Research Article

Some Applications of Second-Order Differential Subordination on a Class of Analytic Functions Defined by Komatu Integral Operator

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We introduce a new class of analytic functions by using Komatu integral operator and obtain some subordination results.

1. Introduction, Definitions, and Preliminaries

Let \( \mathbb{R} = (\infty, \infty) \) be the set of real numbers, \( \mathbb{C} \) the set of complex numbers,
\[
\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}
\]
be the set of positive integers, and
\[
\mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}.
\]

Let \( H \) be the class of analytic functions in the open unit disk
\[
U = \{z \in \mathbb{C} : |z| < 1\}
\]
and \( H[a, n] \) the subclass of \( H \) consisting of the functions of the form
\[
f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots.
\]

Let \( \mathcal{A} \) be the class of all functions of the form
\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,
\]
which are analytic in the open unit disk \( U \) with
\[
\mathcal{A}_1 = \mathcal{A}.
\]

Also let \( S \) denote the subclass of \( \mathcal{A} \) consisting of functions \( f \) which are univalent in \( U \).

A function \( f \) analytic in \( U \) is said to be convex if it is univalent and \( f(U) \) is convex.

Let
\[
\mathcal{K} = \left\{ f \in \mathcal{A} : \mathbb{R} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, z \in U \right\}
\]
denote the class of normalized convex functions in \( U \).

Let \( \mathcal{A} \) be the class of all analytic functions \( f \) in \( U \) such that
\[
f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U)
\]
if there exists a Schwarz function \( w \) which is analytic in \( U \) with
\[
w(0) = 0, \quad |w(z)| < 1
\]
such that
\[
f(z) = g(w(z)) \quad (z \in U).
\]

Indeed, it is known that
\[
f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0), \quad f(U) \subset g(U).
\]

Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence [1, page 4]:
\[
f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0), \quad f(U) \subset g(U).
\]
Let \( \psi : C^3 \times U \to C \) be a function and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the (second-order) differential subordination

\[
\psi \left( p(z), zp'(z), z^2 p''(z); z \right) < h(z), \quad (z \in U),
\]

then \( p \) is called a solution of the differential subordination. The univalent function \( q \) is called a dominant of the solutions of the differential subordination, or more simply a dominant, if \( p < q \) for all \( p \) satisfying (13).

A dominant \( \bar{q} \), which satisfies \( \bar{q} < q \) for all dominants \( q \) of (13), is said to be the best dominant of (13).

Recently, Komatu [2] introduced a certain integral operator \( L^\delta_c \) defined by

\[
L^\delta_c f(z) = \frac{\zeta^\delta}{\Gamma(\delta)} \int_0^1 t^{-2} \left( \log \frac{1}{t} \right)^{\delta-1} f(zt) \, dt,
\]

(14)

\( (z \in U; c > 0; \delta \geq 0; f(z) \in A) \).

Thus, if \( f \in A \) is of the form (5), then it is easily seen from (14) that (see [21])

\[
L^\delta_c f(z) = z + \sum_{k=1}^{\infty} \frac{c}{c+k-1} \delta^k z^k \quad (c > 0; \delta \geq 0).
\]

(15)

Using the relation (15), it is easy verify that

\[
z \left( L^\delta_{c+1} f(z) \right)' = cL^\delta_c f(z) - (c-1) L^\delta_{c+1} f(z),
\]

(16)

\[
L^\delta_c \left( zp'(z) \right) = z \left( L^\delta_c f(z) \right)'.
\]

We note the following.

(i) For \( c = 1 \) and \( \delta = m \) (\( m \) is any integer), the multiplier transformation \( L_1^m f(z) = f^m(z) \) was studied by Flett [3] and Salagean [4].

(ii) For \( c = 1 \) and \( \delta = -m \) (\( m \in \mathbb{N}_0 \)), the differential operator \( L_1^{-m} f(z) = D^{-m} f(z) \) was studied by Salagean [4].

(iii) For \( c = 2 \) and \( \delta = m \) (\( m \) is any integer), the operator \( L_2^m f(z) = f^m(z) \) was studied by Uralegaddi and Somanatha [5].

(iv) For \( c = 2 \), the multiplier transformation \( L_2^\delta f(z) = f^\delta(z) \) was studied by Jung et al. [6].

Using the operator \( L^\delta_c \), we now introduce the following class.

Definition 1. Let \( \mathcal{R}_{c,\delta}(\beta) \) be the class of functions \( f \in A \) satisfying

\[
\mathcal{R} \left\{ \left( L^\delta_c f(z) \right)' \right\} > \beta,
\]

(17)

where \( z \in U, 0 \leq \beta < 1 \), and \( L^\delta_c \) is the Komatu integral operator.

In order to prove our main results, we will make use of the following lemmas.

Lemma 2 (see [7]). Let \( h \) be a convex function with \( h(0) = a \) and let \( \gamma \in C^* := C - \{0\} \) be a complex number with \( \Re \{\gamma\} \geq 0 \). If \( p \in \mathcal{H}[a, n] \) and

\[
p(z) + \frac{1}{\gamma} zp'(z) < h(z) \quad (z \in U),
\]

then

\[
p(z) < q(z) < h(z) \quad (z \in U),
\]

(18)

(19)

where

\[
q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z t^{\gamma/n-1} h(t) \, dt \quad (z \in U).
\]

The function \( q \) is convex and is the best dominant.

Lemma 3 (see [8]). Let \( \Re \{\mu\} > 0, n \in \mathbb{N}, \) and let

\[
\omega = \frac{r^2 + |\mu|^2 - |r^2 - \mu|}{4n \Re \{\mu\}}.
\]

(21)

Let \( h \) be an analytic function in \( U \) with \( h(0) = 1 \) and suppose that

\[
\mathcal{R} \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > -\omega.
\]

(22)

If

\[
p(z) + \frac{1}{\mu} zp'(z) < h(z),
\]

then

\[
p(z) < q(z),
\]

(23)

(24)

(25)

where \( q \) is a solution of the differential equation

\[
q(z) + \frac{n}{\mu} q'(z) = h(z), \quad q(0) = 1,
\]

(26)

given by

\[
q(z) = \frac{\mu}{nz^{\mu/n}} \int_0^z t^{\mu/n-1} h(t) \, dt.
\]

(27)

Moreover \( q \) is the best dominant.

2. Main Results

Theorem 4. The set \( \mathcal{R}_{c,\delta}(\beta) \) is convex.
Proof. Let

\[ f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (z \in U; j = 1, 2, \ldots, l) \tag{28} \]

be in the class \( R_{c,d}(\beta) \). Then, by Definition 1, we have

\[ \Re \left\{ \left( L_c^d f_j(z) \right)' \right\} = \Re \left\{ 1 + \sum_{k=2}^{\infty} k \left( \frac{c}{c + k - 1} \right)^d a_{k,j} z^{k-1} \right\} > \beta. \tag{29} \]

For any nonnegative numbers \( \lambda_1, \lambda_2, \ldots, \lambda_l \) such that

\[ \lambda_1 + \lambda_2 + \cdots + \lambda_l = 1, \tag{30} \]

we must show that the function

\[ h(z) = \sum_{j=1}^{l} \lambda_j f_j(z) \tag{31} \]

is in \( R_{c,d}(\beta) \); that is,

\[ \Re \left\{ \left( L_c^d h(z) \right)' \right\} > \beta. \tag{32} \]

By (28) and (31), we have

\[ h(z) = z + \sum_{k=2}^{\infty} \left( \sum_{j=1}^{l} \lambda_j a_{k,j} \right) z^k. \tag{33} \]

Therefore we get

\[ L_c^d h(z) = z + \sum_{k=2}^{\infty} \left( \sum_{j=1}^{l} \lambda_j a_{k,j} \right) z^k. \tag{34} \]

Differentiating (34) with respect to \( z \), we obtain

\[ (L_c^d h(z))' = 1 + \sum_{k=2}^{\infty} k \left( \frac{c}{c + k - 1} \right)^d \left( \sum_{j=1}^{l} \lambda_j a_{k,j} \right) z^{k-1}. \tag{35} \]

So we get

\[ \Re \left\{ (L_c^d h(z))' \right\} = 1 + \sum_{j=1}^{l} \lambda_j \Re \left\{ \sum_{k=2}^{\infty} k \left( \frac{c}{c + k - 1} \right)^d a_{k,j} z^{k-1} \right\} > 1 + \sum_{j=1}^{l} \lambda_j (\beta - 1) \quad \text{(by (29))} \]

\[ = \beta \tag{36} \]

since \( \lambda_1 + \lambda_2 + \cdots + \lambda_l = 1 \). Therefore we get the desired result.

Theorem 5. Let \( q \) be convex function in \( U \) with \( q(0) = 1 \) and let

\[ h(z) = q(z) + \frac{1}{\gamma + 2} z q'(z) \quad (z \in U), \tag{37} \]

where \( \gamma \) is a complex number with \( \Re \{\gamma\} > -2 \). If \( f \in R_{c,d}(\beta) \) and \( \mathcal{F} = \mathcal{J}_\gamma f \), where

\[ \mathcal{F}(z) = \mathcal{J}_\gamma f(z) = \frac{\gamma + 2}{\gamma} \int_0^z t^\gamma f(t) dt, \tag{38} \]

then

\[ \left( L_c^d f(z) \right)' < h(z) \tag{39} \]

implies

\[ \left( L_c^d \mathcal{F}(z) \right)' < q(z), \tag{40} \]

and this result is sharp.

Proof. From the equality (38), we get

\[ z^{\gamma+1} \mathcal{F}(z) = (\gamma + 2) \int_0^z t^\gamma f(t) dt. \tag{41} \]

Differentiating (41) with respect to \( z \), we have

\[ (\gamma + 1) \mathcal{F}(z) + z \mathcal{F}'(z) = (\gamma + 2) f(z), \tag{42} \]

\[ (\gamma + 1) L_c^d \mathcal{F}(z) + z (L_c^d \mathcal{F}(z))' = (\gamma + 2) L_c^d f(z). \tag{43} \]

Differentiating (43) with respect to \( z \), we obtain

\[ (L_c^d \mathcal{F}(z))' + \frac{1}{\gamma + 2} z (L_c^d \mathcal{F}(z))'' = (L_c^d f(z))'. \tag{44} \]

Using the differential subordination (39) in the equality (44), we get

\[ (L_c^d \mathcal{F}(z))' + \frac{1}{\gamma + 2} z (L_c^d \mathcal{F}(z))'' < h(z). \tag{45} \]

Let us define

\[ p(z) = \left( L_c^d \mathcal{F}(z) \right)' \tag{46} \]

Then a simple computation yields

\[ p(z) = \left[ z + \sum_{k=2}^{\infty} \left( \frac{c}{c + k - 1} \right)^d \frac{\gamma + 2}{\gamma + k + 1} a_k z^k \right]' \tag{47} \]

\[ = 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathcal{H}[1, 1]). \]

Using (46) in the subordination (45), we have

\[ p(z) + \frac{1}{\gamma + 2} z p'(z) < h(z) \tag{48} \]

\[ = q(z) + \frac{1}{\gamma + 2} z q'(z) \quad (z \in U). \]

Using Lemma 2, we obtain

\[ p(z) < q(z) \tag{49} \]

which is desired result. Moreover \( q \) is the best dominant.
Example 6. If we take 
\[ \gamma = i, \quad q(z) = \frac{1}{1 - z} \] (50)
in Theorem 5, then we have
\[ h(z) = \frac{(2+i) - z (1+i)}{(2+i) (1-z)^2} \] (51)
If \( f \in \mathcal{R}_{c, \delta} (\beta) \) and \( F \) is given by
\[ F(z) = \mathcal{I}_{\gamma, f}(z) = \frac{2 + i}{2^{1+\beta}} \int_0^z t^\beta f(t) \, dt, \] (52)
then by Theorem 5, we have
\[ \left( L^\delta f(z) \right)' < \frac{(2+i) - z (1+i)}{(2+i) (1-z)^2} \Rightarrow \left( L^\delta F(z) \right)' < \frac{1}{1 - z} \quad (z \in \mathbb{U}). \] (53)
Theorem 7. Let \( \Re \{ \gamma \} > -2 \) and let
\[ w = \frac{1 + |\gamma + 2|^2 - |\gamma^2 + 4\gamma + 3|}{4 \Re \{ \gamma + 2 \}}. \] (54)
Let \( h \) be an analytic function in \( \mathbb{U} \) with \( h(0) = 1 \) and suppose that
\[ \Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > -w. \] (55)
If \( f \in \mathcal{R}_{c, \delta} (\beta) \) and \( \mathcal{F} = \mathcal{J}_f, \) where \( \mathcal{F} \) is defined by (38), then
\[ \left( L^\delta f(z) \right)' < h(z) \] (56)
implies
\[ \left( L^\delta \mathcal{F}(z) \right)' < q(z), \] (57)
where \( q \) is the solution of the differential equation
\[ h(z) = q(z) + \frac{1}{\gamma + 2} z q'(z), \quad q(0) = 1, \] (58)
given by
\[ q(z) = \frac{\gamma + 2}{2^{1+\beta}} \int_0^z t^{\beta+1} h(t) \, dt. \] (59)
Moreover \( q \) is the best dominant.

Proof. We consider \( n = 1 \) and \( \mu = \gamma + 2 \) in Lemma 3. Then the proof is easily seen by means of the proof of Theorem 5. \( \square \)

Letting
\[ h(z) = \frac{1 - (2\beta - 1) z}{1 - z} \quad (0 \leq \beta < 1) \] (60)
in Theorem 7, we obtain the following interesting result.

Corollary 8. If \( 0 \leq \beta < 1, \ c > 0, \ \delta \geq 0, \ \Re \{ \gamma \} > -2, \) and \( \mathcal{F} = \mathcal{J}_f \) is defined by (38), then
\[ \mathcal{J}_f (\mathcal{R}_{c, \delta}(\beta)) \subset \mathcal{R}_{c, \delta}(\rho), \] (61)
where
\[ \rho = \min_{|z|=1} \Re \{ q(z) \} = \rho(\gamma, \beta) \] (62)
and this result is sharp. Moreover
\[ \rho = \rho(\gamma, \beta) = 2\beta - 1 + 2(\gamma + 2)(1 - \beta) \tau(\gamma), \] (63)
where
\[ \tau(\gamma) = \int_0^1 \frac{t^{\gamma+1}}{1-t} \, dt. \] (64)
Proof. If we let
\[ h(z) = \frac{1 - (2\beta - 1) z}{1 - z} \quad (0 \leq \beta < 1), \] (65)
then \( h \) is convex and by Theorem 7, we deduce
\[ \left( L^\delta \mathcal{F}(z) \right)' < q(z) = \frac{\gamma + 2}{2^{1+\beta}} \int_0^z t^{\beta+1} 1 - (2\beta - 1) t \, dt \]
\[ = 2\beta - 1 + 2(\gamma + 2)(1 - \beta) \tau(\gamma), \] (66)
On the other hand if \( \Re \{ \gamma \} > -2, \) then from the convexity of \( q \) and the fact that \( q(\mathbb{U}) \) is symmetric with respect to the real axis, we get
\[ \Re \left\{ \left( L^\delta \mathcal{F}(z) \right)' \right\} \geq \min_{|z|=1} \Re \{ q(z) \} = \Re \{ q(1) \} = \rho(\gamma, \beta) \]
\[ = 2\beta - 1 + 2(\gamma + 2)(1 - \beta) \tau(\gamma), \] (67)
where \( \tau(\gamma) \) is given by (64). From (66), we have
\[ \mathcal{J}_f (\mathcal{R}_{c, \delta}(\beta)) \subset \mathcal{R}_{c, \delta}(\rho), \] (68)
where \( \rho \) is given by (63). \( \square \)

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References


