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Research Article

Coupled Fixed Point Theorems for (φ, ψ)-Contractive Mixed Monotone Mappings in Partially Ordered Metric Spaces and Applications

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The object of this paper is to establish the existence and uniqueness of coupled fixed points under a (φ, ψ)-contractive condition for mixed monotone operators in the setup of partially ordered metric spaces. Presented work generalizes the recent results of Berinde (2011, 2012) and weakens the contractive conditions involved in the well-known results of Bhaskar and Lakshmikantham (2006), and Luong and Thuan (2011). The effectiveness of our work is validated with the help of a suitable example. As an application, we give a result of existence and uniqueness for the solutions of a class of nonlinear integral equations.

1. Introduction and Preliminaries

Fixed point theory is an important tool for studying the phenomenon of nonlinear analysis and is a bridge bond between pure and applied mathematics. The theory has its wide applications in engineering, computer science, physical and life sciences, economics, and other fields. Banach [1] introduced the well-known classical and valuable theorem in nonlinear analysis, which is named after him, known as the Banach contraction principle. This celebrated principle has been extended and improved by various authors in many ways over the years (see for instance [2–17]). Nowadays, fixed point theory has been receiving much attention in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. Ran and Reurings [17] were the first to establish the results in this direction. The results were then extended by Nieto and Rodríguez-López [10] for nondecreasing mappings. Works noted in [18–24] are some examples in this direction.

The work of Bhaskar and Lakshmikantham [25] is worth mentioning, as they introduced the new notion of fixed points for the mappings having domain the product space $X \times X$, which they called coupled fixed points, and thereby proved some coupled fixed point theorems for mappings satisfying the mixed monotone property in partially ordered metric spaces. As an application, they discussed the existence and uniqueness of a solution for a periodic boundary value problem.

Definition 1 (see [25]). Let $(X, \leq)$ be a partially ordered set. The mapping $F : X \times X \to X$ is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 \quad &\text{implies } F(x_1, y) \leq F(x_2, y), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 \quad &\text{implies } F(x, y_1) \geq F(x, y_2). \end{aligned} \tag{1}$$

Definition 2 (see [25]). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Bhaskar and Lakshmikantham [25] gave the following result.

Theorem 3 (see [25]). *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is*

a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with

$$d(F(x,y), F(u,v)) \le \frac{k}{2}[d(x,u) + d(y,v)] \tag{2}$$

for all $x, y, u, v \in X$ with $x \ge u$ and $y \le v$.

If there exist two elements $x_0, y_0 \in X$ with $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

It has also been shown in [25, Theorem 2.2] that the continuity assumption of F in Theorem 3 can be replaced by an alternative condition imposed on the convergent nondecreasing and nonincreasing sequences in the space X.

Assumption 4. X has the property that

(i) if a nondecreasing sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to x, then $x_n \le x$ for all n,

(ii) if a nonincreasing sequence $\{y_n\}_{n=0}^{\infty} \subset X$ converges to y, then $y \le y_n$ for all n.

These results were then extended and generalized by several authors in the last six years (see [26–40]). Luong and Thuan [26, Theorem 2.1] extended the results of Bhaskar and Lakshmikantham [25] under the following contractive condition:

$$\varphi(d(F(x,y), F(u,v))) \le \frac{1}{2}\varphi(d(x,u) + d(y,v)) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right), \tag{3}$$

with $x \ge u$ and $y \le v$, where $\varphi, \psi : [0,\infty) \to [0,\infty)$ are the functions such that

(φ_1) φ is continuous and nondecreasing;

(φ_2) $\varphi(t) = 0$ if and only if $t = 0$;

(φ_3) $\varphi(t+s) \le \varphi(t) + \varphi(s)$, for all $t, s \in [0,\infty)$,

and the function ψ satisfies

$$(i_\psi) \quad \lim_{t \to r} \psi(t) > 0 \quad \forall r > 0, \qquad \lim_{t \to 0+} \psi(t) = 0. \tag{4}$$

Berinde [27, Theorem 3] in an alternative way generalized the results of Bhaskar and Lakshmikantham [25] under a weaker contraction as follows:

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \le k[d(x,u) + d(y,v)], \tag{5}$$

with $x \ge u$ and $y \le v$, where $k \in [0,1)$.

Berinde [28, Thoerem 2] further generalized and complemented the results noted in [25] and extended the work presented in [27] by considerably weakening the involved contractive condition as

$$\varphi\left(\frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2}\right) \le \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right), \tag{6}$$

with $x \ge u$ and $y \le v$, where φ, ψ: $[0,\infty) \to [0,\infty)$ are the functions such that

(φ_1') φ is continuous and (strictly) increasing;

(φ_2') $\varphi(t) < t$ for all $t > 0$;

(φ_3') $\varphi(t+s) \le \varphi(t) + \varphi(s)$, for all $t, s \in [0,\infty)$,

and the function ψ satisfies (i_ψ).

2. Coupled Fixed Points

In order to proceed with developing of our work and obtain our results, we need to consider the following.

Let Φ denote all functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy the following:

(φ_i) φ is lower semicontinuous and (strictly) increasing;

(φ_{ii}) $\varphi(t) < t$ for all $t > 0$;

(φ_{iii}) $\varphi(t+s) \le \varphi(t) + \varphi(s)$ for all $t, s \in [0,\infty)$.

We note that $\lim_{n\to\infty}\varphi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$ for $t_n \in [0,\infty)$.

Also, for $\varphi \in \Phi$, Ψ_φ denote all functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy the following:

(ψ_i) $\limsup_{n\to\infty}\psi(t_n) < \varphi(r)$ if $\lim_{n\to\infty} t_n = r > 0$;

(ψ_{ii}) $\lim_{n\to\infty}\psi(t_n) = 0$ if $\lim_{n\to\infty} t_n = 0$ for $t_n \in [0,\infty)$.

Now we are ready to prove our first main result as follows.

Theorem 5. *Let $(X, \le)$ be a partially ordered set and there is a metric d on X such that (X,d) is a complete metric space. Suppose $F : X \times X \to X$ is a mapping having mixed monotone property on X. Assume there exist $\varphi \in \Phi$ and $\psi \in \Psi_\varphi$ such that*

$$\varphi\left(\frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2}\right) \le \psi\left(\frac{d(x,u) + d(y,v)}{2}\right) \tag{7}$$

for all $x, y, u, v \in X$ with $x \ge u$ and $y \le v$ (or $x \le u$ and $y \ge v$).

Suppose either (a) F is continuous or (b) X satisfies Assumption 4.

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \le F(x_0, y_0), \qquad y_0 \ge F(y_0, x_0), \tag{8}$$

or

$$x_0 \geq F(x_0, y_0), \qquad y_0 \leq F(y_0, x_0), \tag{9}$$

then there exist $x, y \in X$ such that

$$x = F(x, y), \qquad y = F(y, x), \tag{10}$$

that is, F has a coupled fixed point in X.

Proof. Without loss of generality, assume that there exist two elements $x_0, y_0 \in X$ such that (8) holds; that is, $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Let $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $x_0 \leq x_1$ and $y_0 \geq y_1$. Similarly, let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Since F has the mixed monotone property, then we have $x_1 \leq x_2$ and $y_1 \geq y_2$. Continuing in the same way, we can easily construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n)$, $y_{n+1} = F(y_n, x_n)$ and

$$\begin{aligned} x_0 &\leq x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \leq \cdots, \\ y_0 &\geq y_1 \geq y_2 \geq \cdots y_n \geq y_{n+1} \geq \cdots. \end{aligned} \tag{11}$$

If for some $n \geq 0$, we have $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then $F(x_n, y_n) = x_n$ and $F(y_n, x_n) = y_n$; that is, F has a coupled fixed point. So from now onwards, we suppose $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$, for all $n \geq 0$; that is, we suppose that either $x_{n+1} = F(x_n, y_n) \neq x_n$ or $y_{n+1} = F(y_n, x_n) \neq y_n$.

Now applying inequality (7) with $(x, y) = (x_n, y_n)$ and $(u, v) = (x_{n+1}, y_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} &\varphi\left(\frac{d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})}{2}\right) \\ &\quad = \varphi\left(\left(\begin{array}{l} d(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \\ +d(F(y_n, x_n), F(y_{n+1}, x_{n+1})) \end{array}\right) \times (2)^{-1}\right) \\ &\quad \leq \psi\left(\frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2}\right), \end{aligned} \tag{12}$$

then for all $n \geq 0$, we have

$$\varphi(\delta_{n+1}) \leq \psi(\delta_n), \tag{13}$$

where $\delta_n := (d(x_n, x_{n+1}) + d(y_n, y_{n+1}))/2$.

Clearly, $\delta_n > 0$ for all n. Then for any n, we obtain

$$\varphi(\delta_{n+1}) \leq \psi(\delta_n) < \varphi(\delta_n). \tag{14}$$

By monotonicity of φ, together with (14), we can easily see that $\{\delta_n\}$ is a nonnegative decreasing sequence in $\mathbb{R}$. So we have $\lim_{n\to\infty}\delta_n = \delta$ for some $\delta \geq 0$. If $\delta > 0$, then using the properties of φ, ψ we get

$$\varphi(\delta) \leq \limsup_{n\to\infty} \varphi(\delta_{n+1}) \leq \limsup_{n\to\infty} \psi(\delta_n) < \varphi(\delta), \tag{15}$$

which is a contradiction. Hence $\delta = 0$ and so we have

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2} = 0. \tag{16}$$

We now prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of the sequences $\{x_n\}$, $\{y_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$r_k = \frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2} \geq \varepsilon. \tag{17}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (17). Then,

$$\frac{d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)})}{2} < \varepsilon. \tag{18}$$

By (17), (18), and triangle inequality, we have

$$\begin{aligned} \varepsilon \leq r_k &= \frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2} \\ &\leq \{d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &\qquad +d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)})\} \\ &\quad \times (2)^{-1} \\ &< \frac{d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1})}{2} + \varepsilon. \end{aligned} \tag{19}$$

Letting $k \to \infty$ and using (16) in the last inequality, we have

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \left[\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\right] = \varepsilon. \tag{20}$$

Again, by triangle inequality

$$\begin{aligned} &r_k \\ &\quad = \frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2} \\ &\quad \leq \frac{\left\{\begin{array}{l} d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) \\ +d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) \end{array}\right\}}{2} \\ &\quad = \delta_{n(k)} + \delta_{m(k)} \\ &\qquad + \frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})}{2}. \end{aligned} \tag{21}$$

By monotonicity of φ and property (φ_{iii}), we have

$$\begin{aligned} \varphi(r_k) &\leq \varphi(\delta_{n(k)}) + \varphi(\delta_{m(k)}) \\ &\quad + \varphi\left(\frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})}{2}\right). \end{aligned} \tag{22}$$

Since $n(k) > m(k)$, $x_{n(k)} \geq x_{m(k)}$, and $y_{n(k)} \leq y_{m(k)}$.

Then by (7), we have

$$\begin{aligned}
&\varphi\left(\frac{d\left(x_{n(k)+1}, x_{m(k)+1}\right)+d\left(y_{n(k)+1}, y_{m(k)+1}\right)}{2}\right)\\
&\quad=\varphi\left(\left(\begin{array}{l} d\left(F\left(x_{n(k)}, y_{n(k)}\right), F\left(x_{m(k)}, y_{m(k)}\right)\right)\\ +d\left(F\left(y_{n(k)}, x_{n(k)}\right), F\left(y_{m(k)}, x_{m(k)}\right)\right)\end{array}\right)\times(2)^{-1}\right)\\
&\quad\leq\psi\left(\frac{d\left(x_{n(k)}, x_{m(k)}\right)+d\left(y_{n(k)}, y_{m(k)}\right)}{2}\right)\\
&\quad=\psi\left(r_k\right).
\end{aligned}\tag{23}$$

By (22) and (23), we have

$$\varphi\left(r_k\right)\leq\varphi\left(\delta_{n(k)}\right)+\varphi\left(\delta_{m(k)}\right)+\psi\left(r_k\right).\tag{24}$$

Since φ is lower semicontinuous by taking limit as $k\to\infty$, we get

$$\begin{aligned}
&\varphi(\varepsilon)\\
&\leq\limsup_{k\to\infty}\varphi\left(r_k\right)\\
&\leq\lim_{k\to\infty}\varphi\left(\delta_{n(k)}\right)+\lim_{k\to\infty}\varphi\left(\delta_{n(k)}\right)+\limsup_{k\to\infty}\psi\left(r_k\right)<\varphi(\varepsilon),
\end{aligned}\tag{25}$$

a contradiction. Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in complete metric space X and hence $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$ for some $x, y\in X$.

Now suppose that assumption (a) holds. Then

$$\begin{aligned}
x&=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}F\left(x_n, y_n\right)=F(x, y),\\
y&=\lim_{n\to\infty}y_{n+1}=\lim_{n\to\infty}F\left(y_n, x_n\right)=F(y, x),
\end{aligned}\tag{26}$$

which shows that (x, y) is a coupled fixed point of F.

Suppose now assumption (b) holds. Since $\{x_n\}$ is a nondecreasing sequence that converges to x, we have that $x_n\leq x$ for all n. Similarly, $y_n\geq y$ for all n.

Then

$$\begin{aligned}
d(x, F(x, y))&\leq d\left(x, x_{n+1}\right)+d\left(x_{n+1}, F(x, y)\right)\\
&=d\left(x, x_{n+1}\right)+d\left(F\left(x_n, y_n\right), F(x, y)\right),\\
d(y, F(y, x))&\leq d\left(y, y_{n+1}\right)+d\left(y_{n+1}, F(y, x)\right)\\
&=d\left(y, y_{n+1}\right)+d\left(F\left(y_n, x_n\right), F(y, x)\right).
\end{aligned}\tag{27}$$

So

$$\begin{aligned}
&d(x, F(x, y))-d\left(x, x_{n+1}\right)\leq d\left(F\left(x_n, y_n\right), F(x, y)\right),\\
&d(y, F(y, x))-d\left(y, y_{n+1}\right)\leq d\left(F\left(y_n, x_n\right), F(y, x)\right)
\end{aligned}\tag{28}$$

and hence

$$\begin{aligned}
&\frac{1}{2}\left[d(x, F(x, y))-d\left(x, x_{n+1}\right)+d(y, F(y, x))-d\left(y, y_{n+1}\right)\right]\\
&\quad\leq\frac{1}{2}\left[d\left(F\left(x_n, y_n\right), F(x, y)\right)+d\left(F\left(y_n, x_n\right), F(y, x)\right)\right]
\end{aligned}\tag{29}$$

which imply, by the monotonicity of φ and condition (7),

$$\begin{aligned}
&\varphi\left(\frac{1}{2}\left[d(x, F(x, y))-d\left(x, x_{n+1}\right)+d(y, F(y, x))\right.\right.\\
&\qquad\left.\left.-d\left(y, y_{n+1}\right)\right]\right)\\
&\leq\varphi\left(\frac{1}{2}\left[d\left(F\left(x_n, y_n\right), F(x, y)\right)+d\left(F\left(y_n, x_n\right), F(y, x)\right)\right]\right)\\
&\leq\psi\left(\frac{d\left(x_n, x\right)+d\left(y_n, y\right)}{2}\right).
\end{aligned}\tag{30}$$

Since φ is lower semicontinuous, by taking the limit as $n\to\infty$ in the above inequality, we get

$$\begin{aligned}
&\varphi\left(\frac{1}{2}\left[d(x, F(x, y))+d(y, F(y, x))\right]\right)\\
&\quad\leq\limsup_{n\to\infty}\varphi\left(\frac{1}{2}\left[d(x, F(x, y))-d\left(x, x_{n+1}\right)\right.\right.\\
&\qquad\left.\left.+d(y, F(y, x))-d\left(y, y_{n+1}\right)\right]\right)\\
&\quad\leq\limsup_{n\to\infty}\psi\left(\frac{d\left(x_n, x\right)+d\left(y_n, y\right)}{2}\right)=0.
\end{aligned}\tag{31}$$

Hence, we get $x=F(x, y)$ and $y=F(y, x)$. Therefore, F has a coupled fixed point (x, y). □

Remark 6. In Theorem 5, substituting $\varphi(x)-\psi(x)$ for $\psi(x)$ implies the main result of Berinde [28, Theorem 2]. Note that the function $\varphi(x)-\psi(x)$ satisfies all the conditions of our result. It is easy to verify this as in Theorem 2 [28] φ is a continuous function and

$$\limsup_{n\to\infty}\left(\varphi\left(t_n\right)-\psi\left(t_n\right)\right)\leq\varphi(r)-\liminf_{n\to\infty}\psi\left(t_n\right)<\varphi(r),\tag{32}$$

for all $t_n\in[0,\infty)$ such that $\lim_{n\to\infty}t_n=r>0$. On the other hand,

$$\lim_{t\to0^+}\psi(t)=0\quad\text{implies}\ \lim_{n\to\infty}\psi\left(t_n\right)=0\tag{33}$$

if $\lim_{n\to\infty}t_n=0$ for $t_n\in[0,\infty)$.

Remark 7. In Theorem 5, let $\varphi(x)$ be $x/2$. Then it is easy to see that substituting $kx/2$ for $\psi(x)$, where $k\in[0,1)$, yields the main result of Berinde [27, Theorem 3].

The following example shows that Theorem 5 is more general than Theorem 2.1 [25] (stated as Theorem 3 in the present paper) and Theorem 2.1 in [26], since the contractive condition (7) is more general than the contractive conditions (2) and (3).

Example 8. Let $X=\mathbb{R}$. Then $(X,\leq)$ is a partially ordered set with the natural ordering of real numbers. Let $d: X\times X\to R^+$ be defined by

$$d(x, y)=|x-y|\quad\text{for } x, y\in X.\tag{34}$$

Then (X, d) is a complete metric space.

Define $F : X \times X \to X$ by $F(x, y) = (x - 4y)/8$, $(x, y) \in X \times X$.

Then F is continuous, has mixed monotone property, and satisfies condition (7) but does not satisfy either condition (2) or condition (3). Indeed, assume there exists $k \in [0, 1)$, such that (2) holds. Then, we must have

$$\left|\frac{x-4y}{8} - \frac{u-4v}{8}\right| \le \frac{k}{2}\{|x-u| + |y-v|\}, \quad x \ge u, \quad y \le v, \tag{35}$$

by which, for $x = u$, we get

$$|y - v| \le k|y - v|, \quad y \le v, \tag{36}$$

which for $y < v$ implies $1 \le k$, a contradiction, since $k \in [0, 1)$. Hence F does not satisfy contractive condition (2).

Further, the contractive condition (3) is also not satisfied. Assume, to the contrary, that there exist $\varphi, \psi : [0, \infty) \to [0, \infty)$ satisfying (φ_1)–(φ_3) and (i_ψ), respectively, such that (3) holds. Then, we must have

$$\varphi\left(\left|\frac{x-4y}{8} - \frac{u-4v}{8}\right|\right) \le \frac{1}{2}\varphi(|x-u| + |y-v|) - \psi\left(\frac{|x-u| + |y-v|}{2}\right), \tag{37}$$

for all $x \ge u$ and $y \le v$. On taking $x = u$, $y \ne v$ and letting $\alpha = |y - v|/2$, we obtain

$$\varphi(\alpha) \le \frac{1}{2}\varphi(2\alpha) - \psi(\alpha); \quad \alpha > 0. \tag{38}$$

But by (φ_3) we have $(1/2)\varphi(2\alpha) \le \varphi(\alpha)$ and hence, we deduce that, for all $\alpha > 0$, $\psi(\alpha) \le 0$, that is, $\psi(\alpha) = 0$, which contradicts (i_ψ). This shows that F does not satisfy (3).

Now, we prove that (7) holds. Indeed, for $x \ge u$ and $y \le v$, we have

$$\begin{aligned} \left|\frac{x-4y}{8} - \frac{u-4v}{8}\right| &\le \frac{1}{8}|x-u| + \frac{1}{2}|y-v|, \\ \left|\frac{y-4x}{8} - \frac{v-4u}{8}\right| &\le \frac{1}{8}|y-v| + \frac{1}{2}|x-u|, \end{aligned} \tag{39}$$

and by summing up the last two inequalities we get exactly (7) with $\varphi(t) = (1/2)t$, $\psi(t) = (5/16)t$. Also, $x_0 = -1$, $y_0 = 1$ are the two points in X such that $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$.

So by Theorem 5 we obtain that F has a coupled fixed point $(0, 0)$ but none of Theorem 2.1 in [25] and Theorem 2.1 in [26] can be applied to F in this example.

Next, we prove the uniqueness of coupled fixed point for our main result by defining a partial ordering on $X \times X$. If $(X, \le)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, \quad (x, y) \le (u, v) \Longleftrightarrow x \le u, \quad y \ge v. \tag{40}$$

Then, we say $(x, y) \ge (u, v) \Leftrightarrow x \ge u$, $y \le v$.

We say that (x, y) and (u, v) are comparable if $(x, y) \le (u, v)$ or $(x, y) \ge (u, v)$.

Theorem 9. *Let all the conditions of Theorem 5 be fulfilled and for every (x, y), (x^*, y^*) in $X \times X$, there exists a (u, v) in $X \times X$ that is comparable to (x, y) and (x^*, y^*). Then F has a unique coupled fixed point.*

Proof. From Theorem 5, the set of coupled fixed points of F is nonempty. Assume that (x, y) and (x^*, y^*) are two coupled fixed points of F. We will show that $x = x^*$ and $y = y^*$.

By assumption, there exists $(u, v) \in X \times X$ that is comparable to (x, y) and (x^*, y^*). We define the sequences $\{u_n\}$ and $\{v_n\}$ as follows:

$$\begin{aligned} u_0 &= u, \qquad v_0 = v, \\ u_{n+1} &= F(u_n, v_n), \qquad v_{n+1} = F(v_n, u_n), \quad n \ge 0. \end{aligned} \tag{41}$$

Since (u, v) is comparable to (x, y), we may assume $(x, y) \ge (u, v) = (u_0, v_0)$. Following the proof of Theorem 5 we obtain inductively

$$(x, y) \ge (u_n, v_n), \quad n \ge 0, \tag{42}$$

and therefore, by (7), we get

$$\begin{aligned} &\varphi\left(\frac{d(x, u_{n+1}) + d(y, v_{n+1})}{2}\right) \\ &\quad = \varphi\left(\frac{d(F(x, y), F(u_n, v_n)) + d(F(y, x), F(v_n, u_n))}{2}\right) \\ &\quad \le \psi\left(\frac{d(x, u_n) + d(y, v_n)}{2}\right). \end{aligned} \tag{43}$$

Thus, we have $\varphi(\alpha_{n+1}) \le \psi(\alpha_n)$, where $\alpha_n := (d(x, u_n) + d(y, v_n))/2$. Inspired with the proof of Theorem 5 we can conclude that $\{\alpha_n\}$ converges to α for some $\alpha \ge 0$. If $\alpha > 0$, then we have

$$\varphi(\alpha) \le \limsup_{n\to\infty} \varphi(\alpha_{n+1}) \le \limsup_{n\to\infty} \psi(\alpha_n) < \varphi(\alpha), \tag{44}$$

which shows a contradiction. Thus $\alpha = 0$; that is, $\lim_{n\to\infty}((d(x, u_n) + d(y, v_n))/2) = 0$, which implies

$$\lim_{n\to\infty} d(x, u_n) = \lim_{n\to\infty} d(y, v_n) = 0. \tag{45}$$

Similarly, we obtain that $\lim_{n\to\infty} d(x^*, u_n) = \lim_{n\to\infty} d(y^*, v_n) = 0$, and hence, by the uniqueness of limit $x = x^*$ and $y = y^*$. □

Theorem 10. *Let all the conditions of Theorem 5 be fulfilled and suppose that x_0, $y_0 \in X$ are comparable. Then F has a unique fixed point; that is, there exist $x \in X$ such that $F(x, x) = x$.*

Proof. By Theorem 5, without loss of generality, we assume that

$$x_0 \le F(x_0, y_0), \qquad y_0 \ge F(y_0, x_0). \tag{46}$$

Since x_0 and y_0 are comparable, we have $x_0 \le y_0$ or $x_0 \ge y_0$. We assume the second case. Then, by the mixed monotone property of F, we have

$$x_1 = F(x_0, y_0) \ge F(y_0, x_0) = y_1, \tag{47}$$

and, hence, by making use of induction, we can get

$$x_n \ge y_n, \quad n \ge 0. \tag{48}$$

Also, we have

$$x = \lim_{n\to\infty} F(x_n, y_n), \qquad y = \lim_{n\to\infty} F(y_n, x_n), \tag{49}$$

then, by the continuity of the distance function d, we can obtain

$$\begin{aligned} d(x, y) &= d\left(\lim_{n\to\infty} F(x_n, y_n), \lim_{n\to\infty} F(y_n, x_n)\right) \\ &= \lim_{n\to\infty} d(F(x_n, y_n), F(y_n, x_n)) \\ &= \lim_{n\to\infty} d(x_{n+1}, y_{n+1}). \end{aligned} \tag{50}$$

Further, on setting $(x, y) = (x_n, y_n)$ and $(u, v) = (y_n, x_n)$ in (7), we have

$$\varphi(d(F(x_n, y_n), F(y_n, x_n))) \le \psi(d(x_n, y_n)), \quad n \ge 0. \tag{51}$$

Taking the limit as $n \to \infty$ in the last inequality, we get

$$\begin{aligned} \varphi(d(x, y)) &\le \limsup_{n\to\infty} \varphi(d(F(x_n, y_n), F(y_n, x_n))) \\ &\le \limsup_{n\to\infty} \psi(d(x_n, y_n)). \end{aligned} \tag{52}$$

If $x \ne y$, then by (ψ_i) we get

$$\varphi(d(x, y)) < \varphi(d(x, y)), \tag{53}$$

a contradiction. So $x = y$ and hence $x = F(x, x)$. Similarly, uniqueness of x can be easily established. □

3. Application

As an application of the results proved in Section 2, we study the existence of unique solution of the following integral equation:

$$\begin{aligned} x(t) = &\int_a^b (K_1(s,t) - K_2(s,t))(f_1(s, x(s)) + f_2(s, x(s)))\, ds \\ &+ h(t), \quad t \in [a, b]. \end{aligned} \tag{54}$$

Let Θ denote the class of functions $\theta : [0, \infty) \to [0, \infty)$ which satisfies the following conditions:

(i) θ is nondecreasing; (55)

(ii) there exists some $\psi \in \Psi_\varphi$ such that

$$\theta(r) = \psi\left(\frac{r}{2}\right), \quad \forall r \in [0, \infty), \tag{56}$$

(iii) $\limsup_{n\to\infty} \theta(t_n) < \alpha r$ if $\lim_{n\to\infty} t_n = r > 0$, for some $\alpha \in (0, 1)$, (57)

(iv) $\lim_{n\to\infty} \theta(t_n) = 0$ if $\lim_{n\to\infty} t_n = 0$, (58)

for $t_n \in [0, \infty)$.

Clearly, Θ is nonempty, as for given $\alpha \in (0, 1)$, $\theta_1(r) = kr$ is in Θ, where $0 \le 2k < 1$, $k < \alpha$.

We assume that the functions K_1, K_2, f_1, and f_2 fulfill the following conditions.

Assumption 11. (i) $K_1(s,t), K_2(s,t) \ge 0$, for all $t, s \in [a, b]$.

(ii) There exist the positive numbers λ, μ, and $\theta \in \Theta$, such that for all $t \in [a, b]$ and $x, y \in \mathbb{R}$, with $x \ge y$, the following conditions hold:

$$\begin{aligned} 0 \le f_1(t, x) - f_1(t, y) &\le \lambda\theta(x - y), \\ -\mu\theta(x - y) \le f_2(t, x) - f_2(t, y) &\le 0. \end{aligned} \tag{59}$$

(iii) There exists some $\alpha \in (0, 1)$ satisfying (57), such that

$$\alpha\beta \le 1, \tag{60}$$

where

$$\beta = (\lambda + \mu) \cdot \sup_{t\in I} \int_a^b (K_1(s,t) + K_2(s,t))\, ds. \tag{61}$$

Definition 12. An element $(x, y) \in X \times X$ with $X = C(I, \mathbb{R})$ is called a coupled lower and upper solution of the integral equation (54) if $x(t) \le y(t)$ and

$$\begin{aligned} x(t) \le &\int_a^b K_1(s,t)(f_1(s, x(s)) + f_2(s, y(s)))\, ds \\ &- \int_a^b K_2(s,t)(f_1(s, y(s)) + f_2(s, x(s)))\, ds + h(t), \\ y(t) \ge &\int_a^b K_1(s,t)(f_1(s, y(s)) + f_2(s, x(s)))\, ds \\ &- \int_a^b K_2(s,t)(f_1(s, x(s)) + f_2(s, y(s)))\, ds + h(t) \end{aligned} \tag{62}$$

for all $t \in I = [a, b]$.

Theorem 13. *Consider the integral equation* (54) *with K_i, $f_i \in C(I \times I)$ for $i = 1, 2$ and $h \in C(I, \mathbb{R})$. Suppose that there exists a coupled lower-upper solution $(\overline{x}, \overline{y})$ for* (54) *and the Assumption 11 is satisfied. Then the integral equation* (54) *has a unique solution in $C(I, \mathbb{R})$.*

Proof. Consider the natural order relation on $X = C(I,\mathbb{R})$; that is, for $x, y \in C(I,\mathbb{R})$

$$x \le y \Longleftrightarrow x(t) \le y(t), \quad \forall t \in I. \tag{63}$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,y) = \sup_{t\in I} |x(t) - y(t)|, \quad x, y \in C(I,\mathbb{R}). \tag{64}$$

It is easy to verify that condition (b) in Theorem 5 holds on the complete metric space $C(I,\mathbb{R})$. Also, $X \times X = C(I,\mathbb{R}) \times C(I,\mathbb{R})$ is a partially ordered set under the following order relation in $X \times X$:

$$\begin{gathered}(x,y),(u,v) \in X \times X,\\ (x,y) \le (u,v) \Longleftrightarrow x(t) \le u(t), \quad y(t) \ge v(t), \quad \forall t \in I.\end{gathered} \tag{65}$$

For any $x, y \in X$, max.$\{x(t), y(t)\}$ and min.$\{x(t), y(t)\}$, for each $t \in I$, are in X and are the upper and lower bounds of x, y, respectively. Therefore, for every $(x, y), (u, v) \in X \times X$, there exists a $(\max.\{x,u\}, \min.\{y,v\}) \in X \times X$ that is comparable to (x, y) and (u, v).

Now we define the mapping $F : X \times X \to X$ by

$$\begin{aligned}F(x,y)(t) &= \int_a^b K_1(s,t)\left(f_1(s,x(s)) + f_2(s,y(s))\right)ds\\ &\quad - \int_a^b K_2(s,t)\left(f_1(s,y(s)) + f_2(s,x(s))\right)ds\\ &\quad + h(t) \quad \forall t \in I = [a,b].\end{aligned} \tag{66}$$

First, we show that F has the mixed monotone property. To do this, let $x_1, x_2 \in C(I,\mathbb{R})$ and $x_1 \le x_2$; that is, $x_1(t) \le x_2(t)$ for all $t \in [a,b]$. Then using Assumption 11, for any $y \in C(I,\mathbb{R})$ and all $t \in I = [a,b]$ we obtain

$$\begin{aligned}&F(x_1,y)(t) - F(x_2,y)(t)\\ &= \int_a^b K_1(s,t)\left(f_1(s,x_1(s)) - f_1(s,x_2(s))\right)ds\\ &\quad - \int_a^b K_2(s,t)\left(f_2(s,x_1(s)) - f_2(s,x_2(s))\right)ds \le 0,\end{aligned} \tag{67}$$

which implies that $F(x_1, y) \le F(x_2, y)$. Similarly, if $y_1, y_2 \in C(I,\mathbb{R})$ and $y_1 \le y_2$, then $F(x, y_1) \ge F(x, y_2)$ for any $x \in C(I,\mathbb{R})$. Let $\alpha \in (0,1)$ be as given in Assumption 11. Then for $x, y, u, v \in C(I,\mathbb{R})$ such that $x \ge u$ and $y \le v$ we get

$$\begin{aligned}&F(x,y)(t) - F(u,v)(t)\\ &= \left\{\begin{array}{l}\displaystyle\int_a^b K_1(s,t)\left(f_1(s,x(s)) + f_2(s,y(s))\right)ds\\ \displaystyle -\int_a^b K_2(s,t)\left(f_1(s,y(s)) + f_2(s,x(s))\right)ds + h(t)\end{array}\right\}\\ &\quad - \left\{\begin{array}{l}\displaystyle\int_a^b K_1(s,t)\left(f_1(s,u(s)) + f_2(s,v(s))\right)ds\\ \displaystyle -\int_a^b K_2(s,t)\left(f_1(s,v(s)) + f_2(s,u(s))\right)ds + h(t)\end{array}\right\}\\ &= \int_a^b K_1(s,t)\left(f_1(s,x(s)) - f_1(s,u(s))\right.\\ &\qquad \left. + f_2(s,y(s)) - f_2(s,v(s))\right)ds\\ &\quad + \int_a^b K_2(s,t)\left(f_1(s,v(s)) - f_1(s,y(s))\right.\\ &\qquad \left. + f_2(s,u(s)) - f_2(s,x(s))\right)ds\\ &\le \int_a^b K_1(s,t)\left[\lambda\theta(x(s) - u(s)) + \mu\theta(v(s) - y(s))\right]ds\\ &\quad + \int_a^b K_2(s,t)\left[\lambda\theta(v(s) - y(s)) + \mu\theta(x(s) - u(s))\right]ds.\end{aligned} \tag{68}$$

Since the function θ is nondecreasing and $x \ge u$ and $y \le v$, we have

$$\begin{aligned}\theta(x(s) - u(s)) &\le \theta\left(\sup_{t\in I}|x(t) - u(t)|\right) = \theta(d(x,u)),\\ \theta(v(s) - y(s)) &\le \theta\left(\sup_{t\in I}|v(t) - y(t)|\right) = \theta(d(v,y)),\end{aligned} \tag{69}$$

hence by (68), we obtain

$$\begin{aligned}&|F(x,y)(t) - F(u,v)(t)|\\ &\le \int_a^b K_1(s,t)\left[\lambda\theta(d(x,u)) + \mu\theta(d(v,y))\right]ds\\ &\quad + \int_a^b K_2(s,t)\left[\lambda\theta(d(v,y)) + \mu\theta(d(x,u))\right]ds.\end{aligned} \tag{70}$$

Similarly, we can obtain

$$\begin{aligned}&|F(y,x)(t) - F(v,u)(t)|\\ &\le \int_a^b K_1(s,t)\left[\lambda\theta(d(v,y)) + \mu\theta(d(x,u))\right]ds\\ &\quad + \int_a^b K_2(s,t)\left[\lambda\theta(d(x,u)) + \mu\theta(d(v,y))\right]ds.\end{aligned} \tag{71}$$

By summing up (70) and (71), multiplying by α and dividing by 2, and then taking supremum with respect to t we get, by using (60)-(61),

$$\begin{aligned}&\alpha \cdot \frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\\ &\le \alpha \cdot (\lambda + \mu) \cdot \sup_{t\in I}\int_a^b \left(K_1(s,t) + K_2(s,t)\right)ds\\ &\quad \cdot \frac{\theta(d(x,u)) + \theta(d(v,y))}{2}\\ &\le \frac{\theta(d(x,u)) + \theta(d(v,y))}{2}.\end{aligned} \tag{72}$$

Now, since θ is nondecreasing, we have

$$\theta(d(x,u)) \le \theta(d(x,u)+d(v,y)),$$
$$\theta(d(v,y)) \le \theta(d(x,u)+d(v,y)) \tag{73}$$

and so

$$\frac{\theta(d(x,u))+\theta(d(v,y))}{2} \le \theta(d(x,u)+d(v,y)) = \psi\left(\frac{d(x,u)+d(v,y)}{2}\right), \tag{74}$$

by the definition of θ. Thus we get

$$\alpha \cdot \frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2} \le \psi\left(\frac{d(x,u)+d(v,y)}{2}\right), \tag{75}$$

which is just the contractive condition (7) for $\varphi(t) = \alpha t$, where $\alpha \in (0,1)$. Now, let $(\overline{x},\overline{y}) \in C(I,\mathbb{R}) \times C(I,\mathbb{R})$ be a coupled upper-lower solution of (54). Then we have

$$\overline{x}(t) \le F(\overline{x},\overline{y})(t), \qquad \overline{y}(t) \ge F(\overline{y},\overline{x})(t), \tag{76}$$

for all $t \in I$. Finally, Theorems 5 and 9 give that F has a unique coupled fixed point (x,y). Since $\overline{x} \le \overline{y}$, then the hypotheses of Theorem 10 is satisfied and, therefore, there exists a unique $x \in C(I,\mathbb{R})$ such that $x(t) = F(x,x)(t)$ for all $t \in [a,b]$; that is, the integral equation (54) has a unique solution. □

Remark 14. Trivial choices of the functions $\varphi,\psi,\theta : [0,\infty) \to [0,\infty)$ are $\varphi(t) = \alpha t$, $(0 < \alpha < 1)$, $\psi(t) = \sigma t$, and $\theta(t) = (\sigma/2)t$, $(0 < \sigma < 1)$. Note that here we will assume that $2\alpha \in (0,1)$ and $\sigma < \alpha$.

4. Conclusion

We conclude that the obtained results improve, generalize, and enrich various recent coupled fixed point theorems in the framework of partially ordered metric spaces in a way that is essentially more natural, by considering the more general (symmetric) contractive condition (7). The theoretical results are accompanied by an applied example and an application to the nonlinear integral equation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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