Research Article

On the Logarithmic Regularity Conditions for the Variable Exponent Hardy Type Inequality

Aziz Harman and Mustafa Özgür Keleş

Faculty of Education, Dicle University, 21280 Diyarbakır, Turkey

Correspondence should be addressed to Aziz Harman; aharman@dicle.edu.tr

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We discuss a logarithmic regularity condition in a neighborhood of the origin and infinity on the exponent functions $q(x) \geq p(x)$ and $\beta(x)$ for the variable exponent Hardy inequality $\|x^{\beta(\cdot)-1} \int_0^x f(t) dt\|_{L^{p(\cdot)}(0,l)} \leq C \|x^{\beta(\cdot)} f\|_{L^{p(\cdot)}(0,l)}$ to hold.

1. Introduction

There are several papers devoted to the variable exponent Hardy inequality:

$$\left\| v^{1/q(\cdot)} \int_0^x f(t) dt \right\|_{L^{q(\cdot)}(0,\infty)} \leq \left\| \omega^{1/p(\cdot)} f(\cdot) \right\|_{L^{p(\cdot)}(0,\infty)}. \quad (1)$$

Let $v, w : \mathbb{R} \rightarrow [0, \infty)$ be positive measurable functions and $p^- = \inf\{p(x) : x \in \mathbb{R}\} > 1$, $p^+ = \sup\{p(x) : x \in \mathbb{R}\} < \infty$; then inequality (1) and (2) hold under certain conditions on the weight functions v, w and the exponents $p(\cdot), q(\cdot)$. Two types of conditions arise here: a balance condition on the weights and a regularity condition on the exponents (see below). Necessary and sufficient conditions for the validity of general inequality (1) were found in [1] for the case of $p(x) \leq q(x)$, in [2] for cases $q(0) \geq p(0)$ and $q(\infty) \geq p(\infty)$, in [3] for cases $q(0) < p(0)$ and $q(\infty) < p(\infty)$, and in [4] for mixed cases $q(0) \geq p(0)$ and $q(\infty) < p(\infty)$ ($q(0) < p(0)$ and $q(\infty) \geq p(\infty)$). Some special cases of (1) are studied in [5–10] too.

So, the inequality

$$\left\| x^{\beta(\cdot)-1} Hf(\cdot) \right\|_{L^{p(\cdot)}(0,\infty)} \leq C \left\| x^{\beta(\cdot)} f(\cdot) \right\|_{L^{p(\cdot)}(0,\infty)} \quad (2)$$

is a particular case of (1) when $p(x) = q(x)$, $v(x) = x^{\beta(x)-1}$ and $w(x) = x^{\beta(x)}$, where $Hf(x) = \int_0^x f(t) dt$ is the Hardy operator. For the constant exponents p, β this inequality

holds if $\beta < 1 - (1/p)$, $p > 1$ (see, e.g., [11]). Necessary and sufficient conditions on the $\beta(\cdot), p(\cdot)$ for inequality (2) to hold are $\beta(0) < 1 - (1/p(0))$ and $\beta(\infty) < 1 - (1/p(\infty))$ if the exponents p, q are continuous near the origin and infinity such that the conditions $p, \beta \in \Lambda_0 \cap \Lambda_\infty$ are satisfied (see, e.g., [5–7, 9]). In [8] (see also [10]) it was proved that the condition $p, \beta \in \Lambda_1$ is necessary for inequality (2) to hold if one of these exponents is a constant. Also, it was proved in [10] that the condition Λ_1 is sufficient for inequality (2) to hold on bounded interval $(0, l)$ if a constant A_1 in the condition Λ_1 for the p satisfies $A_1 < p(0)(p(0) - 1)$ and the β is zero. The condition Λ_1 is weaker than Λ_0 . Also the function $p(x) = C/\sqrt{\ln(1/x)}$ satisfies the condition Λ_1 but does not satisfy Λ_0 .

In this note, we will focus on the results of sufficiency and necessity of regularity conditions $\Lambda_0, \Lambda_\infty$, and Λ_1 below for inequality (2) to hold.

The space of functions $L^{p(\cdot)}(0, \infty)$ is introduced as the class of measurable functions $f(x)$ in $(0, \infty)$, which have a finite $I_p(f) := \int_0^\infty |f(x)|^{p(x)} dx$ modular. A norm in $L^{p(\cdot)}(0, \infty)$ is given in the form

$$\|f\|_{L^{p(\cdot)}(0,\infty)} = \inf \left\{ \lambda > 0, I_p \left(\frac{f}{\lambda} \right) \leq 1 \right\}. \quad (3)$$

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [12].

2. Main Results

We will state some sufficiency and necessity assertions concerning inequality (2). Along the way, it will be given a proof for two elementary estimates that we had used.

Let us introduce the following classes of measurable functions. We say, $s(x) : (0, \infty) \rightarrow \mathbb{R}$ is in the class Λ_0 if

$$\Lambda_0 := \limsup_{x \rightarrow 0} |s(x) - s(0)| \ln \frac{1}{|x|} < \infty, \quad (4)$$

is in the class Λ_∞ if

$$\Lambda_\infty := \limsup_{x \rightarrow \infty} |s(x) - s(\infty)| \ln |x| < \infty, \quad (5)$$

and is in the class Λ_1 if

$$\Lambda_1 := \limsup_{x \rightarrow 0} \left| s(x) - s\left(\frac{x}{2}\right) \right| \ln \frac{1}{|x|} < \infty. \quad (6)$$

Theorem 1 (see [8]). Suppose $\beta \in \mathbb{R}$ and $p : (0, l) \rightarrow [1, \infty)$ is an increasing function on $(0, \epsilon)$ such that $p(x)$ is continuous at $x = 0$ and $\beta < 1 - (1/p(0))$, $p^- > 1$; then for the inequality

$$\|x^{\beta(x)-1} Hf\|_{L^{p(\cdot)}(0,l)} \leq C \|x^{\beta(x)} f\|_{L^{p(\cdot)}(0,l)} \quad (7)$$

to hold it is necessary that $p(\cdot) \in \Lambda_1$.

Theorem 2 (see [8]). Suppose $p \in \mathbb{R}$ and $\beta : (0, l) \rightarrow \mathbb{R}$ is a decreasing function on $(0, \epsilon)$ such that $\beta(x)$ is continuous at $x = 0$ and $\beta(0) < 1 - (1/p)$, $p^- > 1$; then for the inequality

$$\|x^{\beta(\cdot)-1} Hf\|_{L^{p(\cdot)}(0,l)} \leq C \|x^{\beta(\cdot)} f\|_{L^{p(\cdot)}(0,l)} \quad (8)$$

to hold it is necessary that $\beta(\cdot) \in \Lambda_1$.

Theorem 3. Suppose $p, \beta : (0, \infty) \rightarrow \mathbb{R}$ is measurable functions such that $1 < \beta^-, \beta^+ < \infty$, $1 < p^-, p^+ < \infty$; then for the inequality

$$\|x^{\beta(\cdot)-1} Hf(\cdot)\|_{L^{p(\cdot)}(0,\infty)} \leq C \|x^{\beta(\cdot)} f(\cdot)\|_{L^{p(\cdot)}(0,\infty)} \quad (9)$$

to hold it is sufficient that $\beta, p \in \Lambda_0 \cap \Lambda_\infty$, whenever $\beta(0) < 1 - (1/p(0))$ and $\beta(\infty) < 1 - (1/p(\infty))$.

Theorem 4. Let $\beta \in \mathbb{R}$, $p : (0, \infty) \rightarrow [1, \infty)$ and let $p^- > 1$. There exists a sequence f_n and a function p , satisfying the conditions $\beta < 1 - (1/p(0))$, $\beta < 1 - (1/p(\infty))$ and

$$\lim_{n \rightarrow \infty} |p(\delta_n) - p(0)| \ln \frac{1}{\delta_n} = \infty : \delta_n = 4^{-n} \quad (10)$$

or

$$\lim_{n \rightarrow \infty} |\beta(\mu_n) - \beta_\infty| \ln \mu_n = \infty : \mu_n = 4^n \quad (11)$$

violating inequality (2).

Theorem 5. Let $p \in \mathbb{R}$, $\beta : (0, \infty) \rightarrow \mathbb{R}$ and let $p > 1$. Then there exists a sequence f_n and a function β satisfying the conditions $\beta_0 < 1 - (1/p)$, $\beta_\infty < 1 - (1/p)$, and

$$\lim_{n \rightarrow \infty} |\beta(\delta_n) - \beta_0| \ln \frac{1}{\delta_n} = \infty : \delta_n = 4^{-n} \quad (12)$$

or

$$\lim_{n \rightarrow \infty} |\beta(\mu_n) - \beta_\infty| \ln \mu_n = \infty : \mu_n = 4^n \quad (13)$$

violating inequality (2).

3. Proof of Main Results

For the proof of Theorems 1 and 2 we refer to [8]. Other proofs of these theorems are given in [10]. The proof of Theorem 3 also is given in [8]. Here we derive an alternative proof of that theorem using the general results of [2, 4].

In the proof of main results we use the following elementary Lemma.

Lemma 6. Suppose $s : \mathbb{R} \rightarrow (0, \infty)$ is a measurable function such that $s \in \Lambda_0 \cap \Lambda_\infty$ and $0 < s^-, s^+ < \infty$; then it holds the estimate

$$C_1^{-1} x^{s(0)} \leq x^{s(x)} \leq C_1 x^{s(0)}, \quad (14)$$

for $0 < x < 1$ and the estimate

$$C_2^{-1} x^{s(\infty)} \leq x^{s(x)} \leq C_2 x^{s(\infty)} \quad (15)$$

for $x \geq 2$.

Proof of Theorem 3. To prove Theorem 3 we apply the results of [2, 4], where it was proved that the following conditions

$$\sup_{x \in (0,1)} V(x)^{1/q(0)} W(x)^{((p(0)-1)/p(0))} < \infty, \quad (16)$$

$$\sup_{x \in (2,\infty)} V(x)^{1/q(\infty)} W(x)^{((p(\infty)-1)/p(\infty))} < \infty$$

are necessary and sufficient for inequality (1) to hold if $q(0) \geq p(0)$, $q(\infty) \geq p(\infty)$, and the regularity conditions are satisfied:

$$\limsup_{x \rightarrow 0} [p(x) - p(0)] \ln \frac{1}{W(x)} < \infty; \quad (17)$$

$$\limsup_{x \rightarrow \infty} [p(x) - p(\infty)] \ln \frac{1}{V(x)} < \infty,$$

where $V(x) = \int_x^\infty v(t) dt$, $W(x) = \int_0^x \omega^{-1/(p(t)-1)} dt$.

In Theorem 3, we have accepted that $q(x) = p(x)$ and $V(x) := V_1(x) = \int_x^\infty t^{(\beta(t)-1)p(t)} dt$, $W(x) := W_1(x) = \int_0^x t^{-(\beta(t)p(t))/(p(t)-1)} dt$. It is easy to show that the conditions $p, \beta \in \Lambda_\infty$ imply $(\beta - 1)p \in \Lambda_\infty$. Therefore, it follows from Lemma 6 that

$$C_3^{-1} x^{(\beta(\infty)-1)p(\infty)} \leq x^{(\beta(x)-1)p(x)} \leq C_3 x^{(\beta(\infty)-1)p(\infty)} \quad (18)$$

for $x \geq 2$ by some $C_3 > 0$. Also the conditions $p, \beta \in \Lambda_0$ imply $-(\beta p/(p-1)) \in \Lambda_0$. Therefore, it follows from Lemma 6 that

$$C_4^{-1} x^{-(\beta(0)p(0)/(p(0)-1))} \leq x^{-(\beta(x)p(x)/(p(x)-1))} \leq C_4 x^{-(\beta(0)p(0)/(p(0)-1))} \quad (19)$$

for $0 < x < 1$ by some $C_4 > 0$. Integrating these inequalities over the intervals (x, ∞) and $(0, x)$, respectively, we get

$$\begin{aligned} C_5^{-1} x^{(\beta(\infty)-1)p(\infty)+1} &\leq V_1(x) \leq C_5 x^{(\beta(\infty)-1)p(\infty)+1}, \quad x > 2, \\ C_6^{-1} x^{1-(\beta(0)p(0)/(p(0)-1))} &\leq W_1(x) \\ &\leq C_6 x^{1-(\beta(0)p(0)/(p(0)-1))}, \quad 0 < x < 1. \end{aligned} \quad (20)$$

To complete the proof of Theorem 3 it suffices to apply estimates (20) to verify conditions (16) and (17). Now, Theorem 3 follows from the upper refereed results of the works [2, 4]. $\square$

Proof of Lemma 6. Let $0 < x \leq \delta$ and let $s(x) \geq s(0)$; then we have $x^{s(x)-s(0)} \leq 1 + \delta^{s^+-s(0)}$, where δ is a certain number from the interval $(0, 1)$. If $0 < x \leq \delta$ and $s(x) < s(0)$ then by condition $s \in \Lambda_0$ we have

$$\begin{aligned} x^{s(x)-s(0)} &= \left(\frac{\delta}{x}\right)^{s(0)-s(x)} \delta^{s(x)-s(0)} \\ &\leq \left(\frac{\delta}{x}\right)^{c_1/\ln(1/x)} \delta^{s(x)-s(0)} \\ &= e^{c_1} \delta^{s(x)-s(0)+c_1/\ln(1/x)} \\ &\leq e^{c_1} \left(1 + \delta^{c_1/\ln(1/\delta)}\right) \left(1 + \left(\frac{1}{\delta}\right)^{s(0)-s^-}\right). \end{aligned} \quad (21)$$

Therefore, for $0 < x \leq \delta$ we have the estimation

$$x^{s(x)-s(0)} \leq C, \quad (22)$$

where the positive constant C depends on $s^-, s^+, s(0), \delta$. Same inequality holds for the function $x^{s(0)-s(x)}$. Indeed, for $0 < x \leq \delta$ and $s(x) \geq s(0)$ we have $x^{s(0)-s(x)} \leq 1 + \delta^{s(0)-s^-}$. If $0 < x \leq \delta$ and $s(x) < s(0)$ by condition $s \in \Lambda_0$ we have

$$\begin{aligned} x^{s(0)-s(x)} &= \left(\frac{\delta}{x}\right)^{s(x)-s(0)} \delta^{s(0)-s(x)} \\ &\leq \left(\frac{\delta}{x}\right)^{c_1/\ln(1/x)} \delta^{s(0)-s(x)} \\ &= e^{c_1} \delta^{s(0)-s(x)+c_1/\ln(1/x)} \\ &\leq e^{c_1} \left(1 + \delta^{c_1/\ln(1/x)}\right) \left(1 + \left(\frac{1}{\delta}\right)^{s^+-s(0)}\right). \end{aligned} \quad (23)$$

By using these inequalities and by the representation

$$x^{s(x)} = x^{s(0)} x^{s(x)-s(0)} \quad (24)$$

we have estimate (14).

To show estimate (15) note that for $x \geq M$ and $s(x) \leq s(\infty)$ we have $x^{s(x)-s(\infty)} \leq 1 + (1/M)^{s(\infty)-s^-}$, where the $M \geq 2$

is a certain number. If $x \geq M$ and $s(x) \geq s(\infty)$ then by the condition $s \in \Lambda_\infty$ we have

$$\begin{aligned} x^{s(x)-s(\infty)} &= \left(\frac{x}{M}\right)^{s(x)-s(\infty)} M^{s(x)-s(\infty)} \\ &\leq \left(\frac{x}{M}\right)^{C_2/\ln x} M^{s(x)-s(\infty)} \\ &= e^{C_2} M^{s(x)-s(\infty)-C_2/\ln x} \\ &\leq e^{C_2} \left(1 + \left(\frac{1}{M}\right)^{C_2/\ln M}\right) \left(1 + M^{s(\infty)-s^+}\right). \end{aligned} \quad (25)$$

Combining the estimates for the functions $x^{s(x)-s(\infty)}$, $x^{s(\infty)-s(x)}$ for $x \geq M$ by the presentation

$$x^{s(x)} = x^{s(\infty)} x^{s(x)-s(\infty)} \quad (26)$$

we get estimate (15). To complete the proof of Lemma 6, note that the condition $s \in \Lambda_0$ is equivalent to

$$-C \leq |s(x) - s(0)| \ln \frac{1}{x} \leq C \quad (27)$$

and the $s \in \Lambda_\infty$ is equivalent to

$$-C \leq |s(x) - s(\infty)| \ln x \leq C, \quad (28)$$

respectively. $\square$

Proof of Theorem 4. Let us assume that $f_k(x) = x^{-(1/\beta(x))-\beta} \chi_{(\delta_k, 2\delta_k)}(x)$; $x \in (0, \infty)$. Fix $k \in \mathbb{N}$. We define the step function

$$p(x) = \begin{cases} p_0 + \alpha_n & \text{if } x \in (2\delta_n, 4\delta_n) \\ p_0 & \text{if } x \in (\delta_n, 2\delta_n), \end{cases} \quad n \in \mathbb{N}. \quad (29)$$

Here α_n is a sequence of positive numbers that satisfies the condition $n\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. Then $\alpha_n \cdot \ln(1/\delta_n) \rightarrow \infty$ as $n \rightarrow \infty$ and condition (10) is fulfilled for the function $p(x)$. We have

$$\begin{aligned} I_{p(\cdot)}(x^{\beta(x)} \cdot f_k(x)) &= \int_{\delta_n}^{2\delta_n} (t^\beta \cdot t^{-(1/p_0)-\beta})^{p_0} dt = \ln 2 < \infty, \\ I_{p(\cdot)}(x^{\beta(x)-1} \cdot Hf_k(x)) &= \int_{2\delta_n}^{4\delta_n} \left[\int_{\delta_n}^{2\delta_n} t^{-(1/p(t))-\beta} dt \right]^{p_0+\alpha_n} \cdot x^{(\beta_0-1) \cdot (p_0+\alpha_n)} dx \\ &\geq \int_{2\delta_n}^{4\delta_n} \delta_n^{(1-(1/p_0)-\beta)(p_0+\alpha_n)} \cdot x^{(\beta_0-1) \cdot (p_0+\alpha_n)} dx \\ &\geq C \cdot \delta_n^{-\alpha_n/p_0} = C \cdot e^{\alpha_n/p_0 \ln(1/\delta_n)} \rightarrow \infty \end{aligned} \quad (30)$$

as $n \rightarrow \infty$. The last relation shows violating of inequality (2) for sufficiently large n .

We define $f_k(x) = x^{-(1/p(x))-\beta} \chi_{(\mu_k, 2\mu_k)}(x)$ for $x \in (0, \infty)$. Fix $k \in \mathbb{N}$. We also define the step function

$$p(x) = \begin{cases} p_\infty - a_n & \text{if } x \in (2\mu_n, 4\mu_n) \\ p_\infty & \text{if } x \in (\mu_n, 2\mu_n), \end{cases} \quad n \in \mathbb{N}, \quad (31)$$

where $n\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. We have $\alpha_n \ln \mu_n \rightarrow \infty$ as $n \rightarrow \infty$ and condition (11) holds for the function $p(x)$. Furthermore,

$$\begin{aligned} I_{p(\cdot)}(x^\beta \cdot f_k(x)) &= \int_{\mu_n}^{2\mu_n} (x^\beta \cdot x^{-(1/p_\infty)-\beta})^{p_\infty} dx = \ln 2 < \infty, \\ I_{p(\cdot)}(x^{\beta-1} \cdot Hf_k(x)) &\geq \int_{2\mu_n}^{4\mu_n} \left[\int_{\mu_n}^{2\mu_n} t^{-(1/p(t))-\beta} dt \right]^{(p_\infty-\alpha_n)} \cdot x^{(\beta-1)(p_\infty-\alpha_n)} dx \\ &\geq C\mu_n^{\alpha_n/p_\infty} = Ce^{(\alpha_n/p_\infty)\ln \mu_n} \rightarrow \infty \end{aligned} \quad (32)$$

as $n \rightarrow \infty$, which contradicts (2) for sufficiently large n . $\square$

Proof of Theorem 5. Let us assume that $f_k(x) = x^{-(1/p)-\beta(x)} \chi_{(\delta_k, 2\delta_k)}(x)$; $x \in (0, \infty)$. Fix $k \in \mathbb{N}$. We define the step function β as

$$\beta(x) = \begin{cases} \beta_0 + a_n & \text{if } x \in (\delta_n, 2\delta_n) \\ \beta_0 & \text{if } x \in (2\delta_n, 4\delta_n), \end{cases} \quad n \in \mathbb{N}, \quad (33)$$

where $a_n \cdot \ln(1/\delta_n) \rightarrow \infty$. Then,

$$\begin{aligned} I_{p(\cdot)}(x^{\beta(x)} \cdot f_k(x)) &= \int_{\delta_n}^{2\delta_n} (x^{\beta_0+\alpha_n} \cdot x^{-(1/p)-\beta_0-\alpha_n})^p dx \\ &= \ln 2 < \infty, \\ I_{p(\cdot)}(x^{\beta(x)-1} \cdot H(f_k(x))) &\geq \int_{2\delta_n}^{4\delta_n} \left[\int_{\delta_n}^{2\delta_n} t^{-(1/p)-\beta_0-\alpha_n} dt \right]^p \cdot x^{(\beta_0-1)p} dx \\ &= C \cdot \delta_n^{-p\alpha_n} = C \cdot e^{p\alpha_n \ln(1/\delta_n)} \rightarrow \infty \end{aligned} \quad (34)$$

as $n \rightarrow \infty$. The last relation contradicts the validity of inequality (2).

We define $f_k(x) = x^{-(1/x)-\beta(x)} \chi_{(\mu_k, 2\mu_k)}(x)$; $x \in (0, \infty)$. Fix $k \in \mathbb{N}$, where the function β is defined as

$$\beta(x) = \begin{cases} \beta_\infty - a_n & \text{if } x \in (\mu_n, 2\mu_n) \\ \beta_\infty & \text{if } x \in (2\mu_n, 4\mu_n), \end{cases} \quad n \in \mathbb{N}, \quad (35)$$

where $\lim_{n \rightarrow \infty} a_n \cdot \ln \mu_n = \infty$. Then,

$$\begin{aligned} I_{p(\cdot)}(x^{\beta(x)} \cdot f_k(x)) &= \int_{\mu_n}^{2\mu_n} (x^{\beta_\infty-\alpha_n} \cdot x^{-(1/p)-\beta_\infty+\alpha_n})^p dx \\ &= \ln 2 < \infty, \\ I_{p(\cdot)}(x^{\beta(x)-1} \cdot H(f_k(x))) &\geq \int_{2\mu_n}^{4\mu_n} \left[\int_{\mu_n}^{2\mu_n} t^{-(1/p)-\beta_\infty+\alpha_n} dt \right]^p \cdot x^{(\beta_\infty-1)p} dx \\ &\geq \int_{2\mu_n}^{3\mu_n} \mu_n^{(1-(1/p)-\beta_\infty+\alpha_n)p} \cdot x^{(\beta_\infty-1)p} dx \\ &\geq C \cdot \mu_n^{p\alpha_n} = C \cdot e^{p\alpha_n \ln \mu} \rightarrow \infty \end{aligned} \quad (36)$$

as $n \rightarrow \infty$ which contradicts inequality (2). $\square$

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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