Research Article

Generalized $\oplus$ -Radical Supplemented Modules

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Çalışıcı and Türkmen called a module M generalized $\oplus$ -supplemented if every submodule has a generalized supplement that is a direct summand of M . Motivated by this, it is natural to introduce another notion that we called generalized $\oplus$ -radical supplemented modules as a proper generalization of generalized $\oplus$ -supplemented modules. In this paper, we obtain various properties of generalized $\oplus$ -radical supplemented modules. We show that the class of generalized $\oplus$ -radical supplemented modules is closed under finite direct sums. We attain that over a Dedekind domain a module M is generalized $\oplus$ -radical supplemented if and only if $M/P(M)$ is generalized $\oplus$ -radical supplemented. We completely determine the structure of these modules over left V -rings. Moreover, we characterize semiperfect rings via generalized $\oplus$ -radical supplemented modules.

1. Introduction

Throughout the whole text, all rings are to be associative; unit and all modules are left unitary. We specially mention [1–4] among books concerning the structures of modules and rings. We shall write $N \leq M$ ($N \ll M$) if N is a submodule of M (small in M). By $\text{Rad}(M)$, we denote the radical of M . Let $U, V \leq M$. V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$. V is a supplement of U in M if and only if $M = U + V$ and $U \cap V \ll V$ [4]. A module M is called *supplemented* if every submodule of M has a supplement, and it is called $\oplus$ -*supplemented* if every submodule of M has a supplement that is a direct summand of M [5]. Clearly every $\oplus$ -supplemented module is supplemented. In [6], Zöschinger introduced a notion of modules whose radical has supplements called *radical supplemented*. Xue defined generalized supplemented modules as another generalization of supplemented modules [7]. Let U be any submodule of M . If there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq \text{Rad}(V)$, V is called a *generalized supplement* of U in M . M is called *generalized supplemented* if every submodule of M has a generalized supplement in M . Also Çalışıcı and Türkmen called a module M *generalized $\oplus$ -supplemented* if every submodule

has a generalized supplement that is a direct summand of M as a generalization of $\oplus$ -supplemented modules [8]. So it is natural to introduce another notion that we called generalized $\oplus$ -radical supplemented modules. A module M is called *generalized $\oplus$ -radical supplemented* if every submodule containing radical has a generalized supplement that is a direct summand of M .

In this paper we obtain various properties of generalized $\oplus$ -radical supplemented modules as a proper generalization of generalized $\oplus$ -supplemented modules. We prove the following indications.

- (i) Every generalized $\oplus$ -radical supplemented module has a radical direct summand.
- (ii) $P(M)$ is a generalized $\oplus$ -radical supplemented module for every R -module M .
- (iii) The class of generalized $\oplus$ -radical supplemented modules is closed under finite direct sums.
- (iv) If M is a generalized $\oplus$ -radical supplemented module, then M/U is a generalized $\oplus$ -radical supplemented module for every fully invariant submodule U of M .
- (v) Let $\{M_i\}_{i \in I}$ be any infinite collection of generalized $\oplus$ -radical supplemented modules, and let $M = \oplus_{i \in I} M_i$.

If M is a duo module, then M is generalized $\oplus$ -radical supplemented.

- (vi) Over a Dedekind domain a module M is generalized $\oplus$ -radical supplemented if and only if $M/P(M)$ is generalized $\oplus$ -radical supplemented.
- (vii) Every generalized $\oplus$ -radical supplemented R -module is injective if and only if R is left Noetherian V -ring.
- (viii) A ring R is semiperfect if and only if every finitely generated free R -module is generalized $\oplus$ -radical supplemented.

2. Generalized $\oplus$ -Radical Supplemented Modules

Definition 1. A module M is called *generalized $\oplus$ -radical supplemented* if every submodule containing radical has a generalized supplement that is a direct summand of M .

Recall that a module M is called *radical* if M has no maximal submodules; that is, $\text{Rad}(M) = M$. For a module M , $P(M)$ will indicate the sum of all radical submodules of M . If $P(M) = 0$, M is called *reduced*. Note that $P(M)$ is the largest radical submodule of M [4].

Now we have the following simple fact, which lays a key role in our study.

Lemma 2. *Let M be an R -module. If $\text{Rad}(M) = M$, then M is a generalized $\oplus$ -radical supplemented module.*

Proof. Since $\text{Rad}(M) = M$, M has the trivial generalized supplement 0 in M . Consequently, M is a generalized $\oplus$ -radical supplemented module. $\square$

Corollary 3. *$P(M)$ is a generalized $\oplus$ -radical supplemented module for every R -module M .*

Proposition 4. *Every generalized $\oplus$ -radical supplemented module has a radical direct summand.*

Proof. Let M be a generalized $\oplus$ -radical supplemented module. Then there exist submodules V and V' of M such that $M = \text{Rad}(M) + V$, $\text{Rad}(M) \cap V \subseteq \text{Rad}(V)$, and $M = V \oplus V'$. Since $\text{Rad}(M) = \text{Rad}(V) \oplus \text{Rad}(V')$ by [4, Theorem 21.6], then $M = \text{Rad}(V') + V$. Applying the modular law, we obtain that $\text{Rad}(V') = V'$. $\square$

Recall from [4] that a submodule U of an R -module M is called *fully invariant* if $f(U)$ is contained in U for every R -endomorphism f of M . A module M is called *duo*, if every submodule of M is fully invariant [9].

Theorem 5. *The following statements hold over a ring R .*

- (1) *Let $\{M_i\}_{i \in \{1, 2, \dots, n\}}$ be any finite collection of modules and $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$. Then M is generalized $\oplus$ -radical supplemented if, for each $1 \leq i \leq n$, M_i is generalized $\oplus$ -radical supplemented.*

- (2) *Let $\{M_i\}_{i \in I}$ be any infinite collection of modules and $= \oplus_{i \in I} M_i$. If M is a duo module, then M is generalized $\oplus$ -radical supplemented if, for each $i \in I$, M_i is generalized $\oplus$ -radical supplemented.*

Proof. (1) Let M_i be a generalized $\oplus$ -radical supplemented module for each $1 \leq i \leq n$. To prove that M is a generalized $\oplus$ -radical supplemented module it is sufficient by induction on n to prove this is the case when $n = 2$. Hence, suppose that $n = 2$. Let N be any submodule of M with $\text{Rad}(M) \subseteq N$. Then $M = M_1 + M_2 + N$ so that $M_1 + M_2 + N$ has a generalized supplement 0 in M . Since $M = M_1 \oplus M_2$, then $\text{Rad}(M_2) \subseteq N + M_1$. It follows that $\text{Rad}(M_2) \subseteq M_2 \cap (N + M_1)$ has a generalized supplement H in M_2 such that H is a direct summand of M_2 . By [10, Proposition 2.4], H is a generalized supplement of $N + M_1$ in M . Moreover, $\text{Rad}(M_1) \subseteq N + H$. Since M_1 is a generalized $\oplus$ -radical supplemented module, $M_1 \subseteq (N + H)$ has a generalized supplement G in M_1 such that G is a direct summand of M_1 . Again applying [10, Proposition 2.4], $H + G$ is a generalized supplement of N in M . It is clear that $H + G$ is a direct summand of M . Therefore, M is a generalized $\oplus$ -radical supplemented module.

(2) Let U be any submodule such that $\text{Rad}(M) \subseteq U$. Then $\text{Rad}(M_i) \subseteq U \cap M_i$ for every $i \in I$. By this hypothesis, there exist submodules V_i and V'_i of M_i such that $M_i = (U \cap M_i) + V_i$, $(U \cap M_i) \cap V_i \subseteq \text{Rad}(V_i)$, and $M_i = V_i \oplus V'_i$ for every $i \in I$. Since U is a fully invariant submodule of M , then $U = \oplus_{i \in I} (U \cap M_i)$. Let $V = \oplus_{i \in I} V_i$ and $V' = \oplus_{i \in I} V'_i$. Then there exist submodules U and V of M such that $M = \oplus_{i \in I} M_i = [\oplus_{i \in I} (U \cap M_i)] + (\oplus_{i \in I} V_i) = U + V$, $U \cap V = \oplus_{i \in I} (U \cap M_i) \cap (\oplus_{i \in I} V_i) = \oplus_{i \in I} [(U \cap M_i) \cap V_i] \subseteq \oplus_{i \in I} \text{Rad}(V_i) = \text{Rad}(V)$ by [4, Theorem 21.6] and $V \oplus V' = (\oplus_{i \in I} V_i) \oplus (\oplus_{i \in I} V'_i) = \oplus_{i \in I} M_i = M$. Thus, M is a generalized $\oplus$ -radical supplemented module. $\square$

Proposition 6. *If M is a generalized $\oplus$ -radical supplemented module, then M/U is a generalized $\oplus$ -radical supplemented module for every fully invariant submodule U of M .*

Proof. Let U be any fully invariant submodule of M and let K/U be any submodule of M/U with $\text{Rad}(M/U) \subseteq K/U$. Since $(\text{Rad}(M) + U)/U \subseteq \text{Rad}(M/U)$, we have $\text{Rad}(M) \subseteq K$. By the hypothesis, we have $M = K + V$, $K \cap V \subseteq \text{Rad}(V)$, and $M = V \oplus V'$ for some submodules V and V' of M . By [11, Lemma 2.3] $(V + U)/U$ is generalized supplement of K/U in M/U . Since U is a fully invariant submodule of M , we have $U = (U \cap V) + (U \cap V')$ and $[(V + U)/U] \cap [(V' + U)/U] = 0$. Then $M/U = [(V + U)/U] \oplus [(V' + U)/U]$. Hence, M/U is a generalized $\oplus$ -radical supplemented module. $\square$

Corollary 7. *Let R be a Dedekind domain and let M be an R -module. M is a generalized $\oplus$ -radical supplemented module if and only if $M/P(M)$ is a generalized $\oplus$ -radical supplemented module.*

Proof. We know that $P(M)$ is a fully invariant submodule of M . So, by Proposition 6, $M/P(M)$ is a generalized $\oplus$ -radical supplemented module. Conversely, suppose that $M/P(M)$ is a generalized $\oplus$ -radical supplemented module. Since R is a Dedekind domain, we have $M = P(M) \oplus N$ for some submodule N of M . By the hypothesis, N is a generalized $\oplus$ -radical

supplemented module. Hence, M is a generalized $\oplus$ -radical supplemented module by Theorem 5(1) and Corollary 3. $\square$

Example 8. Consider the left $\mathbb{Z}$ -module $\mathbb{Q}$. According to Lemma 2, $\mathbb{Q}$ is a generalized $\oplus$ -radical supplemented module. On the other hand, $\mathbb{Q}$ is not $\oplus$ -supplemented because it is not torsion.

Theorem 9. *Let M be a module. Suppose that $\text{Rad}(M)$ is small in M . Then M is a generalized $\oplus$ -radical supplemented module if and only if it is $\oplus$ -supplemented.*

Proof. $(\Rightarrow)$ Let N be any submodule of M . Then $\text{Rad}(M) \subseteq \text{Rad}(M) + N \subseteq M$. Since M is a generalized $\oplus$ -radical supplemented module, we have $M = \text{Rad}(M) + N + L$, $(\text{Rad}(M) + N) \cap L \subseteq \text{Rad}(L)$, and $M = L \oplus L'$ for submodules $L, L' \leq M$. Since $\text{Rad}(M) \ll M$, we have $M = N + L$, $N \cap L \ll L$. Therefore M is $\oplus$ -supplemented.

$(\Leftarrow)$ is trivial. $\square$

Corollary 10. *Every generalized $\oplus$ -radical supplemented module over a left Bass ring is $\oplus$ -supplemented.*

Proposition 11. *Every direct summand of $P(M)$ is a generalized $\oplus$ -radical supplemented module for every R -module M .*

Proof. Let N be any direct summand of $P(M)$. Then there exists a submodule L of $P(M)$ such that $P(M) = N \oplus L$. Since $P(M) = \text{Rad}(P(M)) = \text{Rad}(N) \oplus \text{Rad}(L)$ by [4, Theorem 21.6], then $N = \text{Rad}(N) \oplus [N \cap \text{Rad}(L)] = \text{Rad}(N)$. By Lemma 2, N is generalized $\oplus$ -radical supplemented. $\square$

Proposition 12. *Let M be a generalized $\oplus$ -radical supplemented module. Suppose that a cofinite fully invariant submodule K of M is a direct summand of M . Then K is generalized $\oplus$ -radical supplemented.*

Proof. Let U be any submodule of K with $\text{Rad}(K) \subseteq U$. By the hypothesis, we have $M = K \oplus L$ for some finitely generated submodule L of M . Then $\text{Rad}(L) \ll L$. Clearly $\text{Rad}(M) \subseteq U + \text{Rad}(L)$. So there exist submodules V and V' of M such that $M = U + \text{Rad}(L) + V$, $(U + \text{Rad}(L)) \cap V \subseteq \text{Rad}(V)$, and $M = V \oplus V'$. Since $\text{Rad}(L) \ll L$, we have $M = U + V$, $U \cap V \subseteq \text{Rad}(V)$ and $M = V \oplus V'$. It follows that $K = U + (K \cap V)$ and $U \cap (K \cap V) \subseteq \text{Rad}(M)$. Since K is a fully invariant submodule of M , then $K = (K \cap V) \oplus (K \cap V')$. Note that $U \cap (K \cap V) \subseteq \text{Rad}(K \cap V)$. Therefore, K is generalized $\oplus$ -radical supplemented. $\square$

Lemma 13. *Let M be a module and N a submodule such that $\text{Rad}(M) \subseteq N$. If N is a direct summand of M , then $\text{Rad}(M) = \text{Rad}(N)$. In particular, if $\text{Rad}(M)$ is a direct summand of M , then $\text{Rad}(M) = P(M)$.*

Proof. Let N be any direct summand of M such that $\text{Rad}(M) \subseteq N$. Then there exists a submodule N' of M such that $M = N \oplus N'$. By [4, Theorem 21.6], $\text{Rad}(M) = \text{Rad}(N) \oplus \text{Rad}(N')$. Since $\text{Rad}(M) \subseteq N$, $\text{Rad}(M) = [\text{Rad}(N) \oplus \text{Rad}(N')] \cap N = \text{Rad}(N) \oplus [\text{Rad}(N') \cap N] = \text{Rad}(N)$. Now we take $N = \text{Rad}(M)$ under the similar

condition. $\text{Rad}(M) = \text{Rad}(\text{Rad}(M))$; that is, $\text{Rad}(M)$ is radical. Consequently, $\text{Rad}(M) = P(M)$. $\square$

Let R be a ring and let M be an R -module. We consider the following condition.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M .

Proposition 14. *Let M be a generalized $\oplus$ -radical supplemented module with (D_3) and let N be a submodule with $\text{Rad}(M) \subseteq N$. If N is a direct summand of M , N is generalized $\oplus$ -radical supplemented.*

Proof. Let U be a submodule of N such that $\text{Rad}(N) \subseteq U$. By Lemma 13, $\text{Rad}(M) = \text{Rad}(N)$. Since M is a generalized $\oplus$ -radical supplemented module, there exist submodules V and V' of M such that $M = U + V$, $U \cap V \subseteq \text{Rad}(V)$, and $M = V \oplus V'$. Then $N = U + (N \cap V)$. Since M satisfies (D_3) , $N \cap V$ is a direct summand of M . Then there exists a submodule X of M such that $M = (N \cap V) \oplus X$. It follows that $U \cap (N \cap V) \subseteq \text{Rad}(N \cap V)$ and $N = (N \cap V) \oplus (N \cap X)$. Therefore N is a generalized $\oplus$ -radical supplemented module. $\square$

A module M is called *local* if $M/\text{Rad}(M)$ is simple and $\text{Rad}(M)$ is small in M [10]. The module M is called *w-local* if $M/\text{Rad}(M)$ is simple [12].

The following fact is clear.

Corollary 15. *Every w-local module is generalized $\oplus$ -radical supplemented.*

Lemma 16. *Let M be an indecomposable module. If M is a generalized $\oplus$ -radical supplemented module, then $M = \text{Rad}(M)$ or M is w-local.*

Proof. Suppose that $\text{Rad}(M) \neq M$. Then M contains at least one maximal submodule, say K . Since $\text{Rad}(M) \subseteq K$, then there exist submodules V and V' of M such that $M = K + V$, $K \cap V \subseteq \text{Rad}(V)$, and $M = V \oplus V'$. We have $V = M$ because M is indecomposable. Since V is a generalized supplement of K , V is w-local by [12, Lemma 3.3]. So M is w-local. $\square$

Theorem 17. *Let R be a local commutative ring and M be a uniform R -module. Every submodule of M is generalized $\oplus$ -radical supplemented if and only if it is uniserial.*

Proof. $(\Rightarrow)$ By [13, Lemma 6.2], it suffices to show that every finitely generated submodule of M is local. Let N be any finitely generated submodule of M . By assumption, N is indecomposable. So, by Lemma 16, N is w-local. Since N is finitely generated, N is local.

$(\Leftarrow)$ Since M is uniserial, every submodule of M is hollow by [3, 2.17]. So it is easy to see that every submodule of M is a generalized $\oplus$ -radical supplemented module. $\square$

Corollary 18. *Let R be a local commutative ring. Suppose that every submodule of $E(R/\text{Rad}(R))$ is generalized $\oplus$ -radical supplemented, where $E(R/\text{Rad}(R))$ is the injective hull of the simple module $R/\text{Rad}(R)$. Then R is a uniserial ring.*

Proof. Since $E(R/\text{Rad}(R))$ is uniform, the hypothesis implies that $E(R/\text{Rad}(R))$ is uniserial by Theorem 17. It follows from [13, Lemma 6.2] that R is a uniserial ring. $\square$

Theorem 19. *The following statements are equivalent for a ring R .*

- (1) R is semiperfect.
- (2) Every finitely generated free R -module is generalized $\oplus$ -radical supplemented.

Proof. By [14, Theorem 2.1] and Theorem 9, the proof is clear. $\square$

Lemma 20. *Let M be a module. Suppose that $\text{Rad}(M) = 0$. Then M is a generalized $\oplus$ -radical supplemented module if and only if it is semisimple.*

Proof. This is clear by [15, Proposition 3.3]. $\square$

A ring R is called left V -ring if every simple left R -module is injective. The ring R is called an SSI ring if semisimple left R -module is injective. Let R be a commutative ring. R is regular if and only if every simple left R -module is injective [13, Theorem 2.14].

Proposition 21. *Let R be a left V -ring and M an R -module. Then M is generalized $\oplus$ -radical supplemented if and only if M is semisimple.*

Corollary 22. *Let R be a commutative regular ring and M an R -module. Then M is generalized $\oplus$ -radical supplemented if and only if M is semisimple.*

Proposition 23. *The following statements are equivalent for a ring R .*

- (1) Every generalized $\oplus$ -radical supplemented R -module is injective.
- (2) R is left Noetherian V -ring.

Proof. By [16, Proposition 5.3], the proof is clear. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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