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Research Article

Some Applications of Second-Order Differential Subordination on a Class of Analytic Functions Defined by Komatu Integral Operator

Serap Bulut

Civil Aviation College, Kocaeli University, Arslanbey Campus, 41285 Kartepe-Kocaeli, Turkey

Correspondence should be addressed to Serap Bulut; bulutserap@yahoo.com

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We introduce a new class of analytic functions by using Komatu integral operator and obtain some subordination results.

1. Introduction, Definitions, and Preliminaries

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers,

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\} \tag{1}$$

be the set of positive integers, and

$$\mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}. \tag{2}$$

Let $\mathscr{H}$ be the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \tag{3}$$

and $\mathscr{H}[a, n]$ the subclass of $\mathscr{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots. \tag{4}$$

Let $\mathscr{A}_n$ be the class of all functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \tag{5}$$

which are analytic in the open unit disk $\mathbb{U}$ with

$$\mathscr{A}_1 = \mathscr{A}. \tag{6}$$

Also let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

A function f analytic in $\mathbb{U}$ is said to be convex if it is univalent and $f(\mathbb{U})$ is convex.

Let

$$\mathscr{K} = \left\{ f \in \mathscr{A} : \Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0, z \in \mathbb{U} \right\} \tag{7}$$

denote the class of normalized convex functions in $\mathbb{U}$.

If f and g are analytic in $\mathbb{U}$, then we say that f is subordinate to g, written symbolically as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}) \tag{8}$$

if there exists a Schwarz function w which is analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \tag{9}$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{10}$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{11}$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence [1, page 4]:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{12}$$

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ be a function and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the (second-order) differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \quad (z \in \mathbb{U}), \tag{13}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (13).

A dominant $\tilde{q}$, which satisfies $\tilde{q} \prec q$ for all dominants q of (13), is said to be the best dominant of (13).

Recently, Komatu [2] introduced a certain integral operator L_c^δ defined by

$$L_c^\delta f(z) = \frac{c^\delta}{\Gamma(\delta)} \int_0^1 t^{c-2}\left(\log\frac{1}{t}\right)^{\delta-1} f(zt)\,dt, \quad (z \in \mathbb{U}; c > 0; \delta \geq 0; f(z) \in \mathscr{A}). \tag{14}$$

Thus, if $f \in \mathscr{A}$ is of the form (5), then it is easily seen from (14) that (see [2])

$$L_c^\delta f(z) = z + \sum_{k=2}^{\infty}\left(\frac{c}{c+k-1}\right)^\delta a_k z^k \quad (c > 0; \delta \geq 0). \tag{15}$$

Using the relation (15), it is easy verify that

$$z\left(L_c^{\delta+1} f(z)\right)' = cL_c^\delta f(z) - (c-1)L_c^{\delta+1} f(z), \quad L_c^\delta\left(zf'(z)\right) = z\left(L_c^\delta f(z)\right)'. \tag{16}$$

We note the following.

(i) For $c = 1$ and $\delta = m$ (m is any integer), the multiplier transformation $L_1^m f(z) = I^m f(z)$ was studied by Flett [3] and Sălăgean [4].

(ii) For $c = 1$ and $\delta = -m$ ($m \in \mathbb{N}_0$), the differential operator $L_1^{-m} f(z) = D^m f(z)$ was studied by Sălăgean [4].

(iii) For $c = 2$ and $\delta = m$ (m is any integer), the operator $L_2^m f(z) = L^m f(z)$ was studied by Uralegaddi and Somanatha [5].

(iv) For $c = 2$, the multiplier transformation $L_2^\delta f(z) = I^\delta f(z)$ was studied by Jung et al. [6].

Using the operator L_c^δ, we now introduce the following class.

Definition 1. Let $\mathscr{R}_{c,\delta}(\beta)$ be the class of functions $f \in \mathscr{A}$ satisfying

$$\Re\left\{\left(L_c^\delta f(z)\right)'\right\} > \beta, \tag{17}$$

where $z \in \mathbb{U}$, $0 \leq \beta < 1$, and L_c^δ is the Komatu integral operator.

In order to prove our main results, we will make use of the following lemmas.

Lemma 2 (see [7]). *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^* := \mathbb{C} - \{0\}$ be a complex number with $\Re\{\gamma\} \geq 0$. If $p \in \mathscr{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z) \quad (z \in \mathbb{U}), \tag{18}$$

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in \mathbb{U}), \tag{19}$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z t^{\gamma/n-1} h(t)\,dt \quad (z \in \mathbb{U}). \tag{20}$$

The function q is convex and is the best dominant.

Lemma 3 (see [8]). *Let $\Re\{\mu\} > 0$, $n \in \mathbb{N}$, and let*

$$w = \frac{n^2 + |\mu|^2 - |n^2 - \mu^2|}{4n\Re\{\mu\}}. \tag{21}$$

Let h be an analytic function in $\mathbb{U}$ with $h(0) = 1$ and suppose that

$$\Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w. \tag{22}$$

If

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots \tag{23}$$

is analytic in $\mathbb{U}$ and

$$p(z) + \frac{1}{\mu} zp'(z) \prec h(z), \tag{24}$$

then

$$p(z) \prec q(z), \tag{25}$$

where q is a solution of the differential equation

$$q(z) + \frac{n}{\mu} zq'(z) = h(z), \quad q(0) = 1, \tag{26}$$

given by

$$q(z) = \frac{\mu}{nz^{\mu/n}} \int_0^z t^{\mu/n-1} h(t)\,dt. \tag{27}$$

Moreover q is the best dominant.

2. Main Results

Theorem 4. *The set $\mathscr{R}_{c,\delta}(\beta)$ is convex.*

Proof. Let

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (z \in \mathbb{U}; j = 1, 2, \ldots, l) \tag{28}$$

be in the class $\mathscr{R}_{c,\delta}(\beta)$. Then, by Definition 1, we have

$$\Re\left\{\left(L_c^\delta f_j(z)\right)'\right\} = \Re\left\{1 + \sum_{k=2}^{\infty} k\left(\frac{c}{c+k-1}\right)^\delta a_{k,j} z^{k-1}\right\} > \beta. \tag{29}$$

For any nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_l$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_l = 1, \tag{30}$$

we must show that the function

$$h(z) = \sum_{j=1}^{l} \lambda_j f_j(z) \tag{31}$$

is in $\mathscr{R}_{c,\delta}(\beta)$; that is,

$$\Re\left\{\left(L_c^\delta h(z)\right)'\right\} > \beta. \tag{32}$$

By (28) and (31), we have

$$h(z) = z + \sum_{k=2}^{\infty}\left(\sum_{j=1}^{l} \lambda_j a_{k,j}\right) z^k. \tag{33}$$

Therefore we get

$$L_c^\delta h(z) = z + \sum_{k=2}^{\infty}\left(\frac{c}{c+k-1}\right)^\delta \left(\sum_{j=1}^{l} \lambda_j a_{k,j}\right) z^k. \tag{34}$$

Differentiating (34) with respect to z, we obtain

$$\left(L_c^\delta h(z)\right)' = 1 + \sum_{k=2}^{\infty} k\left(\frac{c}{c+k-1}\right)^\delta \left(\sum_{j=1}^{l} \lambda_j a_{k,j}\right) z^{k-1}. \tag{35}$$

So we get

$$\begin{aligned}\Re\left\{\left(L_c^\delta h(z)\right)'\right\} &= 1 + \sum_{j=1}^{l} \lambda_j \Re\left\{\sum_{k=2}^{\infty} k\left(\frac{c}{c+k-1}\right)^\delta a_{k,j} z^{k-1}\right\} \\ &> 1 + \sum_{j=1}^{l} \lambda_j (\beta - 1) \quad (\text{by } (29)) \\ &= \beta\end{aligned} \tag{36}$$

since $\lambda_1 + \lambda_2 + \cdots + \lambda_l = 1$. Therefore we get the desired result. □

Theorem 5. *Let q be convex function in $\mathbb{U}$ with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{1}{\gamma+2} z q'(z) \quad (z \in \mathbb{U}), \tag{37}$$

where γ is a complex number with $\Re\{\gamma\} > -2$. If $f \in \mathscr{R}_{c,\delta}(\beta)$ and $\mathscr{F} = \mathscr{I}_\gamma f$, where

$$\mathscr{F}(z) = \mathscr{I}_\gamma f(z) = \frac{\gamma+2}{z^{\gamma+1}} \int_0^z t^\gamma f(t)\, dt, \tag{38}$$

then

$$\left(L_c^\delta f(z)\right)' \prec h(z) \tag{39}$$

implies

$$\left(L_c^\delta \mathscr{F}(z)\right)' \prec q(z), \tag{40}$$

and this result is sharp.

Proof. From the equality (38), we get

$$z^{\gamma+1} \mathscr{F}(z) = (\gamma+2) \int_0^z t^\gamma f(t)\, dt. \tag{41}$$

Differentiating (41) with respect to z, we have

$$(\gamma+1)\mathscr{F}(z) + z\mathscr{F}'(z) = (\gamma+2) f(z), \tag{42}$$

$$(\gamma+1) L_c^\delta \mathscr{F}(z) + z\left(L_c^\delta \mathscr{F}(z)\right)' = (\gamma+2) L_c^\delta f(z). \tag{43}$$

Differentiating (43) with respect to z, we obtain

$$\left(L_c^\delta \mathscr{F}(z)\right)' + \frac{1}{\gamma+2} z\left(L_c^\delta \mathscr{F}(z)\right)'' = \left(L_c^\delta f(z)\right)'. \tag{44}$$

Using the differential subordination (39) in the equality (44), we get

$$\left(L_c^\delta \mathscr{F}(z)\right)' + \frac{1}{\gamma+2} z\left(L_c^\delta \mathscr{F}(z)\right)'' \prec h(z). \tag{45}$$

Let us define

$$p(z) = \left(L_c^\delta \mathscr{F}(z)\right)'. \tag{46}$$

Then a simple computation yields

$$\begin{aligned} p(z) &= \left[z + \sum_{k=2}^{\infty}\left(\frac{c}{c+k-1}\right)^\delta \frac{\gamma+2}{\gamma+k+1} a_k z^k\right]' \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathscr{H}[1,1]). \end{aligned} \tag{47}$$

Using (46) in the subordination (45), we have

$$\begin{aligned} p(z) + \frac{1}{\gamma+2} z p'(z) &\prec h(z) \\ &= q(z) + \frac{1}{\gamma+2} z q'(z) \quad (z \in \mathbb{U}). \end{aligned} \tag{48}$$

Using Lemma 2, we obtain

$$p(z) \prec q(z) \tag{49}$$

which is desired result. Moreover q is the best dominant. □

Example 6. If we take

$$\gamma = i, \qquad q(z) = \frac{1}{1-z} \tag{50}$$

in Theorem 5, then we have

$$h(z) = \frac{(2+i) - z(1+i)}{(2+i)(1-z)^2}. \tag{51}$$

If $f \in \mathcal{R}_{c,\delta}(\beta)$ and $\mathcal{F}$ is given by

$$\mathcal{F}(z) = \mathcal{I}_i f(z) = \frac{2+i}{z^{1+i}} \int_0^z t^i f(t)\, dt, \tag{52}$$

then by Theorem 5, we have

$$\begin{aligned} \left(L_c^\delta f(z)\right)' &\prec \frac{(2+i) - z(1+i)}{(2+i)(1-z)^2} \Longrightarrow \left(L_c^\delta \mathcal{F}(z)\right)' \\ &\prec \frac{1}{1-z} \quad (z \in \mathbb{U}). \end{aligned} \tag{53}$$

Theorem 7. *Let $\Re\{\gamma\} > -2$ and let*

$$w = \frac{1 + |\gamma+2|^2 - |\gamma^2 + 4\gamma + 3|}{4\Re\{\gamma+2\}}. \tag{54}$$

Let h be an analytic function in $\mathbb{U}$ with $h(0) = 1$ and suppose that

$$\Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w. \tag{55}$$

If $f \in \mathcal{R}_{c,\delta}(\beta)$ and $\mathcal{F} = \mathcal{I}_\gamma^\delta f$, where $\mathcal{F}$ is defined by (38), *then*

$$\left(L_c^\delta f(z)\right)' \prec h(z) \tag{56}$$

implies

$$\left(L_c^\delta \mathcal{F}(z)\right)' \prec q(z), \tag{57}$$

where q is the solution of the differential equation

$$h(z) = q(z) + \frac{1}{\gamma+2} z q'(z), \quad q(0) = 1, \tag{58}$$

given by

$$q(z) = \frac{\gamma+2}{z^{\gamma+2}} \int_0^z t^{\gamma+1} h(t)\, dt. \tag{59}$$

Moreover q is the best dominant.

Proof. We consider $n = 1$ and $\mu = \gamma + 2$ in Lemma 3. Then the proof is easily seen by means of the proof of Theorem 5. □

Letting

$$h(z) = \frac{1 - (2\beta - 1) z}{1 - z} \quad (0 \le \beta < 1) \tag{60}$$

in Theorem 7, we obtain the following interesting result.

Corollary 8. *If $0 \le \beta < 1$, $c > 0$, $\delta \ge 0$, $\Re\{\gamma\} > -2$, and $\mathcal{F} = \mathcal{I}_\gamma f$ is defined by* (38), *then*

$$\mathcal{I}_\gamma\left(\mathcal{R}_{c,\delta}(\beta)\right) \subset \mathcal{R}_{c,\delta}(\rho), \tag{61}$$

where

$$\rho = \min_{|z|=1} \Re\{q(z)\} = \rho(\gamma, \beta) \tag{62}$$

and this results is sharp. Moreover

$$\rho = \rho(\gamma, \beta) = 2\beta - 1 + 2(\gamma+2)(1-\beta)\tau(\gamma), \tag{63}$$

where

$$\tau(\gamma) = \int_0^1 \frac{t^{\gamma+1}}{1-t} dt. \tag{64}$$

Proof. If we let

$$h(z) = \frac{1 - (2\beta - 1) z}{1 - z} \quad (0 \le \beta < 1), \tag{65}$$

then h is convex and by Theorem 7, we deduce

$$\begin{aligned} \left(L_c^\delta \mathcal{F}(z)\right)' \prec q(z) &= \frac{\gamma+2}{z^{\gamma+2}} \int_0^z t^{\gamma+1} \frac{1 - (2\beta-1)t}{1-t} dt \\ &= 2\beta - 1 + \frac{2(\gamma+2)(1-\beta)}{z^{\gamma+2}} \int_0^z \frac{t^{\gamma+1}}{1-t} dt. \end{aligned} \tag{66}$$

On the other hand if $\Re\{\gamma\} > -2$, then from the convexity of q and the fact that $q(\mathbb{U})$ is symmetric with respect to the real axis, we get

$$\begin{aligned} \Re\left\{\left(L_c^\delta \mathcal{F}(z)\right)'\right\} &\ge \min_{|z|=1} \Re\{q(z)\} = \Re\{q(1)\} = \rho(\gamma, \beta) \\ &= 2\beta - 1 + 2(\gamma+2)(1-\beta)\tau(\gamma), \end{aligned} \tag{67}$$

where $\tau(\gamma)$ is given by (64). From (66), we have

$$\mathcal{I}_\gamma\left(\mathcal{R}_{c,\delta}(\beta)\right) \subset \mathcal{R}_{c,\delta}(\rho), \tag{68}$$

where ρ is given by (63). □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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