at w to $W_u(w)$ the unstable manifold of w for a pseudogradient of J . We then have the next proposition which defines a parameterization of the set $V(p, q, \epsilon, w)$. It follows from the corresponding statement in [19] (see also Proposition 4 in [4]).

Proposition 9. *For any integers p, q , such that $1 \leq q + p$, there exists $\epsilon(p, q) > 0$ such that if $\epsilon < \epsilon(p, q)$ and $u \in V(p, q, \epsilon, w)$, the minimization problem (33) has a unique solution (up to permutation).*

Thus, we can write any $u \in V(p, q, \epsilon, w)$ uniquely as follows:

$$u = \sum_{i=1}^p \bar{\alpha}_i \delta_{(\bar{a}_i, \bar{\lambda}_i)} + \sum_{i=p+1}^{p+q} \bar{\alpha}_i \varphi_{(\bar{a}_i, \bar{\lambda}_i)} + v + \bar{\alpha}_0(w + \bar{h}), \quad (34)$$

where $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{p+q}, \bar{a}_1, \dots, \bar{a}_{p+q}, \bar{\lambda}_1, \dots, \bar{\lambda}_{p+q}, \bar{h})$ is the solution of the minimization problem (33) and $v \in H^1(\mathbb{S}_+^3) \cap T_w(W_s(w))$ is as follows:

$$\begin{aligned} (v, \psi) &= 0 \text{ for } \psi \in \left\{ \delta_i, \frac{\partial \delta_i}{\partial a_i}, \frac{\partial \delta_i}{\partial \lambda_i} \right\}_{i=1}^{i=p} \cup \left\{ \varphi_i, \frac{\partial \varphi_i}{\partial a_i}, \frac{\partial \varphi_i}{\partial \lambda_i} \right\}_{i=p+1}^{i=p+q} \\ (v, w) &= 0, \quad (v, h) = 0 \quad \forall h \in T_w(W_u(w)), \quad \|v\| < \epsilon, \end{aligned} \quad (35)$$

where $\delta_i = \delta_{(a_i, \lambda_i)}$, $\varphi_i = \varphi_{(a_i, \lambda_i)}$, and $T_w(W_s(w))$ is the tangent space at w to $W_s(w)$ the stable manifold of w for a pseudogradient of J .

We also consider the case, where $w = 0$; then the condition (35) becomes

$$\begin{aligned} (v, \psi) &= 0 \text{ for } \psi \in \left\{ \delta_i, \frac{\partial \delta_i}{\partial a_i}, \frac{\partial \delta_i}{\partial \lambda_i} \right\}_{i=1}^{i=p} \cup \left\{ \varphi_i, \frac{\partial \varphi_i}{\partial a_i}, \frac{\partial \varphi_i}{\partial \lambda_i} \right\}_{i=p+1}^{i=p+q}, \\ \|v\| &< \epsilon. \end{aligned} \quad (36)$$

From similar statements in [14, 20, 21] (see also Proposition 2.2 in [3] or Proposition 3 in [4]), we have then the following.

Proposition 10. *There exists a $\mathcal{C}^1$ map which, to each*

$$(\alpha, a, \lambda) = (\alpha_1, \dots, \alpha_{p+q}, a_1, \dots, a_{p+q}, \lambda_1, \dots, \lambda_{p+q}) \quad (37)$$

such that $\sum_{i=1}^p \alpha_i \delta_i + \sum_{i=p+1}^{p+q} \alpha_i \varphi_i + v \in V(p, q, \epsilon, 0)$, with ϵ small, associates $\bar{v} = \bar{v}(\alpha, a, \lambda)$, satisfying

$$\begin{aligned} &J \left(\sum_{i=1}^p \alpha_i \delta_i + \sum_{i=p+1}^{p+q} \alpha_i \varphi_i + \bar{v} \right) \\ &= \min_{\text{verifying (36)}} J \left(\sum_{i=1}^p \alpha_i \delta_i + \sum_{i=p+1}^{p+q} \alpha_i \varphi_i + v \right). \end{aligned} \quad (38)$$

Moreover, there exists $c > 0$ such that

$$\begin{aligned} \|\bar{v}\| &\leq c \left(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j=p+1}^{p+q} \frac{1}{\lambda_j^2 d_j^2} \right. \\ &\quad \left. + \sum_{k \neq l} \epsilon_{kl} (\ln \epsilon_{kl}^{-1})^{1/3} \right). \end{aligned} \quad (39)$$

The next proposition characterizes the critical points at infinity of the associated variational problem. We recall that critical points at infinity are the orbits of the gradient flow of J which remain in $V(p, q, \epsilon(s), w)$, where $\epsilon(s)$ is a function tending to 0 when $s \rightarrow +\infty$ (see [14]).

Proposition 11. *Assume that, for any $(y_1, \dots, y_p) \in \mathcal{F}_p$, $p \geq 1$, the matrix $M(y_1, \dots, y_p)$ is nondegenerate. Under the assumption, (A_1) , the critical points at infinity of J are in $V(p, 0, \epsilon, 0)$. More precisely consider the following. If $p = 1$, they correspond to single bubbles $\delta_{(y, +\infty)}$, $y \in (\mathcal{F}_+^0 \cup \mathcal{F}_+)$. The Morse index of J at its critical point at infinity $\delta_{(y, +\infty)}$ is*

$$\text{ind}(J, \delta_{(y, +\infty)}) = 2 - \text{ind}(K_1, y), \quad (40)$$

where $\text{ind}(K_1, y)$ is the Morse index of K_1 at its critical point y .

- (1) If $p \geq 2$, they correspond to combinations $\sum_{j=1}^p (1/K(y_{i_j}))^4 \delta_{(y_{i_j}, +\infty)}$, where $(y_{i_1}, \dots, y_{i_p}) \in \mathcal{F}_p$ is such that $\rho(y_{i_1}, \dots, y_{i_p}) > 0$, where $\rho(y_{i_1}, \dots, y_{i_p})$ is the smallest eigenvalue of the matrix $M(y_{i_1}, \dots, y_{i_p})$.

Proof. Using Corollary 3 of [4], there are no critical points at infinity for J in $V(p, q, \epsilon, w)$ with $w \neq 0$. Hence they are in $V(p, q, \epsilon, 0)$. Using Corollary 5 of [4], there are no critical points at infinity for J in $V(0, q, \epsilon, w)$ with $q \neq 0$; that is, there are no bubbles (or blow-ups) with all concentration points interior to $\mathbb{S}_+^3$. Then, using Proposition 3.1 of [3], we rule out the existence of critical points at infinity in $V(p, q, \epsilon, w)$ with $pq \neq 0$ (the so-called mixed bubbles), which signifies that bubbles concentrate uniquely in points of the boundary $\partial \mathbb{S}_+^3$; that is to say, critical points at infinity are in $V(p, 0, \epsilon, 0)$, $p \geq 1$. Finally, using Corollary 6 of [4], we derive the result. $\square$

3. Proofs of Theorems 1 and 3

We start the proofs by recalling the following results.

Lemma 12. For $u = \alpha \delta_{(a,\lambda)} \in V(1, 0, \epsilon, 0)$ such that $a \in B(y)$ a small enough neighborhood of $y \in \mathcal{J}_+^0 \cup \mathcal{J}_+$ a critical point of K_1 , there is a change of variables $(\tilde{a}, \tilde{\lambda})$ such that if $y \in \mathcal{J}_+$,

$$\begin{aligned} J(u) &= \psi_1(\tilde{a}, \tilde{\lambda}) \\ &= \left(\frac{S_3}{2}\right)^{2/3} \frac{1}{K(\tilde{a})^{1/3}} \left(1 + \frac{c}{\tilde{\lambda}} \frac{\partial K}{\partial v}(y) + o\left(\frac{1}{\tilde{\lambda}}\right)\right), \end{aligned} \quad (41)$$

and if $y \in \mathcal{J}_+^0$, after another change of variable $\tilde{\lambda} \mapsto \tilde{\tilde{\lambda}}$,

$$\begin{aligned} J(u) &= \psi_2(\tilde{a}, \tilde{\tilde{\lambda}}) \\ &= \left(\frac{S_3}{2}\right)^{2/3} \frac{1}{K(\tilde{a})^{1/3}} \left(1 - \tilde{c} \frac{\Delta_g K(y)}{\tilde{\tilde{\lambda}}^2} + o\left(\frac{1}{\tilde{\tilde{\lambda}}^2}\right)\right), \end{aligned} \quad (42)$$

where c and $\tilde{c}$ are positive constants and S_3 is the Sobolev constant for $\mathbb{S}^3$. (We recall that $\lambda, \tilde{\lambda}$ and $\tilde{\tilde{\lambda}}$ are of the same order of size).

Lemma 12 is easily deduced from Proposition 2.8 in [3].

Lemma 13 (see [3], Lemma 4.1). Let $a_1, a_2 \in \partial \mathbb{S}_+^3$, and $\alpha_1, \alpha_2 > 0$. For λ large enough positive parameter, let $u = \alpha_1 \delta_{(a_1, \lambda)} + \alpha_2 \delta_{(a_2, \lambda)}$. Then

$$\begin{aligned} J(u) &\leq \left(\frac{S_3}{2}\right)^{2/3} \left(\frac{1}{K(a_1)^{1/2}} + \frac{1}{K(a_2)^{1/2}}\right)^{2/3} (1 + o(1)) \\ &= c_\infty(a_1, a_2) (1 + o(1)). \end{aligned} \quad (43)$$

Lemma 14. For $\epsilon > 0$ small enough and $u = (\sum_{i=1}^p \alpha_i \delta_{(a_i, \lambda_i)} + v) \in V(p, 0, \epsilon, 0)$, one has the expansion

$$J(u) = \left(\frac{S_3}{2}\right)^{2/3} \frac{\sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^6 K(a_i))^{1/3}} (1 + o(1)). \quad (44)$$

Lemma 14 is a particular case (for $n = 1$) of Proposition 2.4 of [3].

Proof of Theorem 1. Let

$$c_1 = \frac{3}{2} \left(\frac{S_3}{2}\right)^{2/3}. \quad (45)$$

Using the expansion of J provided by Lemma 14, there exists $c_0 > 0$ independent of K_1 such that if $\|K_1 - 1\|_{L^\infty(\partial \mathbb{S}_+^3)} < c_0$, then for all $u \in V(1, 0, \epsilon, 0)$, $J(u) < c_1$ and $\forall u \in V(p, 0, \epsilon, 0)$ with $p \geq 2$, $J(u) > c_1$. Using Proposition 11, the only critical points at infinity of J under the level c_1 correspond to single bubbles $\delta_{(y, +\infty)}$, where $y \in (\mathcal{J}_+^0 \cup \mathcal{J}_+)$. We let in the sequel $\delta_{(y, +\infty)} = y_\infty$. Now define

$$X = \bigcup_{y \in (\mathcal{J}_+^0 \cup \mathcal{J}_+)} \overline{W}_s(y), \quad (46)$$

where $W_s(y)$ is the stable manifold of y for a decreasing pseudogradient Z of the function K_1 . By hypothesis (A_2) , we have $X \not\subseteq \partial \mathbb{S}_+^3 = \mathbb{S}^2$. Thus, X is contractible in $\mathbb{S}^2$. We denote by $h : [0, 1] \times X \rightarrow \mathbb{S}^2$ the associated contraction. Let

$$X_\infty = \bigcup_{y \in (\mathcal{J}_+^0 \cup \mathcal{J}_+)} \overline{W}_u(y_\infty), \quad (47)$$

where $W_u(y_\infty)$ is the unstable manifolds at infinity, for the pseudogradient V provided by Proposition 3.1 in [3] (or the pseudogradient W provided by Proposition 9 in [4]), of the critical point at infinity y_∞ . It can be described, using the expansion of $J(u)$ given by Lemma 12, as the product of $W_s(y)$, for the pseudogradient Z of K_1 , by $[A, +\infty)$ which is the domain of the variable λ , for some large enough positive real A .

The contraction h gives rise to a contraction

$$\begin{aligned} \tilde{h} : [0, 1] \times X \times [A, +\infty) &\longrightarrow \Sigma^+ \\ (t, x, \lambda) &\longmapsto \frac{\delta_{(h(t, x), \lambda)}}{\|\delta_{(h(t, x), \lambda)}\|}. \end{aligned} \quad (48)$$

Using Lemma 12, $\forall (t, x, \lambda) \in [0, 1] \times X \times [A, +\infty)$, one has

$$\begin{aligned} J(\tilde{h}(t, x, \lambda)) &= \left(\frac{S_3}{2}\right)^{2/3} \frac{1}{K(h(t, x))^{1/3}} \left(1 + O\left(\frac{1}{A}\right)\right). \end{aligned} \quad (49)$$

Since K is close to 1, we derive that $J(\tilde{h}(t, x, \lambda)) < c_1$, for A large enough. Denoting by $R(\tilde{h})$ the range of $\tilde{h}$, that is, $R(\tilde{h}) := \tilde{h}([0, 1] \times X_\infty)$, then we derive that $R(\tilde{h})$ is below the level c_1 ; that is to say, the contraction $\tilde{h}$ is performed under the level c_1 and thus (see [22], Sections 7 and 8)

$$R(\tilde{h}) \simeq \bigcup_{\mathcal{J}_+^0 \cup \mathcal{J}_+} W_u(y_\infty) \bigcup_{\substack{w \in \mathcal{C} \\ R(\tilde{h}) \cap W_s(w) \neq \emptyset}} W_u(w), \quad (50)$$

where $\mathcal{C}$ is the set of critical points of J or, equivalently, of the solutions of (2). Now, denoting by χ the Euler-Poincaré characteristic, and using the fact that $R(\tilde{h})$ is a contractible set, it holds that

$$\begin{aligned} 1 = \chi(R(\tilde{h})) &= \sum_{y \in \mathcal{J}_+^0 \cup \mathcal{J}_+} (-1)^{\text{ind}(J, y_\infty)} \\ &+ \sum_{\substack{w \in \mathcal{C} \\ R(\tilde{h}) \cap W_s(w) \neq \emptyset}} (-1)^{\text{ind}(J, w)}, \end{aligned} \quad (51)$$

where $\text{ind}(J, w)$ (resp., $\text{ind}(J, y_\infty)$) denotes the Morse index of J at w (resp., y_∞). Since $\text{ind}(J, y_\infty) = 2 - \text{ind}(K_1, y)$

and since $\sum_{y \in \mathcal{J}_+^0 \cup \mathcal{J}_+} (-1)^{2-\text{ind}(K_1, y)} \neq 1$, we derive that $\mathcal{C} \neq \emptyset$. Hence

$$\begin{aligned} & \left| 1 - \sum_{y \in \mathcal{J}_+^0 \cup \mathcal{J}_+} (-1)^{2-\text{ind}(K_1, y)} \right| \\ & \leq \left| \sum_{\substack{w \in \mathcal{C} \\ R(\tilde{h}) \cap W_s(w) \neq \emptyset}} (-1)^{\text{ind}(J, w)} \right| \leq \#\mathcal{C}, \end{aligned} \quad (52)$$

where $\#\mathcal{C}$ is the cardinality of $\mathcal{C}$ q.e.d. $\square$

Proof of Theorem 3. Using the same arguments of the proof of Theorem 1, let

$$X_{i\infty} = \bigcup_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{J}_+ \cup \mathcal{J}_+^0}} \overline{W_s}(y_{j\infty}), \quad (53)$$

where X_i is defined by (9). As we remarked it above, $X_{i\infty}$ can be parameterized by $X_i \times [A, +\infty)$. Let

$$c_i^\infty = \left(\frac{S_3}{2}\right)^{2/3} \frac{1}{c_i^{1/3}}, \quad (54)$$

where $c_i = \min K \circ \theta$ and θ is the contraction

$$\begin{aligned} \theta : [0, 1] \times X_i &\longrightarrow \mathbb{S}^2, \\ (t, x) &\longmapsto \theta(t, x) \end{aligned} \quad (55)$$

that is, for all $x \in X_i$, $\theta(0, x) = x$ and $\theta(1, x) = a_0$ a fixed point in X_i .

We now claim that $X_{i\infty}$ is contractible in $J_{c_i^\infty + \epsilon}$. Indeed, the contraction θ induces the following contraction:

$$\begin{aligned} \tilde{\theta} : [0, 1] \times X_i \times [A, +\infty) &\longrightarrow \Sigma^+ \\ (t, x, \lambda) &\longmapsto \frac{\delta_{(\theta(t, x), \lambda)}}{\|\delta_{(\theta(t, x), \lambda)}\|}. \end{aligned} \quad (56)$$

Thus, by Lemma 12, we derive, for λ large enough,

$$J(\delta_{(\theta(t, x), \lambda)}) = \left(\frac{S_3}{2}\right)^{2/3} \frac{1}{K(\theta(t, x))^{1/3}} \left(1 + O\left(\frac{1}{\lambda}\right)\right). \quad (57)$$

Since $\forall t \in [0, 1]$ and $\forall x \in X_i$, we have $K(\theta(t, x)) \geq c_i$; then, for $\lambda \geq A$ large enough,

$$J(\delta_{(\theta(t, x), \lambda)}) \leq \left(\frac{S_3}{2}\right)^{2/3} \frac{1}{c_i^{1/3}} \left(1 + O\left(\frac{1}{A}\right)\right) \leq c_i^\infty + \epsilon, \quad (58)$$

and thus $X_{i\infty}$ is contractible in $J_{c_i^\infty + \epsilon}$. Our claim follows.

On the other side, using Proposition 11, under the assumptions, (A_3) and (A_4) , the critical points at infinity under the level $c_i^\infty + \epsilon$ are $y_{j\infty}$, where $y_j \in (\mathcal{J}_+^0 \cup \mathcal{J}_+)$ and

$0 \leq j \leq i$. Thus, denoting by $R(\tilde{\theta})$ the range of $\tilde{\theta}$, that is, $R(\tilde{\theta}) := \tilde{\theta}([0, 1] \times X_{i\infty})$, we derive that

$$R(\tilde{\theta}) \simeq \bigcup_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{J}_+^0 \cup \mathcal{J}_+}} W_u(y_{j\infty}) \bigcup_{\substack{w \in \mathcal{C} \\ R(\tilde{\theta}) \cap W_s(w) \neq \emptyset}} W_u(w), \quad (59)$$

where $\mathcal{C}$ is the set of the critical points of J . Our claim thus implies that

$$\begin{aligned} 1 = \chi(R(\tilde{\theta})) &= \sum_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{J}_+^0 \cup \mathcal{J}_+}} (-1)^{\text{ind}(J, y_{j\infty})} \\ &+ \sum_{\substack{w \in \mathcal{C} \\ R(\tilde{\theta}) \cap W_s(w) \neq \emptyset}} (-1)^{\text{ind}(J, w)}. \end{aligned} \quad (60)$$

Hence, arguing as above, we derive that $\mathcal{C} \neq \emptyset$, and that

$$\begin{aligned} & \left| 1 - \sum_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{J}_+^0 \cup \mathcal{J}_+}} (-1)^{2-\text{ind}(K_1, y_j)} \right| \\ & \leq \left| \sum_{\substack{w \in \mathcal{C} \\ R(\tilde{\theta}) \cap W_s(w) \neq \emptyset}} (-1)^{\text{ind}(J, w)} \right| \leq \#\mathcal{C}. \end{aligned} \quad (61)$$

Our theorem is thereby proved. $\square$

4. Proofs of Theorem 5 and Corollary 6.

Proof of Theorem 5. Let

$$c_\infty(y_0, y_i) = \left(\frac{S_3}{2}\right)^{2/3} \left(\frac{1}{K(y_0)^{1/2}} + \frac{1}{K(y_i)^{1/2}}\right)^{2/3}. \quad (62)$$

Using Proposition 11, we observe that under the assumptions, (A_1) and (A_5) or (A_6) , there are no critical points at infinity of multiple masses, that is, in $V(p, 0, \epsilon, 0)$ for $p \geq 2$. Indeed, on one hand, under (A_5) , we derive that, for any $p \geq 2$ and for any $(y_{i_1}, \dots, y_{i_p}) \in \mathcal{F}_p$, the matrix $M(y_{i_1}, \dots, y_{i_p})$ is definite negative, and then its smallest eigenvalue $\rho(y_{i_1}, \dots, y_{i_p})$ is < 0 . On the other hand, under (A_6) , $\mathcal{J}_+ = \emptyset$ and therefore $\mathcal{F}_p = \emptyset$ for all p .

Thus the critical points at infinity of our variational problem lie in $V(1, 0, \epsilon, 0)$. Using the assumption (H_1) of Theorem 5, it follows that the only critical points at infinity of J under the level

$$c_i = c_\infty(y_0, y_i) + \epsilon \quad (63)$$

for ϵ small enough are $\delta_{(y_j, +\infty)} = y_{j\infty}$, where $y_j \in (\mathcal{J}_+^0 \cup \mathcal{J}_+)$, for $0 \leq j \leq i$.

The unstable manifolds at infinity, for the pseudogradient V (provided by Proposition 3.1 in [3]), of such critical points at infinity, which we denote by $W_u(y_{j\infty})$, can be described as

the product of $W_s(y_j)$, for the pseudogradient Z of K_1 , by $[A, +\infty)$, domain of the variable λ , for some large enough positive real A . Let

$$X_{i\infty} = \bigcup_{\substack{0 \leq j \leq i \\ y_j \in (\mathcal{J}_+^0 \cup \mathcal{J}_+)}} \overline{W_u}(y_{j\infty}). \quad (64)$$

The set $X_{i\infty}$ can be parameterized by $X_i \times [A, +\infty)$, where X_i is given by (9). We now claim that $X_{i\infty}$ is contractible in J_{c_i} , where

$$J_{c_i} = \{u \in \Sigma^+ \text{ s.t. } J(u) \leq c_i\}. \quad (65)$$

Indeed, let $a_1, a_2 \in \partial \mathbb{S}_+^3$, $\alpha_1, \alpha_2 > 0$, and λ large enough. For $u = \alpha_1 \delta_{(a_1, \lambda)} + \alpha_2 \delta_{(a_2, \lambda)}$, we have, by Lemma 13, the following estimate:

$$\begin{aligned} J(u) &\leq \left(\frac{S_3}{2}\right)^{2/3} \left(\frac{1}{K(a_1)^{1/2}} + \frac{1}{K(a_2)^{1/2}}\right)^{2/3} (1 + o(1)) \\ &= c_{\infty}(a_1, a_2) + o(1). \end{aligned} \quad (66)$$

Let

$$\begin{aligned} \psi : [0, 1] \times X_i \times [A, +\infty) &\longrightarrow \Sigma^+ \\ (t, x, \lambda) &\longmapsto \frac{t\delta_{(y_0, \lambda)} + (1-t)\delta_{(x, \lambda)}}{\|t\delta_{(y_0, \lambda)} + (1-t)\delta_{(x, \lambda)}\|}. \end{aligned} \quad (67)$$

The function ψ is continuous, and it satisfies $\psi(0, x, \lambda) = \delta_{(x, \lambda)} / \|\delta_{(x, \lambda)}\|$, $\psi(1, x, \lambda) = \delta_{(y_0, \lambda)} / \|\delta_{(y_0, \lambda)}\|$. Now, since $K_1(x) \geq K_1(y_i)$, for any $x \in X_i$, it follows from inequality (66) that $J(\psi(t, x, \lambda)) < c_i$ for all $(t, x, \lambda) \in [0, 1] \times X_i \times [A, +\infty)$. Thus, the contraction ψ is performed under the level c_i . We derive that $X_{i\infty}$ is contractible in J_{c_i} , and our claim follows. On the other hand, denoting by $R(\psi)$ the range of ψ , that is, $R(\psi) := \psi([0, 1] \times X_{i\infty})$, we have

$$R(\psi) \simeq \bigcup_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{J}_+^0 \cup \mathcal{J}_+}} W_u(y_{j\infty}) \bigcup_{\substack{w \in \mathcal{C} \\ R(\psi) \cap W_s(w) \neq \emptyset}} W_u(w), \quad (68)$$

where $\mathcal{C}$ is the set of the critical points of J , and since $R(\psi)$ is contractible, it yields

$$\begin{aligned} 1 = \chi(R(\psi)) &= \sum_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{J}_+^0 \cup \mathcal{J}_+}} (-1)^{\text{ind}(J, y_{j\infty})} \\ &+ \sum_{\substack{w \in \mathcal{C} \\ R(\psi) \cap W_s(w) \neq \emptyset}} (-1)^{\text{ind}(J, w)}. \end{aligned} \quad (69)$$

Hence, we derive as above that $\mathcal{C} \neq \emptyset$ and that

$$\begin{aligned} &\left| 1 - \sum_{\substack{0 \leq j \leq i \\ y_j \in \mathcal{J}_+^0 \cup \mathcal{J}_+}} (-1)^{2 - \text{ind}(K_1, y_j)} \right| \\ &\leq \left| \sum_{\substack{w \in \mathcal{C} \\ R(\psi) \cap W_s(w) \neq \emptyset}} (-1)^{\text{ind}(J, w)} \right| \leq \#\mathcal{C}, \end{aligned} \quad (70)$$

where $\#\mathcal{C}$ is the number of solutions of (2). Theorem 5 follows. $\square$

Proof of Corollary 6. Recall that K_1 has only nondegenerate critical points $y_0, y_1, \dots, y_\ell$, such that $K_1(y_0) \geq \dots \geq K_1(y_\ell)$. Observe that the assumption (H_1) of Theorem 5 serves uniquely to exclude points y_j of $\mathcal{J}_+^0 \cup \mathcal{J}_+$ such that $i + 1 \leq j \leq \ell$, from the count-index formula. So, for the index $i = \ell$, the assumption (H_ℓ) vanishes since all the critical points of $\mathcal{J}_+^0 \cup \mathcal{J}_+$ are here taken into consideration. Thus, our Corollary follows from Theorem 5, taking $i = \ell$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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