Research Article

On the Class of (q, α, β) -Metrics

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We study a class of Finsler metrics in the form $F = (\alpha + \beta)^q / \alpha^{q-1}$, where α is a Riemannian metric, β is a 1-form, and $1 < q < 2$. F is called (q, α, β) -metrics. We find the necessary and sufficient conditions under which the class of (q, α, β) -metrics is locally projectively flat and Douglas metrics, respectively.

1. Introduction

It is Hilbert's Fourth Problem in the regular case to study and characterize Finsler metrics on an open domain $\mathcal{U} \subset \mathbb{R}^n$ whose geodesics are straight lines. Finsler metrics with this property are called projectively flat metrics. It is easy to see that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if and only if the spray coefficients are in the form $G^i = Py^i$, where P is a positively homogeneous function of degree one in y . In 1903 G. Hamel found a system of partial differential equations that characterize projectively flat Finsler metrics $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$. That is,

$$F_{x^k y^i} y^k - F_{x^i} = 0. \quad (1)$$

A natural problem is to find projectively flat metrics by solving (1). The Beltrami Theorem tells us that a Riemannian metric $\alpha = \sqrt{a_{ij}(x)} y^i y^j$ is locally projectively flat if and only if it is of constant sectional curvature. Thus this problem has been solved in Riemannian Geometry. However for Finsler metrics, this problem is far from being solved.

In order to find examples of projectively flat Finsler metrics, we consider (α, β) -metrics. An (α, β) -metric is defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ positive scalar function on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)} y^i y^j$ is a Riemannian metric, and $\beta = b_i(x) y^i$ is a 1-form on a manifold M . The simplest (α, β) -metric is Randers metric $F = \alpha + \beta$. In [1]; it is proved that a Randers metric $F = \alpha + \beta$

on a manifold is locally projectively flat if and only if α is projectively flat and β is closed. The other important (α, β) -metric is Berwald metric $F = (\alpha + \beta)^2 / \alpha$ on a manifold M . In [2], Shen and Yildirim find necessary and sufficient conditions under which the Berwald metric $F = (\alpha + \beta)^2 / \alpha$ is projectively flat. They proved that the Berwald metric F is projectively flat if and only if the following conditions hold:

- (1) $b_{i|j} = \tau\{(1 + 2b^2) - 3b_i b_j\}$,
- (2) the spray coefficients G^i_α of α are in the form $G^i_\alpha = \theta y^i - \tau \alpha^2 b^i$,

where $b := \|\beta\|_\alpha$, $b_{i|j}$ denote the covariant derivatives of β with respect to α , $\tau = \tau(x)$ is a scalar function, and $\theta = \theta_i y^i$ is a 1-form on M . Also they determined local structure of F with constant flag curvature. In [3, 4], some other special (α, β) -metrics are studied.

In this paper, we consider a special subclass of (α, β) -metrics in the following form:

$$F = \frac{(\alpha + \beta)^q}{\alpha^{q-1}}, \quad (2)$$

where $\phi = (1 + s)^q$ and $1 < q < 2$. We call it (q, α, β) -metric. We prove the following.

Theorem 1. *Let $F = (\alpha + \beta)^q / \alpha^{q-1}$ ($1 < q < 2$) be a (q, α, β) -metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ ($n \geq 3$). F is locally projectively flat if and only if the following conditions hold:*

- (1) β is parallel with respect to α ;
- (2) α is locally projectively flat; that is, α is of constant curvature.

It is said that a Finsler metric is trivial if it satisfies the conclusion of Theorem 1. Thus the above theorem tells us that in the class of (q, α, β) -metrics, where $1 < q < 2$, there is no nontrivial projectively flat metric.

There are two important non-Riemannian quantities in Finsler Geometry, the Berwald and the Landsberg curvatures. The Berwald and Landsberg tensors are, respectively, defined by

$$B_{jkl}^i := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad L_{jkl} := -\frac{1}{2} F F_{y^i} \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \quad (3)$$

Finsler metrics with $B_{jkl}^i = 0$ and $L_{jkl} = 0$ are called Berwald and Landsberg metrics, respectively. Clearly every Berwald metric is Landsberg, but the converse is not necessarily true.

Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be projectively equivalent if they have the same geodesics as point sets. Projective Finsler Geometry studies these metrics. Douglas curvature $\mathcal{D}$ is an important projective invariant in projective Finsler Geometry. A Finsler metric is called Douglas metric if $\mathcal{D} = 0$. Douglas metrics can be viewed as generalized Berwald metrics. Every Berwald metric is Douglas metric but Douglas metrics are not necessarily Berwald. In [5], it is proved that a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed. For another example a Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ and the exponential metric $F = \alpha \exp(\beta/\alpha) + \varepsilon\beta$ are Douglas metrics if and only if β is parallel with respect to α . In this case both of metrics are Berwald metrics, [4, 6, 7]. In [8], it is proved that the Berwald metric $F = (\alpha + \beta)^2/\alpha$ is a Douglas metric if and only if

$$b_{lij} = \tau \{ (1 + 2b^2) a_{ij} - 3b_i b_j \}, \quad (4)$$

where $\tau = \tau(x)$ is scalar function.

In this paper, we are going to find the conditions under which, on a manifold M of dimension $n \geq 3$, the (q, α, β) -metric $F = (\alpha + \beta)^q/\alpha^{q-1}$ is a Douglas metric. More precisely, we prove following.

Theorem 2. *Let $F = (\alpha + \beta)^q/\alpha^{q-1}$ ($1 < q < 2$) be a (q, α, β) -metric on a manifold M of dimension $n \geq 3$. The Finsler metric F is a Douglas metric if and only if β is parallel with respect to α .*

It is known that a locally projectively flat Finsler metric is a Douglas metric, but the converse is not necessarily true. By Theorems 1 and 2, if $F = (\alpha + \beta)^q/\alpha^{q-1}$ ($1 < q < 2$) is a Douglas metric and α is a locally projectively flat Riemannian metric, then F is a projectively flat Finsler metric.

Corollary 3. *Let $F = (\alpha + \beta)^q/\alpha^{q-1}$ ($1 < q < 2$) be a (q, α, β) -metric on a manifold M of dimension $n \geq 3$. Then F is locally projectively flat if and only if it is Douglas metric and α is locally projectively flat Riemannian metric.*

Corollary 4. *Let $F = (\alpha + \beta)^q/\alpha^{q-1}$ ($1 < q < 2$) be a (q, α, β) -metric on a manifold M of dimension $n \geq 3$. If F is a Douglas metric or projectively flat, then the following properties hold:*

- (1) F is a Landsberg metric;
- (2) F is a Berwald metric.

2. Preliminaries

By definition, an (α, β) -metric is a Finsler metric expressed in the following form:

$$F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha}, \quad (5)$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $b := \|\beta_x\| < b_0$, $x \in M$. The function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(0) = 1, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (6)$$

$$|s| \leq b < b_0.$$

Let G^i and G_α^i denote the spray coefficients of F and α , respectively, given by

$$G^i = \frac{1}{4} g^{ij} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \}, \quad (7)$$

$$G_\alpha^i = \frac{1}{4} a^{ij} \{ [\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^l} \},$$

where $g^{ij} = (g_{ij})^{-1}$ and $a^{ij} = (a_{ij})^{-1}$.

A Finsler metric $F = F(x, y)$ on an open domain $\mathcal{U} \subset \mathbb{R}^n$ is said to be projectively flat in $\mathcal{U}$ if all geodesics are straight lines. This is equivalent to $G^i = P(x, y)y^i$, where $P = F_{x^k} y^k / 2F$ is a 1-homogeneous function on TM_0 . In this case F is of scalar curvature with flag curvature

$$K = \frac{P^2 - P_{x^k} y^k}{F^2}. \quad (8)$$

Let b_{lij} denote the coefficients of the covariant derivative of β with respect to α and define

$$r_{ij} = \frac{1}{2} (b_{lij} + b_{jli}), \quad r_{00} = r_{ij} y^i y^j,$$

$$r_j = b^i r_{ij}, \quad r_0 = r_j y^j, \quad (9)$$

$$s_{ij} = \frac{1}{2} (b_{lij} - b_{jli}), \quad s_j^i = a^{im} s_{mj}, \quad s_0^i = s_j^i y^j,$$

$$s_j = b^i s_{ij}, \quad s_0 = s_j y^j,$$

where $b^i = a^{ij} b_j$ and $b^2 = b^i b_i$. Clearly, β is closed if and only if $s_{ij} = 0$. A 1-form β is said to be parallel with respect to α if $b_{lij} = 0$ and an (α, β) -metric F is said to be trivial if $r_{ij} = s_{ij} = 0$. We have the following.

Lemma 5. The geodesic coefficients of G^i are related to G_α^i by

$$G^i = G_\alpha^i + \alpha Q s_0^i + J(-2\alpha Q s_0 + r_{00}) \frac{y^i}{\alpha} + H(-2\alpha Q s_0 + r_{00}) \left(b^i - s \frac{y^i}{\alpha} \right), \quad (10)$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$

$$J := \frac{(\phi - s\phi')\phi'}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']}, \quad (11)$$

$$H := \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']}.$$

The formula (10) is given in [9, 10]. A different version of (10) is given in [7].

From (10), it is easy to see that if α is projectively flat ($G_\alpha^i = \eta y^i$) and β is parallel with respect to α ($r_{ij} = 0, s_{ij} = 0$), then $G^i = G_\alpha^i = \eta y^i$. Thus $F = \alpha\phi(\beta/\alpha)$ is a projectively flat Berwald metric. If $\mathbf{K} \neq 0$, then F is a Riemannian metric by Numata's Theorem [11]. If $\mathbf{K} = 0$, then F is locally Minkowskian.

Lemma 6 (see [2]). An (α, β) -metric $F = \alpha\phi(s)$, where $s = \beta/\alpha$, is projectively flat on an open subset $\mathcal{U} \subset \mathbb{R}^n$ if and only if

$$(a_{ml}\alpha^2 - y_m y_l) G_\alpha^m + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l \alpha - s y_l) = 0, \quad (12)$$

where $y_m = a_{im} y^i$ and $s_{l0} = s_{lk} y^k$.

3. (q, α, β) -Metrics

In this section, we consider a special subclass of (α, β) -metrics in the following form:

$$F = \frac{(\alpha + \beta)^q}{\alpha^{q-1}}. \quad (13)$$

Let $\phi : (-1, 1) \rightarrow \mathbb{R}$ be a C^∞ function such that $\phi(s) = (1+s)^q$, $1 \leq q \leq 2$ and $b := \|\beta_x\| < 1$. It is easy to see that

$$\phi' = q(1+s)^{q-1}, \quad \phi'' = q(q-1)(1+s)^{q-2} > 0. \quad (14)$$

Since $\phi(s) = (1+s)^q > 0$ and $|s| < 1$, then $\phi - s\phi' = (1+s)^{q-1}\{1 + s(1-q)\} > 0$. Thus $F = (\alpha + \beta)^q/\alpha^{q-1}$ is a Finsler metric. We call it (q, α, β) -metric. When $q = 1$ or $q = 2$, F becomes Randers metric and Berwald metric, respectively. In this paper we assume that $1 < q < 2$.

By Lemma 5, we have

$$Q = \frac{q}{1 + s(1-q)},$$

$$J = \frac{q(1 + s(1-q))}{2[s^2(1-q^2) + s(2-q) + 1 + b^2 q(q-1)]}, \quad (15)$$

$$H = \frac{q(q-1)}{2[s^2(1-q^2) + s(2-q) + 1 + b^2 q(q-1)]}.$$

Lemma 7. If $(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m = 0$, then α is locally projectively flat.

Proof. If $(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m = 0$, then

$$a_{ml}\alpha^2 G_\alpha^m = y_m y_l G_\alpha^m. \quad (16)$$

Contracting (16) with a^{il} , we get

$$\alpha^2 G_\alpha^i = y_m y^i G_\alpha^m. \quad (17)$$

Let $P := (y_m/\alpha^2)G_\alpha^m$, then

$$G_\alpha^i = P y^i. \quad (18)$$

Thus, α is projectively flat. $\square$

Theorem 8. Let $F = (\alpha + \beta)^q/\alpha^{q-1}$ ($1 < q < 2$) be a (q, α, β) -metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ ($n \geq 3$). F is locally projectively flat if and only if the following conditions hold:

- (1) β is parallel with respect to α .
- (2) α is locally projectively flat; that is, α is of constant curvature.

Proof. If F is projectively flat. First, we rewrite (12) as a polynomial in y^i and α , which is linear in α . This gives

$$2[\alpha^2(1 + b^2 q(q-1)) + \alpha\beta(2-q) + \beta^2(1-q^2)]$$

$$\times [\alpha + \beta(1-q)] [a_{lm}\alpha^2 - y_m y_l] G_\alpha^m$$

$$+ 2\alpha^4 q [\alpha^2(1 + b^2 q(q-1)) + \alpha\beta(2-q)$$

$$+ \beta^2(1-q^2)] s_{l0} + \alpha^2 q(q-1)$$

$$\times [-2\alpha^2 q s_0 + r_{00}(\alpha + \beta(1-q))] [b_l \alpha^2 - \beta y_l] = 0. \quad (19)$$

Contracting (19) with b^l yields

$$2[\alpha^2(1 + b^2 q(q-1)) + \alpha\beta(2-q) + \beta^2(1-q^2)]$$

$$\times [\alpha + \beta(1-q)] [b_m \alpha^2 - y_m \beta] G_\alpha^m$$

$$+ 2\alpha^4 q [\alpha^2(1 + b^2 q(q-1)) + \alpha\beta(2-q)$$

$$+ \beta^2(1-q^2)] s_0 + \alpha^2 q(q-1)$$

$$\times [-2\alpha^2 q s_0 + r_{00}(\alpha + \beta(1-q))] [b^2 \alpha^2 - \beta^2] = 0. \quad (20)$$

Namely,

$$\begin{aligned}
 & 2 \left[\alpha^2 (1 + b^2 q (q - 1)) + \alpha \beta (2 - q) + \beta^2 (1 - q^2) \right] \\
 & \times [\alpha + \beta (1 - q)] [b_m \alpha^2 - y_m \beta] G_\alpha^m \\
 & + 2\alpha^4 q [\alpha^2 + \alpha \beta (2 - q) + \beta^2 (1 - q)] s_0 \\
 & + \alpha^2 q (q - 1) r_{00} [\alpha + \beta (1 - q)] [b^2 \alpha^2 - \beta^2] = 0.
 \end{aligned} \quad (21)$$

We have

$$\begin{aligned}
 & [\alpha^2 + \alpha \beta (2 - q) + \beta^2 (1 - q)] \\
 & = [\alpha + \beta (1 - q)] [\alpha + \beta].
 \end{aligned} \quad (22)$$

Plugging (22) into (21) yields

$$\begin{aligned}
 & 2 \left[\alpha^2 (1 + b^2 q (q - 1)) + \alpha \beta (2 - q) + \beta^2 (1 - q^2) \right] \\
 & \times [b_m \alpha^2 - y_m \beta] G_\alpha^m + 2\alpha^4 q [\alpha + \beta] s_0 \\
 & + \alpha^2 q (q - 1) r_{00} [b^2 \alpha^2 - \beta^2] = 0.
 \end{aligned} \quad (23)$$

Replacing y with $-y$, we get

$$\begin{aligned}
 & 2 \left[\alpha^2 (1 + b^2 q (q - 1)) - \alpha \beta (2 - q) + \beta^2 (1 - q^2) \right] \\
 & \times [b_m \alpha^2 - y_m \beta] G_\alpha^m - 2\alpha^4 q [\alpha - \beta] s_0 \\
 & + \alpha^2 q (q - 1) r_{00} [b^2 \alpha^2 - \beta^2] = 0.
 \end{aligned} \quad (24)$$

Equations (23)-(24) yield

$$4\alpha\beta(2-q)[b_m\alpha^2 - y_m\beta]G_\alpha^m + 4\alpha^5qs_0 = 0. \quad (25)$$

Namely,

$$\beta(2-q)[b_m\alpha^2 - y_m\beta]G_\alpha^m = -\alpha^4qs_0. \quad (26)$$

Note that α^4 is not divisible by β and β is not divisible by α^4 . Thus $[b_m\alpha^2 - y_m\beta]G_\alpha^m$ is divisible by α^4 and s_0 is divisible by β . Therefore, there are two scalar functions $\tau = \tau(x)$ and $\lambda = \lambda(x)$ such that

$$\begin{aligned}
 & [b_m \alpha^2 - y_m \beta] G_\alpha^m = \tau \alpha^4, \\
 & s_0 = \lambda \beta.
 \end{aligned} \quad (27)$$

Then (26) becomes

$$(2-q)\tau = -\lambda q. \quad (28)$$

Plugging (27) and (28) into (23) yields

$$\begin{aligned}
 & 2\alpha^2 [\alpha^2 (1 + b^2 q (q - 1)) - \beta^2 (q^2 - q + 1)] \tau \\
 & = -q(q-1)r_{00}[b^2\alpha^2 - \beta^2].
 \end{aligned} \quad (29)$$

Note that $[b^2\alpha^2 - \beta^2]$ is not divisible by α^2 and $[\alpha^2(1 + b^2q(q-1)) - \beta^2(q^2 - q + 1)]$. Thus $\tau = 0$; then $\lambda = 0$ and

$$r_{00} = 0, \quad s_0 = 0, \quad [b_m\alpha^2 - y_m\beta]G_\alpha^m = 0. \quad (30)$$

Thus β is closed and parallel with respect to α . By Lemma 7, α is locally projectively flat.

Conversely, if β is parallel with respect to α and α is locally projectively flat, by Lemma 5 we get $G^i = G_\alpha^i = \eta y^i$, where $\eta = \eta(x, y)$ is y -homogeneous function of degree one; thus F is locally projectively flat. $\square$

Lemma 9. Suppose that $F = (\alpha + \beta)^q / \alpha^{q-1}$ ($1 < q < 2$) is locally projectively flat with constant flag curvature $\mathbf{K} = \lambda$; then $\lambda = 0$.

Proof. By Theorem 8, $G^i = \eta y^i$; then by (8), the equation $\mathbf{K} = \lambda$ yields

$$\alpha^2 \left(1 + \frac{\beta}{\alpha} \right)^{2q} \lambda = \eta^2 - \eta_{x^k} y^k. \quad (31)$$

Replacing y with $-y$ yields

$$\alpha^2 \left(1 - \frac{\beta}{\alpha} \right)^{2q} \lambda = \eta^2 - \eta_{x^k} y^k. \quad (32)$$

From (31) + (32), we get

$$\alpha^2 \lambda \left\{ \left(1 + \frac{\beta}{\alpha} \right)^{2q} + \left(1 - \frac{\beta}{\alpha} \right)^{2q} \right\} = 2(\eta^2 - \eta_{x^k} y^k). \quad (33)$$

Using Taylor expansion of $(1 + \beta/\alpha)^{2q}$ and $(1 - \beta/\alpha)^{2q}$, we get

$$2\alpha^2 \lambda = 2(\eta^2 - \eta_{x^k} y^k). \quad (34)$$

Because α^2 is not divisible by $(\eta^2 - \eta_{x^k} y^k)$, then $\lambda = 0$. $\square$

Proposition 10. Let $F = (\alpha + \beta)^q / \alpha^{q-1}$ be projectively flat with zero flag curvature; then α is flat metric and β is parallel. In this case F is locally Minkowskian.

Proof. By Theorem 8, β is parallel. By Lemma 9, F has zero flag curvature; thus α has zero sectional curvature. Thus α is flat metric; that is, α is locally isometric to the Euclidean metric, so $G_\alpha^i = 0$. Thus $G^i = 0$; that is, F is locally Minkowskian. $\square$

4. Douglas Metric

In this section, we consider $F = (\alpha + \beta)^q / \alpha^{q-1}$ and determine under which conditions Finsler metric F is a Douglas metric.

Let

$$D_{jkl}^i := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right). \quad (35)$$

It is easy to verify that $\mathcal{D} = D_{jkl}^i dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is well-defined tensor on slit tangent bundle TM_0 . We call $\mathcal{D}$

the Douglas curvature. A Finsler metric with $\mathcal{D} = 0$ is called a Douglas metric. In particular, Riemannian metrics are Douglas metrics. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant; namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + P y^i$, where $P = P(x, y)$ is positively y -homogeneous of degree one; then the Douglas tensor of F is the same as that of $\bar{F}$. The notion of Douglas metric was proposed by Bácsó and Matsumoto as a generalization of Berwald metrics [5].

It is easy to verify the following lemma.

Lemma 11. *A Finsler metric F is a Douglas metric if and only if*

$$G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i = \gamma_{jk}^i(x) y^j y^k, \quad (36)$$

for some set of local functions $\gamma_{jk}^i(x)$.

For an (α, β) -metric, the Douglas tensor is determined by

$$D_{jkl}^i := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right), \quad (37)$$

where

$$\begin{aligned} T^i &:= \alpha Q s_0^i + H(r_{00} - 2\alpha Q s_0) b^i, \\ \frac{\partial T^m}{\partial y^m} &= Q' s_0 + H' \alpha^{-1} (b^2 - s^2) (r_{00} - 2\alpha Q s_0) \\ &\quad + 2H[r_0 - Q'(b^2 - s^2) s_0 - Q s s_0]. \end{aligned} \quad (38)$$

Lemma 12 (see [8]). *Suppose that Q/s is not constant. If an (α, β) -metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is a Douglas metric on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ ($n \geq 3$) and $b = \|\beta_x\|_\alpha \neq 0$, then β is closed.*

Theorem 13. *Suppose that $F = (\alpha + \beta)^q / \alpha^{q-1}$ ($1 < q < 2$) is a (q, α, β) -metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ ($n \geq 3$). The Finsler metric F is a Douglas metric in $\mathcal{U}$ if and only if β is parallel with respect to α .*

Proof. Let F be a Douglas metric. It is easy to verify that Q/s is not constant; then by Lemma 12, β is closed. Thus $s_0 = 0$ and $s_0^i = 0$. By Lemma 11 and (37), we have

$$T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i = \gamma_{jk}^i(x) y^j y^k, \quad (39)$$

where $\gamma_{jk}^i(x)$ are some set of local scalar functions and

$$\begin{aligned} T^i &= H r_{00} b^i, \quad \frac{\partial T^m}{\partial y^m} = H' \alpha^{-1} (b^2 - s^2) r_{00} + 2H r_0, \\ H' &= \frac{-q(q-1)[2s(1-q^2) + (2-q)]}{2[s^2(1-q^2) + s(2-q) + 1 + b^2 q(q-1)]^2}. \end{aligned} \quad (40)$$

We rewrite (39) as a polynomial in y^i and α , which is linear in α . This gives

$$\begin{aligned} & \frac{q(q-1)\alpha^2}{2[\beta^2(1-q^2) + \alpha\beta(2-q) + \alpha^2(1+b^2q(q-1))]} r_{00} b^i \\ & + \frac{1}{n+1} \\ & \times \frac{q(q-1)[2\beta(1-q^2) + \alpha(2-q)]}{2[\beta^2(1-q^2) + \alpha\beta(2-q) + \alpha^2(1+b^2q(q-1))]^2} \\ & \times (b^2\alpha^2 - \beta^2) r_{00} y^i \\ & - \frac{1}{n+1} \\ & \times \frac{q(q-1)\alpha^2}{[\beta^2(1-q^2) + \alpha\beta(2-q) + \alpha^2(1+b^2q(q-1))]} r_0 y^i \\ & = \gamma_{jk}^i(x) y^j y^k. \end{aligned} \quad (41)$$

Namely,

$$\begin{aligned} & [n+1]q(q-1)\alpha^2 \\ & \times [\beta^2(1-q^2) + \alpha\beta(2-q) + \alpha^2(1+b^2q(q-1))] r_{00} b^i \\ & + q(q-1)[2\beta(1-q^2) + \alpha(2-q)] \\ & \times (b^2\alpha^2 - \beta^2) r_{00} y^i - 2q(q-1)\alpha^2 \\ & \times [\beta^2(1-q^2) + \alpha\beta(2-q) + \alpha^2(1+b^2q(q-1))] r_0 y^i \\ & = 2\gamma_{jk}^i(x) y^j y^k [n+1][\beta^2(1-q^2) + \alpha\beta(2-q) \\ & \quad + \alpha^2(1+b^2q(q-1))]^2. \end{aligned} \quad (42)$$

Contracting (42) with b_i yields

$$\begin{aligned} & [n+1]q(q-1)\alpha^2 \\ & \times [\beta^2(1-q^2) + \alpha\beta(2-q) + \alpha^2(1+b^2q(q-1))] r_{00} b^2 \\ & + q(q-1)[2\beta(1-q^2) + \alpha(2-q)] \\ & \times (b^2\alpha^2 - \beta^2) r_{00} \beta - 2q(q-1)\alpha^2 \\ & \times [\beta^2(1-q^2) + \alpha\beta(2-q) + \alpha^2(1+b^2q(q-1))] r_0 \beta \\ & - T[n+1][\beta^2(1-q^2) + \alpha\beta(2-q) \\ & \quad + \alpha^2(1+b^2q(q-1))]^2 = 0, \end{aligned} \quad (43)$$

where $T := 2\gamma_{jk}^i(x) y^j y^k b_i$.

In the following proof we consider the following four cases.

Case 1. Suppose r_{00} and T are divisible by α^2 ; then there are two scalar functions $\lambda = \lambda(x)$ and $\tau = \tau(x)$ such that

$$T = \lambda\alpha^2, \quad r_{00} = \tau\alpha^2, \quad r_0 = \tau\beta. \quad (44)$$

Plugging (44) into (43), we get

$$A_1\alpha^6 + A_2\alpha^5\beta + A_3\alpha^4\beta^2 + A_4\alpha^3\beta^3 + A_5\alpha^2\beta^4 = 0, \quad (45)$$

where

$$\begin{aligned} A_1 &= [n+1]q(q-1)[1+b^2q(q-1)]b^2\tau \\ &\quad - [n+1][1+b^2q(q-1)]^2\lambda, \\ A_2 &= [n+2]q(q-1)(2-q)b^2\tau \\ &\quad - 2[n+1](2-q)[1+b^2q(q-1)]\lambda, \\ A_3 &= [n+2]q(q-1)(1-q^2)b^2\tau \\ &\quad - 2q(q-1)[1+b^2q(q-1)]\tau \\ &\quad - [n+1](2-q)^2\lambda - 2[n+1] \\ &\quad \times (1-q^2)[1+b^2q(q-1)]\lambda, \\ A_4 &= -3q(q-1)(2-q)\tau \\ &\quad - 2[n+1](1-q^2)(2-q)\lambda, \\ A_5 &= -4q(q-1)(1-q^2)\tau - [n+1](1-q^2)^2\lambda. \end{aligned} \quad (46)$$

Because α^{even} is polynomial in y^i , from (45), we obtain

$$\begin{aligned} A_1\alpha^6 + A_3\alpha^4\beta^2 + A_5\alpha^2\beta^4 &= 0, \\ A_2\alpha^5\beta + A_4\alpha^3\beta^3 &= 0. \end{aligned} \quad (47)$$

By (47), we get

$$\begin{aligned} \alpha^2[A_1\alpha^2 + A_3\beta^2] &= -A_5\beta^4, \\ A_2\alpha^2 &= -A_4\beta^2. \end{aligned} \quad (48)$$

Because β^4 is not divisible by α^2 and $[A_1\alpha^2 + A_3\beta^2]$, we get $A_5 = 0$, $A_1\alpha^2 + A_3\beta^2 = 0$. β^2 is not divisible by α^2 ; we get

$$A_1 = A_2 = A_3 = A_4 = 0. \quad (49)$$

By $A_4 = A_5 = 0$, we get $\tau = \lambda = 0$. Thus $r_{00} = r_0 = T = 0$; that is, β is parallel with respect to α .

Case 2. Suppose that r_{00} is not divisible by α^2 , but T is divisible by α^2 ; namely, there is a scalar function $\lambda = \lambda(x)$ such that

$$T = \lambda\alpha^2. \quad (50)$$

Plugging (50) into (43) yields

$$\begin{aligned} B_1\alpha^6 + B_2\beta\alpha^5 + (B_3r_{00} + B_4\beta r_0)\alpha^4 \\ + (B_5\beta r_{00} + B_6\beta^2 r_0 + B_7\beta^3)\alpha^3 \\ + (B_8\beta^2 r_{00} + B_9\beta^3 r_0 + B_{10}\beta^4)\alpha^2 \\ + B_{11}\beta^3 r_{00}\alpha + B_{12}\beta^4 r_{00} = 0, \end{aligned} \quad (51)$$

where

$$\begin{aligned} B_1 &= -[n+1][1+b^2q(q-1)]^2\lambda, \\ B_2 &= -2[n+1](2-q)[1+b^2q(q-1)]\lambda, \\ B_3 &= [n+1]q(q-1)[1+b^2q(q-1)]b^2, \\ B_4 &= -2q(q-1)[1+b^2q(q-1)], \\ B_5 &= [n+2]q(q-1)(2-q)b^2, \\ B_6 &= -2q(q-1)(2-q), \\ B_7 &= -2[n+1](2-q)(1-q^2)\lambda, \\ B_8 &= [n+3]q(q-1)(1-q^2)b^2, \\ B_9 &= -2q(q-1)(1-q^2), \\ B_{10} &= -[n+1](1-q^2)^2\lambda, \\ B_{11} &= -q(q-1)(2-q), \\ B_{12} &= -q(q-1)(1-q^2). \end{aligned} \quad (52)$$

Because α^{even} is polynomial in y^i , from (51), we obtain

$$\begin{aligned} B_1\alpha^6 + (B_3r_{00} + B_4\beta r_0)\alpha^4 \\ + (B_8\beta^2 r_{00} + B_9\beta^3 r_0 + B_{10}\beta^4)\alpha^2 \\ + B_{12}\beta^4 r_{00} = 0, \end{aligned} \quad (53)$$

$$B_2\alpha^5\beta + (B_5\beta r_{00} + B_6\beta^2 r_0 + B_7\beta^3)\alpha^3 + B_{11}\beta^3 r_{00}\alpha = 0. \quad (54)$$

From (54), we have

$$\alpha^2[B_2\alpha^2 + (B_5r_{00} + B_6\beta r_0 + B_7\beta^2)] = -B_{11}\beta^2 r_{00}. \quad (55)$$

Because β^2 is not divisible by α^2 , r_{00} must be divisible by α^2 . This contradicts our assumption. Hence, this case is impossible.

Case 3. Suppose that T is not divisible by α^2 , but r_{00} is divisible by α^2 ; namely, there is a scalar function $\tau = \tau(x)$ such that

$$r_{00} = \tau\alpha^2, \quad r_0 = \tau\beta. \quad (56)$$

Plugging (56) into (43) yields

$$\begin{aligned} C_1\alpha^6 + C_2\beta\alpha^5 + (C_3\beta^2 + C_4T)\alpha^4 \\ + (C_5\beta^3 + C_6\beta T)\alpha^3 + (C_7\beta^4 + C_8\beta^2T)\alpha^2 \\ + C_9\beta^3T\alpha + C_{10}\beta^4T = 0, \end{aligned} \quad (57)$$

where

$$\begin{aligned} C_1 &= [n+1]q(q-1)[1+b^2q(q-1)]b^2\tau, \\ C_2 &= [n+2]q(q-1)(2-q)b^2\tau, \\ C_3 &= [n+3]q(q-1)(1-q^2)b^2\tau \\ &\quad - 2q(q-1)[1+b^2q(q-1)]\tau, \\ C_4 &= [n+1][1+b^2q(q-1)]^2, \\ C_5 &= -3q(q-1)(2-q)\tau, \\ C_6 &= -2[n+1](2-q)[1+b^2q(q-1)], \\ C_7 &= -4q(q-1)(1-q^2)\tau, \\ C_8 &= -2[n+1](1-q^2)[1+b^2q(q-1)], \\ C_9 &= -2[n+1](2-q)(1-q^2), \\ C_{10} &= [n+1](1-q^2)^2. \end{aligned} \quad (58)$$

Because α^{even} is polynomial in y^i , from (57), we obtain

$$\begin{aligned} C_1\alpha^6 + (C_3\beta^2 + C_4T)\alpha^4 + (C_7\beta^4 + C_8\beta^2T)\alpha^2 \\ + C_{10}\beta^4T = 0, \end{aligned} \quad (59)$$

$$C_2\beta\alpha^5 + (C_5\beta^3 + C_6\beta T)\alpha^3 + C_9\beta^3T\alpha = 0. \quad (60)$$

From (60), we have

$$\alpha^2 [C_2\alpha^2 + (C_5\beta^2 + C_6T)] = -C_9\beta^2T. \quad (61)$$

Because β^2 is not divisible by α^2 , then T must be divisible by α^2 . This contradicts our assumption. Hence, this case is impossible, too.

Case 4. Suppose that T and r_{00} are not divisible by α^2 . From (43), we have

$$\begin{aligned} (D_1r_{00} + D_2r_0\beta + D_3T)\alpha^4 + (D_4r_{00}\beta + D_5r_0\beta^2 + D_6\beta T)\alpha^3 \\ + (D_7r_{00}\beta^2 + D_8r_0\beta^3 + D_9\beta^2T)\alpha^2 \\ + (D_{10}r_{00}\beta^3 + D_{11}\beta^3T)\alpha + (D_{12}r_{00}\beta^4 + D_{13}\beta^4T) = 0, \end{aligned} \quad (62)$$

where

$$\begin{aligned} D_1 &= [n+1]q(q-1)[1+b^2q(q-1)]b^2, \\ D_2 &= -2q(q-1)[1+b^2q(q-1)], \\ D_3 &= -[n+1][1+b^2q(q-1)]^2, \\ D_4 &= [n+2]q(q-1)(2-q)b^2, \\ D_5 &= -2q(q-1)(2-q), \\ D_6 &= -2[n+1][1+b^2q(q-1)], \\ D_7 &= [n+3]q(q-1)(1-q^2), \\ D_8 &= -2q(q-1)(1-q^2), \\ D_9 &= -[n+1][2(1-q^2)[1+b^2q(q-1)] + (2-q)^2], \\ D_{10} &= -q(q-1)(2-q), \\ D_{11} &= -2[n+1](2-q)(1-q^2), \\ D_{12} &= -2q(q-1)(1-q^2), \\ D_{13} &= -[n+1](1-q^2)^2. \end{aligned} \quad (63)$$

Because α^{even} is polynomial in y^i , from (62), we get

$$\begin{aligned} (D_1r_{00} + D_2r_0\beta + D_3T)\alpha^4 \\ + (D_7r_{00}\beta^2 + D_8r_0\beta^3 + D_9\beta^2T)\alpha^2 \\ + (D_{12}r_{00}\beta^4 + D_{13}\beta^4T) = 0, \\ (D_4r_{00}\beta + D_5r_0\beta^2 + D_6\beta T)\alpha^3 \\ + (D_{10}r_{00}\beta^3 + D_{11}\beta^3T)\alpha = 0. \end{aligned} \quad (64)$$

We get

$$\begin{aligned} \alpha^2 [(D_1r_{00} + D_2r_0\beta + D_3T)\alpha^2 \\ + (D_7r_{00}\beta^2 + D_8r_0\beta^3 + D_9\beta^2T)] \\ = -\beta^4 [D_{12}r_{00} + D_{13}T], \end{aligned} \quad (65)$$

$$\alpha^2 [D_4r_{00} + D_5r_0\beta + D_6T] = -\beta^2 [D_{10}r_{00} + D_{11}T].$$

Because β^4 and β^2 are not divisible by α^2 , $[D_{12}r_{00} + D_{13}T]$ and $[D_{10}r_{00} + D_{11}T]$ are divisible by α^2 ; hence there are two scalar functions $K_1 = K_1(x)$ and $K_2 = K_2(x)$ such that

$$\begin{aligned} D_{12}r_{00} + D_{13}T &= K_1\alpha^2, \\ D_{10}r_{00} + D_{11}T &= K_2\alpha^2. \end{aligned} \quad (66)$$

Let

$$A := \begin{bmatrix} D_{12} & D_{13} \\ D_{10} & D_{11} \end{bmatrix}. \quad (67)$$

We have $\det A = 3[n+1]q(q-1)(2-q)(1-q^2)^2 \neq 0$; then, we get

$$A^{-1} := \begin{bmatrix} \frac{-2}{3q(q-1)(1-q^2)} & \frac{1}{3q(q-1)(2-q)} \\ \frac{1}{3[n+1](1-q^2)^2} & \frac{-2}{3[n+1](2-q)(1-q^2)} \end{bmatrix}. \quad (68)$$

From (66) and (68), we obtain

$$\begin{aligned} r_{00} &= \left[\frac{-2}{3q(q-1)(1-q^2)} K_1 + \frac{1}{3[n+1](1-q^2)^2} K_2 \right] \alpha^2, \\ T &= \left[\frac{1}{3q(q-1)(2-q)} K_1 \right. \\ &\quad \left. + \frac{-2}{3[n+1](2-q)(1-q^2)} K_2 \right] \alpha^2. \end{aligned} \quad (69)$$

Thus r_{00} and T are divisible by α^2 . This contradicts our assumption; hence, this case is impossible too. So only Case 1 is possible and in this case we proved that $r_{00} = 0$; that is, β is parallel with respect to α .

Conversely, if β is parallel with respect to α , by Lemma 5, we get $G^i = G^i_\alpha$. Because α is a Riemannian metric then $D^i_{jkl} = 0$. $\square$

By Theorems 1 and 2, we have following Corollary.

Corollary 14. Let $F = (\alpha + \beta)^q / \alpha^{q-1}$ ($1 < q < 2$) be a (q, α, β) -metric. Then F is locally projectively flat if and only if it is Douglas metric and α is locally projectively flat Riemannian metric.

By Theorems 1 and 2 and Lemma 5, we have following corollary.

Corollary 15. Let $F = (\alpha + \beta)^q / \alpha^{q-1}$ ($1 < q < 2$) be a (q, α, β) -metric. If F is a Douglas metric or projectively flat then the following properties hold:

- (1) F is a Landsberg metric;
- (2) F is a Berwald metric.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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